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Well-Posedness of the Mixed Problem for the Degenerate Multi-Dimensional Elliptic Equations

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Abstract. The boundary-value problems for elliptic PDEs are of fundamental importance for mathematical physics. Some of their applications lead to the analysis of degenerate PDEs of elliptic type. The well-posedness (correctness) of boundary-value problems for elliptic equations on the plane has been well studied using the methods of analytic functions of a complex variable. However, fundamental problems arise when investigating similar problems if the number of independent variables exceeds two. In this paper, we prove the unique solvability and obtain the explicit form of the classical solution of the mixed boundary-value problem for degenerate elliptic PDEs with the Chaplygin operator.

Key Words and Phrases: well-posedness, mixed problem, degenerate elliptic PDEs, Bessel functions.

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1. Introduction

The boundary-value problems for elliptic PDEs are of fundamental importance for mathematical physics, mainly because the analysis of stationary processes of various physical phenomena (such as oscillations, heat transfer, diffusion, etc.) generally leads to obtaining a PDE of elliptic type. Key examples of their applications are the diffusion of radiowaves through hollow metallic tubes, electromagnetic oscillations in hollow resonators (widely used in electrotechnics), and the distribution of variable electric current through the section of a conductor (the so-called "skin effect"). Some of these applications lead to the analysis of degenerate PDEs of elliptic type (see [11]).

The well-posedness (correctness) of boundary-value problems for elliptic equations on the plane has been well studied using the methods of analytic functions

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of a complex variable (see [1, 2]). However, fundamental problems arise when investigating similar problems under the number of independent variables greater than two. The attractive and convenient method of singular integral equations loses its validity due to the absence of a complete theory of multi-dimensional singular integral equations.

This paper uses an alternative method to prove the unique solvability and to obtain the explicit form of the classical solution of the mixed boundary-value problem for degenerate elliptic PDEs with the Chaplygin operator.

2. Problem statement and main result

Let D_α be a cylindric domain in the Euclidean space E_{m+1} of points $(x_1, ..., x_m, t)$, bounded by the cylinder $\Gamma = \{(x,t) : |x| = 1\}$ and the planes $t = \alpha > 0$ and $t = 0$, where $|x|$ is the length of the vector $x = (x_1, ..., x_m)$.

Let's denote by Γ_α, S_α, and S_0, respectively, the parts of the surfaces that form the boundary ∂D_α of the domain D_α.

We study, in the domain D_α, the mutually adjoint degenerate multi-dimensional elliptic equations:

$$Lu \equiv g(t)\Delta_x u + u_{tt} + \sum_{i=1}^{m} a_i(x,t)u_{x_i} + b(x,t)u_t + c(x,t)u = 0, \tag{1}$$

$$L^*\upsilon \equiv g(t)\Delta_x \upsilon + \upsilon_{tt} - \sum_{i=1}^{m} a_i \upsilon_{x_i} - b\upsilon_t + d\upsilon = 0, \tag{1*}$$

where $g(t) > 0$ for $t > 0$, $g(0) = 0$, $g(t) \in C([0,\alpha]) \cap C^2((0,\alpha))$, Δ_x is the Laplace operator of the variables $x_1, ..., x_m,\ m \geq 2$, and $d(x,t) = c - \sum\limits_{i=1}^{m} a_{ix_i} - b_t$.

As a preliminary step in our analysis, let us switch from the Cartesian coordinates $x_1, ..., x_m, t$ to the spherical ones $r, \theta_1, ..., \theta_{m-1}, t,\ r \geq 0,\ 0 \leq \theta_1 < 2\pi,\ 0 \leq \theta_i \leq \pi,\ i = 2, 3, ..., m-1$.

Problem 1. Find the solution of the equation (1) in the domain D_α, belonging to the the class $C^1(\overline{D}_\alpha) \cap C^2(D_\alpha)$, that satisfies the following boundary conditions:

$$u|_{S_0} = \tau(r,\theta), \quad u_t|_{S_0} = \nu(r,\theta) \quad u|_{\Gamma_\alpha} = \psi(t,\theta), \tag{2}$$

where $\tau(1,\theta) = \psi(0,\theta)$, $\nu(1,\theta) = \psi_t(0,\theta)$.

Let $\left\{Y_{n,m}^k(\theta)\right\}$ be a system of linearly independent spherical functions of order $n,\ 1 \leq k \leq k_n,\ (m-2)!n!k_n = (n+m-3)!(2n+m-2)$, $\theta = (\theta_1, ..., \theta_{m-1})$. Let also $W_2^l(S_0), l = 0, 1, ...$ be the Sobolev spaces.

We will need the following useful lemmas ([3]).

Lemma 1. *Let $f(r,\theta) \in W_2^l(S_0)$. If $l \geq m-1$, then the series*

$$f(r,\theta) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} f_n^k(r) Y_{n,m}^k(\theta), \tag{3}$$

as well as the series obtained by its differentiation of order $p \leq l-m+1$, converge absolutely and uniformly.

Lemma 2. *For $f(r,\theta) \in W_2^l(S_0)$, it is necessary and sufficient that the coefficients of the series (3) satisfy the inequalities*

$$|f_0^1(r)| \leq c_1, \ \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} n^{2l}|f_n^k(r)|^2 \leq c_2, \ c_1, c_2 = const.$$

Let's denote by $\tilde{a}_{in}^k(r,t)$, $a_{in}^k(r,t)$, $\tilde{b}_n^k(r,t)$, $\tilde{c}_n^k(r,t)$, $\tilde{d}_n^k(r,t)$, ρ_n^k, $\bar{\tau}_n^k(r)$, $\bar{\nu}_n^k(r)$, $\psi_n^k(t)$, the coefficients of the series (3) of the functions $a_i(r,\theta,t)\rho(\theta)$, $a_i\frac{x_i}{r}\rho$, $b(r,\theta,t)\rho$, $c(r,\theta,t)\rho$, $d(r,\theta,t)\rho$, $\rho(\theta)$, $i = 1,...,m$, $\tau(r,\theta)$, $\nu(t,\theta)$, $\psi(r,\theta)$, respectively, where $\rho(\theta) \in C^\infty(H)$ and H is a unit sphere in E_m.

Let $a_i(r,\theta,t)$, $b(r,\theta,t)$, $c(r,\theta,t) \in W_2^l(D_\alpha) \subset C(\overline{D}_\alpha)$, $l \geq m+1$, $i = 1,...,m$ and $c(r,\theta,t) \leq 0$, $\forall (r,\theta,t) \in D_\alpha$.

The following theorem is the main result of this paper:

Theorem 1. *Let the functions $\tau(r,\theta)$, $\nu(r,\theta) \in W_2^l(S_0)$, $\psi(t,\theta) \in W_2^l(\Gamma_\alpha)$, $l > \frac{3m}{2}$. Then, the Problem 1 has a unique solution.*

3. Solvability of Problem 1

First, we show that Problem 1 has a solution. In the spherical coordinates, the equation (1) takes the form

$$Lu \equiv g(t)\left(u_{rr} + \frac{m-1}{r}u_r - \frac{\delta u}{r^2}\right) +$$

$$+u_{tt} + \sum_{i=1}^{m} a_i(r,\theta,t)u_{x_i} + b(r,\theta,t)u_t + c(r,\theta,t)u = 0, \tag{4}$$

$$\delta \equiv -\sum_{j=1}^{m-1} \frac{1}{g_j \sin^{m-j-1}\theta_j}\frac{\partial}{\partial\theta_j}\left(\sin^{m-j-1}\theta_j\frac{\partial}{\partial\theta_j}\right),$$

$$g_1 = 1, \ g_j = (\sin\theta_1 ... \sin\theta_{j-1})^2, \ j > 1.$$

A well-known result (see [3]) states that the spectrum of the operator δ consists of eigenvalues $\lambda_n = n(n+m-2),\ n = 0,1,...,$ to each of which correspond k_n orthonormalized eigenfunctions $Y^k_{n,m}(\theta)$.

Hence, we can search for the solution of Problem 1 in the form of the series

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\overline{u}^k_n(r,t)Y^k_{n,m}(\theta), \tag{5}$$

where we need to determine the functions $\overline{u}^k_n(r,t)$.

Let's substitute (5) into (4) and multiply the obtained expression by $\rho(\theta) \neq 0$. If we then integrate over the unit sphere H, we obtain for $\overline{u}^k_n$ (see [4]-[6] for the detailed derivation):

$$\begin{gathered} g(t)\rho^1_0\overline{u}^1_{0rr} + \rho^1_0\overline{u}^1_{0tt} + \left(\frac{m-1}{r}g(t)\rho^1_0 + \sum_{i=1}^{m} a^1_{i0}\right)\overline{u}^1_{0r} + \widetilde{b}^1_0\overline{u}^1_{0t} + \widetilde{c}^1_0\overline{u}^1_0 + \\ +\sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\left\{g(t)\rho^k_n\overline{u}^k_{nrr} + \rho^k_n\overline{u}^k_{ntt} + \left(\frac{m-1}{r}g(t)\rho^k_n + \sum_{i=1}^{m} a^k_{in}\right)\overline{u}^k_{nr} + \widetilde{b}^k_n\overline{u}^k_{nt} + \right. \\ \left. + \left[\widetilde{c}^k_n - \lambda_n\frac{\rho^k_n}{r^2}g(t) + \sum_{i=1}^{m}(\widetilde{a}^k_{in-1} - na^k_{in})\right]\overline{u}^k_n\right\} = 0. \end{gathered} \tag{6}$$

Next, let's analyze the infinite system of differential equations

$$\rho^1_0 g(t)\overline{u}^1_{0rr} + \rho^1_0\overline{u}^1_{0tt} + \frac{m-1}{r}g(t)\rho^1_0\overline{u}^1_{0r} = 0, \tag{7}$$

$$\begin{gathered} \rho^k_1 g(t)\overline{u}^k_{1rr} + \rho^k_1\overline{u}^k_{1tt} + \frac{m-1}{r}g(t)\rho^k_1\overline{u}^k_{1r} - \frac{\lambda_1}{r^2}g(t)\rho^k_1\overline{u}^k_1 = \\ = -\frac{1}{k_1}\left(\sum_{i=1}^{m} a^1_{i0}\overline{u}^1_{0r} + \widetilde{b}^1_0\overline{u}^1_{0t} + \widetilde{c}^1_0\overline{u}^1_0\right),\ n = 1,\ k = \overline{1,k_1}, \end{gathered} \tag{8}$$

$$\begin{gathered} \rho^k_n g(t)\overline{u}^k_{nrr} + \rho^k_n\overline{u}^k_{ntt} + \frac{m-1}{r}g(t)\rho^k_n\overline{u}^k_{nr} - \frac{\lambda_n}{r^2}g(t)\rho^k_n\overline{u}^k_n = \\ = -\frac{1}{k_n}\sum_{k=1}^{k_{n-1}}\left\{\sum_{i=1}^{m} a^k_{in-1}\overline{u}^k_{n-1r} + \widetilde{b}^k_{n-1}\overline{u}^k_{n-1t} + \right. \\ \left. + \left[\widetilde{c}^k_{n-1} + \sum_{i=1}^{m}(\widetilde{a}^k_{in-2} - (n-1)a^k_{in-1})\right]\overline{u}^k_{n-1}\right\}, \\ k = \overline{1,k_n},\ n = 2,3,.... \end{gathered} \tag{9}$$

Clearly, if $\{\overline{u}^k_n\},\ k = \overline{1,k_n},\ n = 0,1,...$ is the solution of the system (7)-(9), then it is also the solution of the equation (6).

It is easy to see that each equation of the system (7)-(9) can be represented in the form

$$g(t)\left(\overline{u}^k_{nrr}+\frac{m-1}{r}\overline{u}^k_{nr}-\frac{\lambda_n}{r^2}\overline{u}^k_n\right)+\overline{u}^k_{ntt}=f^k_n(r,t), \tag{10}$$

where $f^k_n(r,t)$ are determined from the previous equations of this system, with $f^1_0(r,t)\equiv 0$.

Next, from the boundary conditions (2), taking into account (5) and Lemma 1, we obtain

$$\overline{u}^k_n(r,0)=\overline{\tau}^k_n(r),\quad \overline{u}^k_{nt}(r,0)=\overline{\nu}^k_n(r),\quad \overline{u}^k_n(1,t)=\psi^k_n(t),\quad k=\overline{1,k_n},\ n=0,1,... \tag{11}$$

In (10), (11), making a change of variables

$$\overline{\upsilon}^k_n(r,t)=\overline{u}^k_n(r,t)-\psi^k_n(t),$$

we get

$$g(t)\left(\overline{\upsilon}^k_{nrr}+\frac{m-1}{r}\overline{\upsilon}^k_{nr}-\frac{\lambda_n}{r^2}\overline{\upsilon}^k_n\right)+\overline{\upsilon}^k_{ntt}=\overline{f}^k_n(r,t), \tag{12}$$

$$\overline{\upsilon}^k_n(r,0)=\tau^k_n(r),\ \overline{\upsilon}^k_{nt}(r,0)=\nu^k_n(r),\ \overline{\upsilon}^k_n(1,t)=0,\quad k=\overline{1,k_n},\ n=0,1,..., \tag{13}$$

$$\overline{f}^k_n(r,t)=f^k_n(r,t)+\frac{\lambda_n}{r^2}g(t)\psi^k_n-\psi^k_{ntt},\ \tau^k_n(r)=\overline{\tau}^k_n(r)-\psi^k_n(0),$$
$$\nu^k_n(r)=\overline{\nu}^k_n(r)-\psi^k_{nt}(0).$$

Making a change of variable $\overline{\upsilon}^k_n(r,t)=r^{(1-m)/2}\upsilon^k_n(r,t)$, we can reduce the problem (12), (13) to the following problem:

$$L\upsilon^k_n \equiv g(t)\left(\upsilon^k_{nrr}+\frac{\overline{\lambda}_n}{r^2}\upsilon^k_n\right)+\upsilon^k_{ntt}=\widetilde{f}^k_n(r,t), \tag{14}$$

$$\upsilon^k_n(r,0) = \widetilde{\tau}^k_n(r),\ \upsilon^k_{nt}(r,0)=\widetilde{\nu}^k_n(r),\ \upsilon^k_n(1,t)=0, \tag{15}$$

$$\overline{\lambda}_n = \frac{[(m-1)(3-m)-4\lambda_n]}{4},\ \widetilde{f}^k_n(r,t)=r^{\frac{(m-1)}{2}}\overline{f}^k_n(r,t),$$

$$\widetilde{\tau}^k_n(r) = r^{(m-1)/2}\tau^k_n(r),\ \widetilde{\nu}^k_n(r)=r^{(m-1)/2}\nu^k_n(r).$$

We search for the solution of the problem (14), (15) in the form

$$\upsilon^k_n(r,t)=\upsilon^k_{1n}(r,t)+\upsilon^k_{2n}(r,t), \tag{16}$$

where $\upsilon^k_{1n}(r,t)$ is the solution of the problem

$$L\upsilon^k_{1n}=\widetilde{f}^k_n(r,t), \tag{17}$$

$$\upsilon_{1n}^k(r,0) = \upsilon_{1nt}^k(r,0) = 0,\ \upsilon_{1n}^k(1,t) = 0, \tag{18}$$

and $\upsilon_{2n}^k(r,t)$ is the solution of the problem

$$L\upsilon_{2n}^k = 0, \tag{19}$$

$$\upsilon_{2n}^k(r,0) = \widetilde{\tau}_n^k(r),\ \upsilon_{2nt}^k(r,0) = \widetilde{\nu}_n^k(r),\ \upsilon_{2n}^k(1,t) = 0. \tag{20}$$

We analyze the solutions of the above problems in the form

$$\upsilon_n^k(r,t) = \sum_{s=1}^{\infty} R_s(r)T_s(t). \tag{21}$$

Moreover, let

$$\widetilde{f}_n^k(r,t) = \sum_{s=1}^{\infty} a_{ns}^k(t)R_s(r),\ \widetilde{\tau}_n^k(r) = \sum_{s=1}^{\infty} b_{ns}^k R_s(r),\ \widetilde{\nu}_n^k(r) = \sum_{s=1}^{\infty} e_{ns}^k R_s(r). \tag{22}$$

Substituting (21) into (17), (18), and taking into account (22), we come to the problem

$$R_{srr} + \frac{\overline{\lambda}_n}{r^2}R_s + \mu R_s = 0,\ 0 < r < 1, \tag{23}$$

$$R_s(1) = 0,\ \ |R_s(0)| < \infty, \tag{24}$$

$$T_{stt} - \mu g(t)T_s(t) = a_{ns}(t),\ 0 < t < \alpha, \tag{25}$$

$$T_s(0) = 0,\ T_{st}(0) = 0. \tag{26}$$

The bounded solution of the problem (24), (25) is (see [7])

$$R_s(r) = \sqrt{r}J_\nu(\mu_{s,n}r), \tag{27}$$

where $\nu = \frac{n+(m-2)}{2}$, and $\mu_{s,n}$ are the zeros of the Bessel function of the first kind, $\mu = \mu_{s,n}^2$.

The problem (25), (26) reduces to the integral Volterra equation of the second kind with respect to $T_{s,n}(t)$ (see [8]):

$$T_{s,n}(t) - \mu_{s,n}^2 \int_0^t (t-\xi)g(\xi)T_{s,n}(\xi)d\xi = \int_0^t (t-\xi)a_{ns}(\xi)d\xi. \tag{28}$$

This equation has a solution; moreover, it is unique.

Next, substituting (27) into (22), we obtain

$$r^{-\frac{1}{2}}\widetilde{f}_n^k(r,t) \ = \ \sum_{s=1}^{\infty} a_{ns}^k(t)J_\nu(\mu_{s,n}r),\ r^{-\frac{1}{2}}\widetilde{\tau}_n^k(r) = \sum_{s=1}^{\infty} b_{ns}^k J_\nu(\mu_{s,n}r), \tag{29}$$

$$r^{-\frac{1}{2}}\widetilde{\nu}_n^k(r) = \sum_{s=1}^{\infty} e_{ns}^k J_\nu(\mu_{s,n}r),\ 0 < r < 1.$$

The series (29) are the decompositions into the Fourier-Bessel series (see [9]), if

$$a_{ns}^k(t) = \frac{2}{[J_{\nu+1}(\mu_{s,n})]^2}\int_0^1 \sqrt{\xi}\widetilde{f}_n^k(\xi,t)J_\nu(\mu_{s,n}\xi)d\xi, \tag{30}$$

$$b_{ns}^k = \frac{2}{[J_{\nu+1}(\mu_{s,n})]^2}\int_0^1 \sqrt{\xi}\widetilde{\tau}_n^k(\xi)J_\nu(\mu_{s,n}\xi)d\xi, \tag{31}$$

$$e_{ns}^k = \frac{2}{[J_{\nu+1}(\mu_{s,n})]^2}\int_0^1 \sqrt{\xi}\widetilde{\nu}_n^k(\xi)J_\nu(\mu_{s,n}\xi)d\xi,$$

where $\mu_{s,n}$, $s = 1, 2, ...$ are the positive zeros of the Bessel functions $J_\nu(z)$, put in the increasing order.

From (27), (28) we get the solution of the problem (17), (18) in the form

$$\upsilon_{1n}^k(r,t) = \sum_{s=1}^{\infty} \sqrt{r}T_{s,n}(t)J_\nu(\mu_{s,n}r), \tag{32}$$

where $a_{ns}^k(t)$ is determined from (30).

Next, substituting (27) into (19), (20) and taking into account (22), we get the problem

$$\begin{aligned} V_{stt} - \mu_{s,n}^2 g(t)V_s(t) &= 0,\ 0 < t < \alpha, \\ V_s(0) &= b_{ns}^k,\ V_{st}(0) = e_{ns}^k, \end{aligned}$$

from which, making the substitution

$$T_s(t) = V_s(t) - b_{ns}^k - te_{ns}^k, \tag{33}$$

we obtain the following, simpler, problem:

$$T_{stt} - \mu_{s,n}^2 g(t)T_s = q_{ns}^k(t), \tag{34}$$

$$T_s(0) = 0,\ T_{st}(0) = 0 \tag{35}$$

$$q_{ns}^k(t) = \mu_{s,n}^2 g(t)(b_{ns}^k + te_{ns}^k).$$

The problem (34), (35) also reduces to the integral equation (28), where instead of $a_{ns}^k(t)$ we use $q_{ns}^k(t)$.

From (27), (28), (33) we find the solution of the problem (19), (20):

$$\upsilon_{2n}^k(r,t) = \sum_{s=1}^{\infty} \sqrt{r} V_{s,n}(t) J_\nu(\mu_{s,n} r), \tag{36}$$

where b_{ns}^k, e_{ns}^k are determined from (31).

Therefore, having first solved the problem (7), (11) (for $n = 0$), and then (8), (11) (for $n = 1$), and so on, we can consecutively find all functions $\upsilon_n^k(r,t)$ from (16), where $\upsilon_{1n}^k(r,t)$, $\upsilon_{2n}^k(r,t)$ are determined from (32), (36), $k = \overline{1, k_n}$, $n = 0, 1, \ldots.$

Hence, we have shown that, in the domain D_α,

$$\int_H \rho(\theta) Lu dH = 0. \tag{37}$$

Next, let $f(r,\theta,t) = R(r)\rho(\theta)T(t)$. Moreover, $R(r) \in V_0$, V_0 is dense in $L_2((0,1))$, $\rho(\theta) \in C^\infty(H)$ is dense in $L_2(H)$, and $T(t) \in V_1$, V_1 is dense in $L_2((0,\alpha))$. Then, $f(r,\theta,t) \in V$, $V = V_0 \otimes H \otimes V_1$ is dense in $L_2(D_\alpha)$ (see [10]).

From here and from (37) it follows that

$$\int_{D_\alpha} f(r,\theta,t) Lu dD_\alpha = 0$$

and

$$Lu = 0, \forall (r,\theta,t) \in D_\alpha \ .$$

Therefore, the solution of Problem (1) is the series

$$u(r,\theta,t) = \sum_{n=0}^{\infty} \sum_{k=1}^{k_n} \left\{ \psi_n^k(t) + r^{\frac{(1-m)}{2}} \left[\upsilon_{1n}^k(r,t) + \upsilon_{2n}^k(r,t) \right] \right\} Y_{n,m}^k(\theta), \tag{38}$$

where $\upsilon_{1n}^k(r,t)$, $\upsilon_{2n}^k(r,t)$ are determined from the expressions (32) and (36).

Taking into account the formula (see [9])

$$2J'_\nu(z) = J_{\nu-1}(z) - J_{\nu+1}(z),$$

the estimates [11, 3]

$$|k_n| \le c_1 n^{m-2}, \quad \left| \frac{\partial^q}{\partial \theta_j^q} Y_{n,m}^k(\theta) \right| \le c_2 n^{\frac{m}{2}-1+q}, \ j = \overline{1, m-1}, \ q = 0, 1, \ldots, \tag{39}$$

Lemmas 1 and 2, and the bounds on the coefficients of the equation (1) and on the given functions $\tau(r,\theta)$, $\nu(r,\theta)$, $\psi(t,\theta)$, we can show, using the same procedure as in [4]-[6], that the obtained solution (38) belongs to the class $C^1(\overline{D}_\alpha) \cap C^2(D_\alpha)$.

The solvability of Problem 1 is thus established.

4. Uniqueness of the solution of Problem 1

Now we can proceed to prove the uniqueness of the solution. For this, first, let's build the solution of the boundary-value problem (1*) with the conditions

$$\upsilon|_{\Gamma_\alpha} = 0,\ \ \upsilon|_{S_\alpha} = 0,\ \ \upsilon_t|_{S_\alpha} = \nu(r,\theta) = \overline{\nu}_n^k(r) Y_{n,m}^k(\theta),\quad k = \overline{1,k_n},\ \ n = 0,1,..., \tag{40}$$

where $\overline{\nu}_n^k(r) \in G$, G is the set of functions $\nu(r)$ from the class $C([0,1]) \cap C^1([0,1])$. Obviously, the set G is dense everywhere in $L_2((0,1))$ (see [10]).

We will search for the solution of the problem (1*),(40) in the form (5), where we have to determine the functions $\overline{\upsilon}_n^k(r,t)$. Then, analogously to the previous section, the functions $\overline{\upsilon}_n^k(r,t)$ satisfy the system of equations (8)-(10), where $\widetilde{a}_{in}^k, a_{in}^k, \widetilde{b}_n^k$ are replaced with $-\widetilde{a}_{in}^k, -a_{in}^k, -\widetilde{b}_n^k$, respectively, and $\widetilde{c}_n^k$ is replaced with $\widetilde{d}_n^k$, $i = 1,...,m$, $k = \overline{1,k_n}$, $n = 0,1,...$

Next, from the boundary condition (40), given the form (5), we obtain

$$\overline{\upsilon}_n^k(1,t) = 0,\ \ \overline{\upsilon}_n^k(r,\alpha) = 0,\ \ \overline{\upsilon}_{nt}^k(r,\alpha) = \overline{\nu}_n^k(r),\ \ k = \overline{1,k_n},\quad n = 0,1,... \tag{41}$$

As we explained earlier, each equation of the system (7)-(9) can be represented in the form (10). Similarly to the previous section, it is easy to show that the problem (10), (41) also has a unique solution.

We have thus built the solution of the problem (2), (40) in the form of the series (38). Furthermore, given the estimates (39), this solution belongs to the class $C^1(\overline{D}_\alpha) \cap C^2(D_\alpha)$.

From the definition of the adjoint operators L, L^*(see [12]), we have

$$\upsilon Lu - uL^*\upsilon = -\upsilon P(u) + uP(\upsilon) - u\upsilon Q,$$

where

$$\begin{aligned} P(u) &= g(t)\sum_{i=1}^m u_{x_i}\cos(N^\perp, x_i) - u_t\cos(N^\perp, t), \\ Q &= \sum_{i=1}^m a_i\cos(N^\perp, x_i) - b\cos(N^\perp, t), \end{aligned}$$

and $N^\perp$ is the inner normal to the boundary ∂D_α. Using the Green's formula, we obtain the equality

$$\int\limits_{D_\alpha} (\upsilon Lu - uL^*\upsilon) dD_\alpha = \int\limits_{\partial D_\alpha} \left[\left(\upsilon\frac{\partial u}{\partial N} - u\frac{\partial \upsilon}{\partial N}\right) M + u\upsilon Q\right] ds, \tag{42}$$

where

$$\frac{\partial}{\partial N} = g(t)\sum_{i=1}^{m} \cos(N^{\perp}, x_i) - \cos(N^{\perp}, t)\frac{\partial}{\partial t},$$

$$M^2 = g^2(t)\sum_{i=1}^{m} \cos^2(N^{\perp}, x_i) + \cos^2(N^{\perp}, t).$$

From (42), taking into account the homogeneous boundary conditions (2) and the conditions (40), we obtain

$$\int\limits_{S_\alpha} \nu(r,\theta)u(r,\theta,\alpha)ds = 0. \tag{43}$$

Note that the linear hull of the system of the functions $\left\{\overline{\nu}_n^k(r)Y_{n,m}^k(\theta)\right\}$ is dense in $L_2(S)$ (see [10]). Then, from (43), we can conclude that $u(r,\theta,\alpha) = 0$, $\forall(r,\theta) \in S_\alpha$.

Hence, we have come to the Dirichlet problem

$$Lu = 0, \quad u|_{S_0} = 0, \ u|_{\Gamma_\alpha} = 0, \quad u|_{S_\alpha} = 0,$$

which has the null solution (see [13]).

Thus, the uniqueness of the solution of Problem 1 is established.

This completes the proof of the theorem.

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