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Degree of Convergence of Functions of Multiple Fourier Series in Sobolev Spaces

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Abstract. In this paper, the degree of convergence of a function of two dimensional variable of double Fourier series in Sobolev spaces using double Riesz means is obtained. The degree of convergence of a function of N-dimensional variable of N-multiple Fourier series in Sobolev spaces using N-dimensional Riesz means is also obtained in this paper.

Key Words and Phrases: degree of convergence, modulus of smoothness, Sobolev spaces, N-dimensional Riesz means, multiple Fourier series.

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1. Introduction

Sobolev spaces are vector spaces whose elements are functions defined on domains in N-dimensional Euclidean space $\mathbb{R}^{\mathbb{N}}$ and whose partial derivatives satisfy certain integrability conditions. In order to develop and elucidate the properties of these spaces and mappings between them, we require some machinery of general topology and real and functional analysis.

In one of the classical approximation theories, the properties of approximation of orthogonal function systems, polynomials and trigonometric functions have been studied in L^q-norm, and mostly in the maximum norm in [4, 8, 9, 22, 16, 18, 23, 21].

The L^q-norm for $q < \infty$, capture the "*height*" and "*width*" of a function. In mathematical terms "*width*" is the same as the measure of the support of the function. The Sobolev norm captures the "*height*", "*width*" and "*oscillations*". The Fourier transform measures oscillation (or frequency or wavelength) by decay of the Fourier transform i.e. the "*oscillation*" of a function is translated to "*decay*"

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of its Fourier transform. Sobolev norm measures "*oscillation*" via its derivatives (or regularity).

The degree of approximation of a function of single variable in Lipschitz, generalized Lipschitz, Hölder, generalized Hölder and Besov spaces using different means of Fourier series, has been studied in [11, 12, 13, 14, 15, 25, 27], etc.

The degree of approximation of a function of double variables in Lipschitz and Hölder spaces using different means of double Fourier series, has been studied in [2, 3, 7, 17, 26].

In this paper, the degree of convergence of a function of two dimensional variable of double Fourier series in Sobolev spaces using double Riesz means is obtained. The degree of convergence of a function of N-dimensional variable of N-multiple Fourier series in Sobolev spaces using N-dimensional Riesz means is also obtained in this paper.

The organization of the paper is as follows: In Section 2, we give important definitions and known results related to the present work. In Section 3, we obtain the degree of convergence of double Fourier series in Sobolev spaces using double Riesz means. In Section 4, we obtain the degree of convergence of multiple Fourier series in Sobolev spaces using N-dimensional Riesz means. Conclusion is given in Section 5.

2. Preliminaries

In this section, we present definitions and known results.

2.1. Double Fourier Series and Double Derived Fourier Series

Let $f(x_1, x_2)$ be a periodic function with period 2π in both x_1 and x_2, Lebesgue integrable and summable in the square $A(-\pi, \pi; -\pi, \pi)$.

The double Fourier series of the function $f(x_1, x_2)$ is given by

$$\begin{aligned} f(x_1, x_2) \sim \sum_{\nu_1, \nu_2 \in \mathbb{N}} \alpha_{\nu_1,\nu_2} [&\zeta_{\nu_1,\nu_2} \cos \nu_1 x_1 \cos \nu_2 x_2 + \lambda_{\nu_1,\nu_2} \sin \nu_1 x_1 \cos \nu_2 x_2 \\ &+ \xi_{\nu_1,\nu_2} \cos \nu_1 x_1 \sin \nu_2 x_2 + \chi_{\nu_1,\nu_2} \sin \nu_1 x_1 \sin \nu_2 x_2], \end{aligned} \tag{1}$$

where

$$\alpha_{\nu_1,\nu_2} = \begin{cases} \frac{1}{4}, & \nu_1 = \nu_2 = 0; \\ \frac{1}{2}, & \nu_1 > 0, \nu_2 = 0 \text{ and } \nu_1 = 0, \nu_2 > 0; \\ 1, & \nu_1 > 0, \nu_2 > 0; \end{cases}$$

and $\zeta_{\nu_1,\nu_2}, \lambda_{\nu_1,\nu_2}, \xi_{\nu_1,\nu_2}$, χ_{ν_1,ν_2} are the Fourier coefficients of $f(x_1, x_2)$.

We know that the quantities

$$s_{\nu_1\nu_2}(x_1,x_2)=\sum_{i_1=0}^{\nu_1}\sum_{i_2=0}^{\nu_2}[\zeta_{i_1,i_2}\cos i_1x_1\cos i_2x_2+\lambda_{i_1,i_2}\sin i_1x_1\cos i_2x_2 + \xi_{i_1,i_2}\cos i_1x_1\sin i_2x_2+\chi_{i_1,i_2}\sin i_1x_1\sin i_2x_2] \tag{2}$$

for $\nu_1=0,1,\cdots;\ \nu_2=0,1,\cdots$, are called the partial sums of the double Fourier series.

We know that

$$s_{\nu_1\nu_2}(x_1,x_2)=\frac{1}{\pi^2}\iint\limits_{A^2} f(x_1+t_1,x_2+t_2)\frac{[\sin(\nu_1+\frac{1}{2})t_1][\sin(\nu_2+\frac{1}{2})t_2]}{4\sin(\frac{t_1}{2})\sin(\frac{t_2}{2})}d_{t_1}d_{t_2}, \tag{3}$$

and

$$s_{\nu_1\nu_2}(x_1,x_2)-f(x_1,x_2)=\frac{1}{\pi^2}\iint\limits_{A^2}\Phi(t_1,t_2)\frac{[\sin(\nu_1+\frac{1}{2})t_1][\sin(\nu_2+\frac{1}{2})t_2]}{4\sin(\frac{t_1}{2})\sin(\frac{t_2}{2})}d_{t_1}d_{t_2}, \tag{4}$$

where

$$\Phi(t_1,t_2)=\phi(x_1,x_2;t_1,t_2)=\frac{1}{4}\Big[f(x_1+t_1,x_2+t_2)+f(x_1-t_1,x_2+t_2) + f(x_1+t_1,x_2-t_2)+f(x_1-t_1,x_2-t_2)-4f(x_1,x_2)\Big]. \tag{5}$$

The double derived Fourier series of (1) with respect to x_1 is given by

$$f'_{x_1}(x_1,x_2)\sim\sum_{\nu_1,\nu_2\in\mathbb{N}}\nu_1\big[-\zeta_{\nu_1,\nu_2}\sin\nu_1x_1\cos\nu_2x_2+\lambda_{\nu_1,\nu_2}\cos\nu_1x_1\cos\nu_2x_2 - \xi_{\nu_1,\nu_2}\sin\nu_1x_1\sin\nu_2x_2+\chi_{\nu_1,\nu_2}\cos\nu_1x_1\sin\nu_2x_2\big]. \tag{6}$$

The partial sums of (6) is given by

$$s'_{\nu_1\nu_2}(x_1,x_2)=\frac{1}{\pi^2}\iint\limits_{A^2}\frac{[\sin(\nu_1+\frac{1}{2})t_1][\sin(\nu_2+\frac{1}{2})t_2]}{4\sin(\frac{t_1}{2})\sin(\frac{t_2}{2})}d\{f(x_1+t_1,x_2+t_2)\}, \tag{7}$$

and

$$s'_{\nu_1\nu_2}(x_1,x_2)-f'_{x_1}(x_1,x_2)=\frac{1}{\pi^2}\iint\limits_{A^2}\frac{[\sin(\nu_1+\frac{1}{2})t_1][\sin(\nu_2+\frac{1}{2})t_2]}{4\sin(\frac{t_1}{2})\sin(\frac{t_2}{2})}dg_{x_1}(t_1,t_2), \tag{8}$$

where

$$dg_{x_1}(t_1,t_2) = \frac{1}{4}\Big[d\Big(f(x_1+t_1,x_2+t_2)-f(x_1-t_1,x_2+t_2) - f(x_1+t_1,x_2-t_2)+f(x_1-t_1,x_2-t_2)\Big) - 4\frac{df}{dx_1}(x_1,x_2)dt_1dt_2\Big]. \quad (9)$$

The double derived Fourier series of (1) with respect to x_2 is given by

$$f'_{x_2}(x_1,x_2) \sim \sum_{\nu_1,\nu_2\in\mathbb{N}} \nu_2\big[-\zeta_{\nu_1,\nu_2}\cos\nu_1x_1\sin\nu_2x_2 - \lambda_{\nu_1,\nu_2}\sin\nu_1x_1\sin\nu_2x_2 + \xi_{\nu_1,\nu_2}\cos\nu_1x_1\cos\nu_2x_2 + \chi_{\nu_1,\nu_2}\sin\nu_1x_1\cos\nu_2x_2\big]. \quad (10)$$

The partial sums of (10) is given by

$$s'_{\nu_1\nu_2}(x_1,x_2) = \frac{1}{\pi^2}\iint_{A^2}\frac{[\sin(\nu_1+\frac{1}{2})t_1][\sin(\nu_2+\frac{1}{2})t_2]}{4\sin(\frac{t_1}{2})\sin(\frac{t_2}{2})}\frac{d}{dx_2}\{f(x_1+t_1,x_2+t_2)\}, \quad (11)$$

and

$$s'_{\nu_1\nu_2}(x_1,x_2) - f'_{x_2}(x_1,x_2) = \frac{1}{\pi^2}\iint_{A^2}\frac{[\sin(\nu_1+\frac{1}{2})t_1][\sin(\nu_2+\frac{1}{2})t_2]}{4\sin(\frac{t_1}{2})\sin(\frac{t_2}{2})}dg_{x_2}(t_1,t_2), \quad (12)$$

where

$$dg_{x_2}(t_1,t_2) = \frac{1}{4}\Big[d\Big(f(x_1+t_1,x_2+t_2)-f(x_1-t_1,x_2+t_2) - f(x_1+t_1,x_2-t_2)+f(x_1-t_1,x_2-t_2)\Big) - 4\frac{df}{dx_2}(x_1,x_2)dt_1dt_2\Big]. \quad (13)$$

2.2. Sobolev Spaces

Assume that $\mathbf{X}$ is an open subset of $\mathbb{R}^{\mathbb{N}}$.
For $1 \le q < \infty$, the space $L^q(\mathbf{X})$ consists of all measurable functions on $\mathbf{X}$ such that

$$\int_{\mathbf{X}} |f(t)|^q dt < \infty$$

and the norm is defined by

$$||f||_q = \begin{cases} \left(\int_{\mathbf{X}} |f(t)|^q dt \right)^{\frac{1}{q}}, & 1 \le q < \infty; \\ ess \sup_{f \in \mathbf{X}} |f(t)|, & q = \infty. \end{cases}$$

When $q = 2$, then

$$||f||_2 = \left(\int_{\mathbf{X}} |f(t)|^2 dt \right)^{\frac{1}{2}}.$$

The ν-th order modulus of smoothness of a function $f : A \to \mathbb{R}$ is defined by

$$\omega_\nu(f, t) = \sup_{0 \le h \le t} \{\sup\{|\Delta_h^\nu f(t)| : t, t + \nu h \in A\}\}, \quad t \ge 0, \tag{14}$$

where

$$\Delta_h^\nu f(t) = \sum_{j=0}^{\nu} (-1)^{\nu - j} \binom{\nu}{j} f(t + jh), \quad \nu \in \mathbb{N}.$$

For $\nu = 1$, $\omega(f, t)$ is called the modulus of continuity of f ([1]).

The Sobolev space $W^{\nu,q}(\mathbf{X}), \nu = 1, 2, 3, \cdots$ consists of functions $u \in L^q(\mathbf{X})$ such that for every multi-index β with $|\beta| \le \nu$, the weak derivative $D^\beta u$ exists and $D^\beta u \in L^q(\mathbf{X})$.

Thus,

$$W^{\nu,q}(\mathbf{X}) = \{u \in L^q(\mathbf{X}) : D^\beta u \in L^q(\mathbf{X}), |\beta| \le \nu\} \text{ ([20])}. \tag{15}$$

The norm of (15) is defined by

$$||u||_{W^{\nu,q}(\mathbf{X})} = \sum_{|\beta| \le \nu} ||D^\beta u||^q_{L^q(\mathbf{X})}. \tag{16}$$

The equivalent norm of (16) is given by

$$||u||_{W^{\nu,q}(\mathbf{X})} = \left(\sum_{|\beta| \le \nu} ||D^\beta u||^q_{L^q(\mathbf{X})} \right)^{\frac{1}{q}}, \quad 1 \le q < \infty \tag{17}$$

and

$$||u||_{W^{\nu,\infty}(\mathbf{X})} = \max_{|\beta| \le \nu} ||D^\beta u||_{L^\infty(\mathbf{X})}. \tag{18}$$

The semi-norm of (15) is defined by

$$|u|_{W^{\nu,q}(\mathbf{X})} = \left(\sum_{|\beta| = \nu} ||D^\beta u||^q_{L^q(\mathbf{X})} \right)^{\frac{1}{q}}, \quad 1 \le q < \infty \tag{19}$$

and

$$|u|_{W^{\nu,\infty}(\mathbf{X})} = \max_{|\beta|=\nu} ||D^{\beta}u||_{L^{\infty}(\mathbf{X})}. \tag{20}$$

When $q = 2$, the Sobolev space $W^{\nu,2}(\mathbf{X})$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{W^{\nu,2}(\mathbf{X})} = \sum_{|\beta|\leq\nu} \langle D^{\beta}u, D^{\beta}v\rangle_{L^2(\mathbf{X})},$$

where

$$\langle D^{\beta}u, D^{\beta}v\rangle_{L^2(\mathbf{X})} = \int_{\mathbf{X}} D^{\beta}uD^{\beta}vdt$$

and

$$||u||_{W^{\nu,2}(\mathbf{X})} = \langle u, u\rangle^{\frac{1}{2}}_{W^{\nu,2}(\mathbf{X})}.$$

Note that for $\nu = 1$ the Sobolev space is denoted by $W^{1,q}(\mathbf{X})$ and its norm is defined by

$$||u||_{W^{1,q}(\mathbf{X})} = ||u||_q + \sum_{i=1}^{N} \left|\left|\frac{\partial u}{\partial x_i}\right|\right|_q. \tag{21}$$

The norm of Sobolev space in two dimensional case is defined by

$$||u||_{W^{1,q}(\mathbf{X})} = ||u||_q + \sum_{i=1}^{2} \left|\left|\frac{\partial u}{\partial x_i}\right|\right|_q. \tag{22}$$

Remark 1. *Here, we discuss some important properties of Sobolev space.*

(i) For $1 \leq q \leq \infty$ and $\nu = 1, 2, \cdots$, the Sobolev space $W^{\nu,q}(\mathbf{X})$ is a Banach space.

(ii) For $1 \leq q < \infty$ and $\nu = 1, 2, \cdots$, the Sobolev space $W^{\nu,q}(\mathbf{X})$ is separable.

Remark 2. *(i) For $\nu = 0$, the Sobolev space reduces to L^q space, i.e. $W^{0,q}(\mathbf{X}) = L^q(\mathbf{X})$.*

(ii) For $\nu = 1, 2, 3, \cdots, W^{\nu,q}(\mathbf{X}) = Lip(\nu, q)$.

(iii) For $\beta = 1 = n$, we have $W^{1,p}(\mathbf{X}) = Lip(1, q)$.

(iv) For $\nu = 1, q \to \infty, Lip(1, q) = Lip(1)$.

Theorem 1. *([10]) Let $u \in L^q(\mathbf{X})$ with $1 < q \leq \infty$. The following properties are equivalent:*

(i) $u \in W^{1,q}(\mathbf{X})$,

(ii) $\exists$ *a constant* C *such that*

$$\int_{\mathbf{X}} u\frac{\partial\phi}{\partial x_i} \leq C||\phi||_{L^{q'}(\mathbf{X})}\ \forall\ \phi \in\ C_c^{\infty}(\mathbf{X}),\ i = 1, 2, \cdots, \mathbb{N},$$

(iii) $\exists$ *a constant* C *such that for all* $w \subset\subset \mathbf{X}$*, and all* $h \in (\mathbb{R}^{\mathbb{N}})$ *with* $|h| < dis(w, \partial\mathbf{X})$

$$||\tau_h u - u||_{L^q}(\mathbb{R}^{\mathbb{N}}) \leq C|h|.$$

Moreover, one can choose $C = ||u^{'}||_{L^q}(\mathbb{R}^{\mathbb{N}})$ *in (ii) and* $(\tau_h(u))(t) = u(t+h)$*.*

2.3. Double Riesz $(R, p_{\nu_1}p_{\nu_2})$ Means

Let $\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} u_{i_1i_2}$ be a double infinite series such that $\sum_{i_1=0}^{\nu_1}\sum_{i_2=0}^{\nu_2} s_{i_1i_2}$ is its partial sum. Let $p_{\nu_1}p_{\nu_2}$ be a non-negative, non-decreasing sequence of numbers such that

$$P_{\nu_1}P_{\nu_2} = \sum_{i_1=0}^{\nu_1}\sum_{i_2=0}^{\nu_2} p_{i_1}p_{i_2} \neq 0\ \forall\ \nu_1, \nu_2 \geq 0$$

and $P_{\nu_1}P_{\nu_2} \to \infty$ as $\nu_1, \nu_2 \to \infty$.
The sequence-to-sequence transformation defined by

$$T_{\nu_1\nu_2} = \frac{1}{P_{\nu_1}P_{\nu_2}}\sum_{i_1=0}^{\nu_1}\sum_{i_2=0}^{\nu_2} p_{i_1}p_{i_2}s_{i_1i_2}$$

is called double Riesz $(R, p_{\nu_1}p_{\nu_2})$ means of the sequence $s_{\nu_1\nu_2}$. The series $\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} u_{i_1i_2}$ is said to be summable to the sum s by double Riesz method if we can write $t_{\nu_1,\nu_2} \to s$ as $\nu_1, \nu_2 \to \infty$.

The necessary and sufficient conditions for $(R, p_{\nu_1}p_{\nu_2})$ method to be regular are given by

$$\sum_{i_1=0}^{\nu_1}\sum_{i_2=0}^{\nu_2} |p_{i_1}p_{i_2}| < c|P_{\nu_1}P_{\nu_2}|,\ |P_{\nu_1}P_{\nu_2}| \to \infty.$$

2.4. Degree of Convergence

The degree of convergence of a summation method to a given function f is a measure that shows how fast T_ν converges to f and is given by

$$||f - T_\nu|| = \mathcal{O}\left(\frac{1}{\lambda_\nu}\right)\ ([5]),$$

where $\lambda_\nu \to \infty$ as $\nu \to \infty$.

3. Degree of convergence of a function of two dimensional variable of double Fourier series in Sobolev spaces

In this section, we study the degree of convergence of a function of two dimensional variable of double Fourier series in Sobolev spaces using double Riesz means. In fact, we establish a following theorem.

Theorem 2. *Let $f(x_1,x_2)$ be a periodic function with period 2π in both x_1 and x_2 and Lebesgue integrable on $A(-\pi,\pi;-\pi,\pi)$. Then, the degree of convergence of the function $f(x_1,x_2)$ in Sobolev spaces $W^{1,2}(\mathbf{X})$ using double Riesz means of double Fourier series is given by*

$$||\mathcal{T}_{\nu_1\nu_2}(x_1,x_2)||_{1,2}=$$
$$\mathcal{O}\Bigg[\frac{2Cp_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\Bigg(\frac{1}{\nu_1+1}+\frac{1}{\nu_2+1}+\log\pi(\nu_2+1)+\frac{1}{\nu_1+1}$$
$$\Bigg((\nu_2+1)-\frac{1}{\pi}\Bigg)+\log\pi(\nu_1+1)+\frac{1}{\nu_2+1}\Bigg((\nu_1+1)-\frac{1}{\pi}\Bigg)$$
$$+\log\pi(\nu_1+1)\Bigg((\nu_2+1)-\frac{1}{\pi}\Bigg)+\log\pi(\nu_2+1)\Bigg((\nu_1+1)-\frac{1}{\pi}\Bigg)\Bigg)$$
$$+\frac{p_{\nu_1}p_{\nu_2}(\nu_1+1)(\nu_2+1)}{P_{\nu_1}P_{\nu_2}}\int_0^{\frac{1}{\nu_1+1}}\int_0^{\frac{1}{\nu_2+1}}|dg_{x_1}(t_1,t_2)|$$
$$+\frac{p_{\nu_1}p_{\nu_2}(\nu_1+1)}{P_{\nu_1}P_{\nu_2}}\int_0^{\frac{1}{\nu_1+1}}\int_{\frac{1}{\nu_2+1}}^{\pi}\frac{|dg_{x_1}(t_1,t_2)|}{t_2^2}+\frac{p_{\nu_1}p_{\nu_2}(\nu_2+1)}{P_{\nu_1}P_{\nu_2}}$$
$$\int_{\frac{1}{\nu_1+1}}^{\pi}\int_0^{\frac{1}{\nu_2+1}}\frac{|dg_{x_1}(t_1,t_2)|}{t_1^2}+\frac{p_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\int_{\frac{1}{\nu_1+1}}^{\pi}\int_{\frac{1}{\nu_2+1}}^{\pi}\frac{|dg_{x_1}(t_1,t_2)|}{t_1^2t_2^2}\Bigg]$$
$$+\frac{p_{\nu_1}p_{\nu_2}(\nu_1+1)(\nu_2+1)}{P_{\nu_1}P_{\nu_2}}\int_0^{\frac{1}{\nu_1+1}}\int_0^{\frac{1}{\nu_2+1}}|dg_{x_2}(t_1,t_2)|+\frac{p_{\nu_1}p_{\nu_2}(\nu_1+1)}{P_{\nu_1}P_{\nu_2}}$$
$$\int_0^{\frac{1}{\nu_1+1}}\int_{\frac{1}{\nu_2+1}}^{\pi}\frac{|dg_{x_2}(t_1,t_2)|}{t_2^2}+\frac{p_{\nu_1}p_{\nu_2}(\nu_2+1)}{P_{\nu_1}P_{\nu_2}}\int_{\frac{1}{\nu_1+1}}^{\pi}\int_0^{\frac{1}{\nu_2+1}}\frac{|dg_{x_2}(t_1,t_2)|}{t_1^2}$$
$$+\frac{p_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\int_{\frac{1}{\nu_1+1}}^{\pi}\int_{\frac{1}{\nu_2+1}}^{\pi}\frac{|dg_{x_2}(t_1,t_2)|}{t_1^2t_2^2}\Bigg].$$

Following lemmas are required for the proof of Theorem 2.

Lemma 1. *Let $\{p_{\nu_1}\}$ and $\{p_{\nu_2}\}$ be non-negative and non-decreasing sequences. Then*

$$|\mathcal{K}_{\nu_1}(t_1)| = \mathcal{O}\left(\frac{p_{\nu_1}(\nu_1+1)}{P_{\nu_1}}\right) \text{ for } 0 < t_1 \leq \frac{1}{\nu_1+1}$$

and

$$|\mathcal{K}_{\nu_2}(t_2)| = \mathcal{O}\left(\frac{p_{\nu_2}(\nu_2+1)}{P_{\nu_2}}\right) \text{ for } 0 < t_2 \leq \frac{1}{\nu_2+1}.$$

Proof. For $0 < t_1 \leq \frac{1}{\nu_1+1}$, $\sin(\frac{t_1}{2}) \geq \frac{t_1}{\pi}$ and $\sin(i_1+\frac{1}{2})t_1 \leq (i_1+\frac{1}{2})t_1$.

$$\begin{aligned}|\mathcal{K}_{\nu_1}(t_1)| =& \left|\frac{1}{2\pi P_{\nu_1}}\sum_{i_1=0}^{\nu_1} p_{i_1}\frac{\sin(i_1+\frac{1}{2})t_1}{\sin\frac{t_1}{2}}\right| \\ \leq& \frac{1}{2t_1 P_{\nu_1}}\left|\sum_{i_1=0}^{\nu} p_{i_1}\sin\left(i_1+\frac{1}{2}\right)t_1\right|. \\ \leq& \frac{1}{4P_{\nu_1}}\left|\sum_{i_1=0}^{\nu_1} p_{i_1}(2i_1+1)\right| \\ \leq& \frac{1}{4P_{\nu_1}}\mathcal{O}\big(p_{\nu_1}(\nu_1+1)\big).\end{aligned}$$

Thus,

$$|\mathcal{K}_{\nu_1}(t_1)| = \mathcal{O}\left(\frac{p_{\nu_1}(\nu_1+1)}{P_{\nu_1}}\right).$$

Similarly, for $0 < t_2 \leq \frac{1}{\nu_2+1}$, $\sin(\frac{t_2}{2}) \geq \frac{t_2}{\pi}$ *and* $\sin(i_2+\frac{1}{2})t_2 \leq (i_2+\frac{1}{2})t_2$, *we can find*

$$|\mathcal{K}_{\nu_2}(t_2)| = \mathcal{O}\left(\frac{p_{\nu_2}(\nu_2+1)}{P_{\nu_2}}\right).$$

◄

Lemma 2. *Let $\{p_{\nu_1}\}$ and $\{p_{\nu_2}\}$ be non-negative and non-decreasing sequences. Then*

$$|\mathcal{K}_{\nu_1}(t_1)| = \mathcal{O}\left(\frac{p_{\nu_1}}{t_1^2 P_{\nu_1}}\right) \text{ for } \frac{1}{\nu_1+1} < t_1 \leq \pi$$

and

$$|\mathcal{K}_{\nu_2}(t_2)| = \mathcal{O}\left(\frac{p_{\nu_2}}{t_2^2 P_{\nu_2}}\right) \text{ for } \frac{1}{\nu_2+1} < t_2 \leq \pi.$$

Proof. For $\frac{1}{\nu_1+1} < t_1 \leq \pi$, $\sin(\frac{t_1}{2}) \geq \frac{t_1}{\pi}$, $|\sin t_1| \leq 1$.

$$|\mathcal{K}_{\nu_1}(t_1)| = \left|\frac{1}{2\pi P_{\nu_1}}\sum_{i_1=0}^{\nu_1} p_{i_1}\frac{\sin(i_1+\frac{1}{2})t_1}{\sin\frac{t_1}{2}}\right|$$

$$\leq \frac{1}{2t_1 P_{\nu_1}} \left| \sum_{i_1=0}^{\nu} p_{i_1} \sin\left(i_1 + \frac{1}{2}\right) t_1 \right|.$$

Using Abel's transformation we have

$$\left| \sum_{i_1=0}^{\nu_1} p_{i_1} \sin\left(i_1 + \frac{1}{2}\right) t_1 \right| =$$

$$\left| \sum_{i_1=0}^{\nu_1 - 1} (p_{i_1} - p_{i_1+1}) \sum_{r=0}^{i_1} \sin\left(r + \frac{1}{2}\right) t_1 + p_{\nu_1} \sum_{i_1=0}^{\nu_1} \sin\left(i_1 + \frac{1}{2}\right) t_1 \right|$$

$$= \mathcal{O}\left(\frac{1}{t_1}\right) \left[\sum_{i_1=0}^{\nu_1 - 1} |p_{i_1} - p_{i_1+1}| + |p_{\nu_1}| \right] = \mathcal{O}\left(\frac{p_{\nu_1}}{t_1}\right).$$

Thus,

$$|\mathcal{K}_{\nu_1}(t_1)| = \mathcal{O}\left(\frac{p_{\nu_1}}{t_1^2 P_{\nu_1}}\right).$$

Similarly, for $\frac{1}{\nu_1+1} < t_2 \leq \pi$, $\sin(\frac{t_2}{2}) \geq \frac{t_2}{\pi}$, $|\sin t_2| \leq 1$, *we can find*

$$|\mathcal{K}_{\nu_2}(t_2)| = \mathcal{O}\left(\frac{p_{\nu_2}}{t_2^2 P_{\nu_2}}\right).$$

◀

Proof of Theorem 2. Using (4), the double Riesz transform of the sequence $\{s_{\nu_1\nu_2}(x_1, x_2)\}$ is given by

$$\begin{aligned}
& t_{\nu_1\nu_2}(x_1, x_2) - f(x_1, x_2) \\
&= \frac{1}{P_{\nu_1} P_{\nu_2}} \sum_{i_1=0}^{\nu_1} \sum_{i_2=0}^{\nu_2} p_{i_1} p_{i_2} \{ s_{i_1 i_2}(x_1, x_2) - f(x_1, x_2) \} \\
&= \frac{1}{P_{\nu_1} P_{\nu_2}} \sum_{i_1=0}^{\nu_1} \sum_{i_2=0}^{\nu_2} p_{i_1} p_{i_2} \left[\frac{1}{\pi^2} \iint_{A^2} \Phi(t_1, t_2) \frac{[\sin(i_1 + \frac{1}{2}) t_1][\sin(i_2 + \frac{1}{2}) t_2]}{4 \sin(\frac{t_1}{2}) \sin(\frac{t_2}{2})} d_{t_1} d_{t_2} \right] \\
&= \iint_{A^2} \Phi(t_1, t_2) \left[\frac{1}{\pi^2} \frac{1}{P_{\nu_1} P_{\nu_2}} \sum_{i_1=0}^{\nu_1} \sum_{i_2=0}^{\nu_2} p_{i_1} p_{i_2} \frac{[\sin(i_1 + \frac{1}{2}) t_1][\sin(i_2 + \frac{1}{2}) t_2]}{4 \sin(\frac{t_1}{2}) \sin(\frac{t_2}{2})} d_{t_1} d_{t_2} \right].
\end{aligned} \tag{23}$$

Thus,

$$t_{\nu_1\nu_2}(x_1,x_2)-f(x_1,x_2)=\iint\limits_{A^2}\Phi(t_1,t_2)K_{\nu_1}(t_1)K_{\nu_2}(t_2)d_{t_1}d_{t_2}. \tag{24}$$

Let

$$\begin{aligned}\mathcal{T}_{\nu_1\nu_2}(x_1,x_2)=&t_{\nu_{1,}\nu_2}(x_1,x_2)-f(x_1,x_2)\\=&\iint\limits_{A^2}\Phi(t_1,t_2)K_{\nu_1}(t_1)K_{\nu_2}(t_2)d_{t_1}d_{t_2}.\end{aligned} \tag{25}$$

By the definition of Sobolev norm given in (17), we have

$$||\mathcal{T}_{\nu_1\nu_2}(x_1,x_2)||_{1,2}=||\mathcal{T}_{\nu_1\nu_2}(x_1,x_2)||_2+||\mathcal{T}^{'}_{\nu_1}(x_1,x_2)||_2+||\mathcal{T}^{'}_{\nu_2}(x_1,x_2)||_2. \tag{26}$$

Using generalized Minkowski's inequality ([6]), we have

$$||\mathcal{T}_{\nu_1\nu_2}(x_1,x_2)||_2\leq\int_0^\pi\int_0^\pi||\Phi(t_1,t_2)||_2|K_{\nu_1}(t_1)||K_{\nu_2}(t_2)|d_{t_1}d_{t_2}.$$

Using Theorem 1(iii), we have

$$\begin{aligned}||\mathcal{T}_{\nu_1\nu_2}(x_1,x_2)||_2\leq&\int_0^\pi\int_0^\pi 2C|t_1-t_2||K_{\nu_1}(t_1)||K_{\nu_2}(t_2)|d_{t_1}d_{t_2}\\\leq&\Big[\int_0^{\frac{1}{\nu_1+1}}\int_0^{\frac{1}{\nu_2+1}}+\int_0^{\frac{1}{\nu_1+1}}\int_{\frac{1}{\nu_2+1}}^\pi+\int_{\frac{1}{\nu_1+1}}^\pi\int_0^{\frac{1}{\nu_2+1}}+\int_{\frac{1}{\nu_1+1}}^\pi\int_{\frac{1}{\nu_2+1}}^\pi\Big]\\&\times\left(2C|t_1-t_2||K_{\nu_1}(t_1)||K_{\nu_2}(t_2)|d_{t_1}d_{t_2}\right)\\=&L_1+L_2+L_3+L_4.\end{aligned} \tag{27}$$

Using Lemma 1, we have

$$\begin{aligned}L_1=&\int_0^{\frac{1}{\nu_1+1}}\int_0^{\frac{1}{\nu_2+1}}2C|t_1-t_2||K_{\nu_1}(t_1)||K_{\nu_2}(t_2)|d_{t_1}d_{t_2}\\\leq&\int_0^{\frac{1}{\nu_1+1}}\int_0^{\frac{1}{\nu_2+1}}2C(|t_1|+|t_2|)|K_{\nu_1}(t_1)||K_{\nu_2}(t_2)|d_{t_1}d_{t_2}\\=&\mathcal{O}\Big[\frac{2Cp_{\nu_1}p_{\nu_2}(\nu_1+1)(\nu_2+1)}{P_{\nu_1}P_{\nu_2}}\int_0^{\frac{1}{\nu_1+1}}\int_0^{\frac{1}{\nu_2+1}}(|t_1|+|t_2|)d_{t_1}d_{t_2}\Big]\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{O}\left[\frac{2Cp_{\nu_1}p_{\nu_2}(\nu_1+1)}{P_{\nu_1}P_{\nu_2}}\int_0^{\frac{1}{\nu_1+1}}\left(|t_1|+\frac{1}{\nu_2+1}\right)d_{t_1}\right]\\
&= \mathcal{O}\left[\frac{2Cp_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\left(\frac{1}{\nu_1+1}+\frac{1}{\nu_2+1}\right)\right]. \qquad (28)
\end{aligned}
$$

Using Lemmas 1 and 2, we have

$$
\begin{aligned}
L_2 &= \int_0^{\frac{1}{\nu_1+1}}\int_{\frac{1}{\nu_2+1}}^{\pi} 2C|t_1-t_2||K_{\nu_1}(t_1)||K_{\nu_2}(t_2)|d_{t_1}d_{t_2}\\
&\leq \int_0^{\frac{1}{\nu_1+1}}\int_{\frac{1}{\nu_2+1}}^{\pi} 2C(|t_1|+|t_2|)|K_{\nu_1}(t_1)||K_{\nu_2}(t_2)|d_{t_1}d_{t_2}\\
&= \mathcal{O}\left[\frac{2Cp_{\nu_1}p_{\nu_2}(\nu_1+1)}{P_{\nu_1}P_{\nu_2}}\int_0^{\frac{1}{\nu_1+1}}\int_{\frac{1}{\nu_2+1}}^{\pi}\left(\frac{|t_1|}{t_2^2}+\frac{1}{|t_2|}\right)d_{t_1}d_{t_2}\right]\\
&= \mathcal{O}\left[\frac{2Cp_{\nu_1}p_{\nu_2}(\nu_1+1)}{P_{\nu_1}P_{\nu_2}}\int_0^{\frac{1}{\nu_1+1}}\left(|t_1|\left((\nu_2+1)-\frac{1}{\pi}\right)+\log\pi(\nu_2+1)\right)d_{t_1}\right]\\
&= \mathcal{O}\left[\frac{2Cp_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\left(\log\pi(\nu_2+1)+\frac{1}{\nu_1+1}\left((\nu_2+1)-\frac{1}{\pi}\right)\right)\right]. \qquad (29)
\end{aligned}
$$

Using Lemmas 1 and 2, we have

$$
\begin{aligned}
L_3 &= \int_{\frac{1}{\nu_1+1}}^{\pi}\int_0^{\frac{1}{\nu_2+1}} 2C|t_1-t_2||K_{\nu_1}(t_1)||K_{\nu_2}(t_2)|d_{t_1}d_{t_2}\\
&\leq \int_{\frac{1}{\nu_1+1}}^{\pi}\int_0^{\frac{1}{\nu_2+1}} 2C(|t_1|+|t_2|)|K_{\nu_1}(t_1)||K_{\nu_2}(t_2)|d_{t_1}d_{t_2}\\
&= \mathcal{O}\left[\frac{2Cp_{\nu_1}p_{\nu_2}(\nu_2+1)}{P_{\nu_1}P_{\nu_2}}\int_{\frac{1}{\nu_1+1}}^{\pi}\int_0^{\frac{1}{\nu_2+1}}\left(\frac{1}{|t_1|}+\frac{|t_2|}{t_1^2}\right)d_{t_1}d_{t_2}\right]\\
&= \mathcal{O}\left[\frac{2Cp_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\int_{\frac{1}{\nu_1+1}}^{\pi}\left(\frac{1}{|t_1|}+\frac{1}{t_1^2(\nu_2+1)}\right)d_{t_1}\right]\\
&= \mathcal{O}\left[\frac{2Cp_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\left(\log\pi(\nu_1+1)+\frac{1}{\nu_2+1}\left((\nu_1+1)-\frac{1}{\pi}\right)\right)\right]. \qquad (30)
\end{aligned}
$$

Using Lemma 2, we have

$$
L_4 = \int_{\frac{1}{\nu_1+1}}^{\pi}\int_{\frac{1}{\nu_2+1}}^{\pi} 2C|t_1-t_2||K_{\nu_1}(t_1)||K_{\nu_2}(t_2)|d_{t_1}d_{t_2}
$$

$$
\begin{aligned}
&\leq \int_{\frac{1}{\nu_1+1}}^{\pi}\int_{\frac{1}{\nu_2+1}}^{\pi} 2C(|t_1|+|t_2|)|K_{\nu_1}(t_1)||K_{\nu_2}(t_2)|d_{t_1}d_{t_2}\\
&=\mathcal{O}\Bigg[\frac{2Cp_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\int_{\frac{1}{\nu_1+1}}^{\pi}\int_{\frac{1}{\nu_2+1}}^{\pi}\bigg(\frac{|t_1|+|t_2|}{t_1^2t_2^2}\bigg)d_{t_1}d_{t_2}\Bigg]\\
&=\mathcal{O}\Bigg[\frac{2Cp_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\int_{\frac{1}{\nu_1+1}}^{\pi}\int_{\frac{1}{\nu_2+1}}^{\pi}\bigg(\frac{1}{t_1t_2^2}+\frac{1}{t_1^2t_2}\bigg)d_{t_1}\Bigg]\\
&=\mathcal{O}\Bigg[\frac{2Cp_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\int_{\frac{1}{\nu_1+1}}^{\pi}\bigg(\frac{\big((\nu_2+1)-\frac{1}{\pi}\big)}{t_1}+\frac{\log\pi(\nu_1+1)}{t_1^2}\bigg)d_{t_1}\Bigg]\\
&=\mathcal{O}\Bigg[\frac{2Cp_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\bigg(\log\pi(\nu_1+1)\bigg((\nu_2+1)-\frac{1}{\pi}\bigg)\\
&\quad+\log\pi(\nu_2+1)\bigg((\nu_1+1)-\frac{1}{\pi}\bigg)\bigg)\Bigg]. \qquad (31)
\end{aligned}
$$

Combining (27) to (31), we have

$$
\begin{aligned}
&||\mathcal{T}_{\nu_1\nu_2}(x_1,x_2)||_2\\
&=\mathcal{O}\Bigg[\frac{2Cp_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\bigg(\frac{1}{\nu_1+1}+\frac{1}{\nu_2+1}+\log\pi(\nu_2+1)+\frac{(\nu_2+1)-\frac{1}{\pi}}{\nu_1+1}\\
&+\log\pi(\nu_1+1)+\frac{1}{\nu_2+1}\bigg((\nu_1+1)-\frac{1}{\pi}\bigg)+\log\pi(\nu_1+1)\bigg((\nu_2+1)\\
&-\frac{1}{\pi}\bigg)+\log\pi(\nu_2+1)\bigg((\nu_1+1)-\frac{1}{\pi}\bigg)\bigg)\Bigg]. \qquad (32)
\end{aligned}
$$

Using (8), the double Riesz transform of the sequence $\{s'_{\nu_1\nu_2}(x_1,x_2)\}$ is given by

$$
\begin{aligned}
&t'_{\nu_1\nu_2}(x_1,x_2)-f'_{x_1}(x_1,x_2)\\
&=\frac{1}{P_{\nu_1}P_{\nu_2}}\sum_{i_1=0}^{\nu_1}\sum_{i_2=0}^{\nu_2}p_{i_1}p_{i_2}\{s'_{i_1i_2}(x_1,x_2)-f'_{x_1}(x_1,x_2)\}\\
&=\frac{1}{P_{\nu_1}P_{\nu_2}}\sum_{i_1=0}^{\nu_1}\sum_{i_2=0}^{\nu_2}p_{i_1}p_{i_2}\Bigg[\frac{1}{\pi^2}\iint_{A^2}\frac{[\sin(i_1+\frac{1}{2})t_1][\sin(i_2+\frac{1}{2})t_2]}{4\sin(\frac{t_1}{2})\sin(\frac{t_2}{2})}dg_{x_1}(t_1,t_2)\Bigg]\\
&=\iint_{A^2}\Bigg[\frac{1}{\pi^2}\frac{1}{P_{\nu_1}P_{\nu_2}}\sum_{i_1=0}^{\nu_1}\sum_{i_2=0}^{\nu_2}p_{i_1}p_{i_2}\frac{[\sin(i_1+\frac{1}{2})t_1][\sin(i_2+\frac{1}{2})t_2]}{4\sin(\frac{t_1}{2})\sin(\frac{t_2}{2})}dg_{x_1}(t_1,t_2)\Bigg]. \qquad (33)
\end{aligned}
$$

Thus,

$$
t'_{\nu_1\nu_2}(x_1,x_2)-f'_{x_1}(x_1,x_2)=\iint_{A^2}K_{\nu_1}(t_1)K_{\nu_2}(t_2)dg_{x_1}(t_1,t_2). \qquad (34)
$$

Using generalized Minkowski's inequality ([6]), we have

$$
\begin{aligned}
||\mathcal{T}_{\nu_1}'(x_1,x_2)||_2 \leq & \iint_{A^2} |K_{\nu_1}(t_1)|K_{\nu_2}(t_2)||dg_{x_1}(t_1,t_2)| \\
\leq & \Bigg[\int_0^{\frac{1}{\nu_1+1}}\int_0^{\frac{1}{\nu_2+1}} + \int_0^{\frac{1}{\nu_1+1}}\int_{\frac{1}{\nu_2+1}}^{\pi} + \int_{\frac{1}{\nu_1+1}}^{\pi}\int_0^{\frac{1}{\nu_2+1}} + \int_{\frac{1}{\nu_1+1}}^{\pi}\int_{\frac{1}{\nu_2+1}}^{\pi}\Bigg] \\
& \Bigg(|K_{\nu_1}(t_1)||K_{\nu_2}(t_2)||dg_{x_1}(t_1,t_2)|\Bigg) \\
= & J_1 + J_2 + J_3 + J_4. \qquad (35)
\end{aligned}
$$

Using Lemma 1, we have

$$
\begin{aligned}
J_1 = & \Bigg[\int_0^{\frac{1}{\nu_1+1}}\int_0^{\frac{1}{\nu_2+1}} |K_{\nu_1}(t_1)||K_{\nu_2}(t_2)||dg_{x_1}(t_1,t_2)|\Bigg] \\
= & \mathcal{O}\Bigg[\frac{p_{\nu_1}p_{\nu_2}(\nu_1+1)(\nu_2+1)}{P_{\nu_1}P_{\nu_2}}\int_0^{\frac{1}{\nu_1+1}}\int_0^{\frac{1}{\nu_2+1}} |dg_{x_1}(t_1,t_2)|\Bigg]. \qquad (36)
\end{aligned}
$$

Using Lemmas 1 and 2, we have

$$
\begin{aligned}
J_2 = & \int_0^{\frac{1}{\nu_1+1}}\int_{\frac{1}{\nu_2+1}}^{\pi} |K_{\nu_1}(t_1)||K_{\nu_2}(t_2)||dg_{x_1}(t_1,t_2)| \\
= & \mathcal{O}\Bigg[\frac{p_{\nu_1}p_{\nu_2}(\nu_1+1)}{P_{\nu_1}P_{\nu_2}}\int_0^{\frac{1}{\nu_1+1}}\int_{\frac{1}{\nu_2+1}}^{\pi} \frac{|dg_{x_1}(t_1,t_2)|}{t_2^2}\Bigg]. \qquad (37)
\end{aligned}
$$

Using Lemmas 1 and 2, we have

$$
\begin{aligned}
J_3 = & \int_{\frac{1}{\nu_1+1}}^{\pi}\int_0^{\frac{1}{\nu_2+1}} |K_{\nu_1}(t_1)||K_{\nu_2}(t_2)||dg_{x_1}(t_1,t_2)| \\
= & \mathcal{O}\Bigg[\frac{p_{\nu_1}p_{\nu_2}(\nu_2+1)}{P_{\nu_1}P_{\nu_2}}\int_{\frac{1}{\nu_1+1}}^{\pi}\int_0^{\frac{1}{\nu_2+1}} \frac{|dg_{x_1}(t_1,t_2)|}{t_1^2}\Bigg]. \qquad (38)
\end{aligned}
$$

Using Lemma 2, we have

$$
\begin{aligned}
J_4 = & \int_{\frac{1}{\nu_1+1}}^{\pi}\int_{\frac{1}{\nu_2+1}}^{\pi} |K_{\nu_1}(t_1)||K_{\nu_2}(t_2)||dg_{x_1}(t_1,t_2)| \\
= & \mathcal{O}\Bigg[\frac{p_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\int_{\frac{1}{\nu_1+1}}^{\pi}\int_{\frac{1}{\nu_2+1}}^{\pi} \frac{|dg_{x_1}(t_1,t_2)|}{t_1^2t_2^2}\Bigg]. \qquad (39)
\end{aligned}
$$

Combining (35) to (39), we have

$$||\mathcal{T}'_{\nu_1}(x_1,x_2)||_2 = \mathcal{O}\Bigg[\frac{p_{\nu_1}p_{\nu_2}(\nu_1+1)(\nu_2+1)}{P_{\nu_1}P_{\nu_2}}\int_0^{\frac{1}{\nu_1+1}}\int_0^{\frac{1}{\nu_2+1}}|dg_{x_1}(t_1,t_2)| + \frac{p_{\nu_1}p_{\nu_2}(\nu_1+1)}{P_{\nu_1}P_{\nu_2}}\int_0^{\frac{1}{\nu_1+1}}\int_{\frac{1}{\nu_2+1}}^{\pi}\frac{|dg_{x_1}(t_1,t_2)|}{t_2^2} + \frac{p_{\nu_1}p_{\nu_2}(\nu_2+1)}{P_{\nu_1}P_{\nu_2}} \int_{\frac{1}{\nu_1+1}}^{\pi}\int_0^{\frac{1}{\nu_2+1}}\frac{|dg_{x_1}(t_1,t_2)|}{t_1^2} + \frac{p_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\int_{\frac{1}{\nu_1+1}}^{\pi}\int_{\frac{1}{\nu_2+1}}^{\pi}\frac{|dg_{x_1}(t_1,t_2)|}{t_1^2t_2^2}\Bigg]. \tag{40}$$

Similarly, we have

$$||\mathcal{T}'_{\nu_1}(x_1,x_2)||_2 = \mathcal{O}\Bigg[\frac{p_{\nu_1}p_{\nu_2}(\nu_1+1)(\nu_2+1)}{P_{\nu_1}P_{\nu_2}}\int_0^{\frac{1}{\nu_1+1}}\int_0^{\frac{1}{\nu_2+1}}|dg_{x_2}(t_1,t_2)| + \frac{p_{\nu_1}p_{\nu_2}(\nu_1+1)}{P_{\nu_1}P_{\nu_2}}\int_0^{\frac{1}{\nu_1+1}}\int_{\frac{1}{\nu_2+1}}^{\pi}\frac{|dg_{x_2}(t_1,t_2)|}{t_2^2} + \frac{p_{\nu_1}p_{\nu_2}(\nu_2+1)}{P_{\nu_1}P_{\nu_2}} \int_{\frac{1}{\nu_1+1}}^{\pi}\int_0^{\frac{1}{\nu_2+1}}\frac{|dg_{x_2}(t_1,t_2)|}{t_1^2} + \frac{p_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\int_{\frac{1}{\nu_1+1}}^{\pi}\int_{\frac{1}{\nu_2+1}}^{\pi}\frac{|dg_{x_2}(t_1,t_2)|}{t_1^2t_2^2}\Bigg]. \tag{41}$$

Combining (26), (32) (40) and (41), we have

$$\begin{aligned}
&||\mathcal{T}_{\nu_1\nu_2}(x_1,x_2)||_{1,2} \\
&= \mathcal{O}\Bigg[\frac{2Cp_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\Bigg(\frac{1}{\nu_1+1}+\frac{1}{\nu_2+1}+\log\pi(\nu_2+1)+\frac{1}{\nu_1+1} \\
&\Bigg((\nu_2+1)-\frac{1}{\pi}\Bigg)+\log\pi(\nu_1+1)+\frac{1}{\nu_2+1}\Bigg((\nu_1+1)-\frac{1}{\pi}\Bigg) \\
&+\log\pi(\nu_1+1)\Bigg((\nu_2+1)-\frac{1}{\pi}\Bigg)+\log\pi(\nu_2+1)\Bigg((\nu_1+1)-\frac{1}{\pi}\Bigg)\Bigg) \\
&+\frac{p_{\nu_1}p_{\nu_2}(\nu_1+1)(\nu_2+1)}{P_{\nu_1}P_{\nu_2}}\int_0^{\frac{1}{\nu_1+1}}\int_0^{\frac{1}{\nu_2+1}}|dg_{x_1}(t_1,t_2)| \\
&+\frac{p_{\nu_1}p_{\nu_2}(\nu_1+1)}{P_{\nu_1}P_{\nu_2}}\int_0^{\frac{1}{\nu_1+1}}\int_{\frac{1}{\nu_2+1}}^{\pi}\frac{|dg_{x_1}(t_1,t_2)|}{t_2^2}+\frac{p_{\nu_1}p_{\nu_2}(\nu_2+1)}{P_{\nu_1}P_{\nu_2}} \\
&\int_{\frac{1}{\nu_1+1}}^{\pi}\int_0^{\frac{1}{\nu_2+1}}\frac{|dg_{x_1}(t_1,t_2)|}{t_1^2}+\frac{p_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\int_{\frac{1}{\nu_1+1}}^{\pi}\int_{\frac{1}{\nu_2+1}}^{\pi}\frac{|dg_{x_1}(t_1,t_2)|}{t_1^2t_2^2}\Bigg]
\end{aligned}$$

$$+\frac{p_{\nu_1}p_{\nu_2}(\nu_1+1)(\nu_2+1)}{P_{\nu_1}P_{\nu_2}}\int_0^{\frac{1}{\nu_1+1}}\int_0^{\frac{1}{\nu_2+1}}|dg_{x_2}(t_1,t_2)|$$
$$+\frac{p_{\nu_1}p_{\nu_2}(\nu_1+1)}{P_{\nu_1}P_{\nu_2}}\int_0^{\frac{1}{\nu_1+1}}\int_{\frac{1}{\nu_2+1}}^{\pi}\frac{|dg_{x_2}(t_1,t_2)|}{t_2^2}+\frac{p_{\nu_1}p_{\nu_2}(\nu_2+1)}{P_{\nu_1}P_{\nu_2}}$$
$$\int_{\frac{1}{\nu_1+1}}^{\pi}\int_0^{\frac{1}{\nu_2+1}}\frac{|dg_{x_2}(t_1,t_2)|}{t_1^2}+\frac{p_{\nu_1}p_{\nu_2}}{P_{\nu_1}P_{\nu_2}}\int_{\frac{1}{\nu_1+1}}^{\pi}\int_{\frac{1}{\nu_2+1}}^{\pi}\frac{|dg_{x_2}(t_1,t_2)|}{t_1^2t_2^2}\Bigg].$$

4. Degree of convergence of a function of N dimensional variable of N-Multiple Fourier series in Sobolev spaces

In this section, we study the degree of convergence of a function of N dimensional variable of N-Multiple Fourier series in Sobolev spaces using N dimensional Riesz means.

Let $f(x_1,\cdots,x_N)$ be integrable over the N dimensional cube A^N and of period 2π in each variable. The N-multiple Fourier series of $f(x_1,x_2,\cdots,x_N)$ can be written in the form

$$f(x_1,\cdots,x_N)\sim\sum_{\nu_1\in\mathbb{Z}}\sum_{\nu_2\in\mathbb{Z}}\cdots\sum_{\nu_N\in\mathbb{Z}}c_{\nu_1,\nu_2,\cdots,\nu_N}e^{i(\nu_1x_1+\nu_2x_2,\cdots,\nu_Nx_N)},$$

where $c_{\nu_1,\nu_2,\cdots,\nu_N}$ are the Fourier coefficients of f. The series is denoted by $S[f]$ and the partial sums of it are given by

$$s_{\nu_1\nu_2\cdots\nu_N}(x_1,\cdots,x_N)=\pi^{-N}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}f(x_1+t_1,\cdots,x_N+t_N)\prod_{i=1}^{N}D_{\nu_i}(t_i)dt_1\cdots dt_N,$$

where $D_{\nu_i}(t_i)$ are the Dirichlet kernels for each i. Moreover, similar to the two dimensional case, we can write

$$t_{\nu_1\nu_2\cdots\nu_N}(x_1,\cdots,x_N)=\frac{1}{P_{\nu_1}P_{\nu_2}\cdots P_{\nu_N}}\sum_{i_1=0}^{\nu_1}\sum_{i_2=0}^{\nu_2}\cdots\sum_{i_N=0}^{\nu_N}p_{i_1}p_{i_2}\cdots p_{i_N}s_{i_1i_2\cdots i_N}$$

for all $\nu_k\geq 0$, which is called multiple Riesz $(R,p_{\nu_1},\cdots,p_{\nu_N})$ means of the sequence $s_{\nu_1\nu_2}\cdots\nu_N$. The series $\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_N=0}^{\infty}u_{i_1i_2\cdots i_N}$ is said to be summable to the sum s by multiple Riesz method if we can write $t_{\nu_1\nu_2\cdots\nu_N}\to s$ as $\nu_1,\nu_2,\cdots,\nu_N\to\infty$.

The necessary and sufficient conditions for $(R, p_{\nu_1} \cdots p_{\nu_N})$ method to be regular are given by

$$\sum_{i_1=0}^{\nu_1} \sum_{i_2=0}^{\nu_2} \cdots \sum_{i_1=0}^{\nu_1} |p_{i_1} p_{i_2} \cdots p_{i_N}| < c|P_{\nu_1} P_{\nu_2} \cdots P_{\nu_N}|, \ |P_{\nu_1} P_{\nu_2} \cdots P_{\nu_N}| \to \infty.$$

Now we extend Theorem 2 to N-dimensional case.

Theorem 3. *Let $f(x_1, x_2, \cdots, x_N)$ be a periodic function with period 2π in each variable and Lebesgue integrable on A^N. Then, the degree of convergence of the function $f(x_1, x_2, \cdots, x_N)$ in Sobolev spaces $W^{1,2}(\mathbf{X})$ using multiple Riesz means of multiple Fourier series is given by*

$$\begin{aligned} ||\mathcal{T}_{\nu_1 \nu_2 \cdots \nu_N}(x_1, x_2, \cdots, x_N)||_{1,2} =& \mathcal{O}\Bigg[2C\left(\prod_{i=1}^{N} \frac{p_{\nu_i}}{P_{\nu_i}}\right)\Bigg\{\left(\sum_{i=1}^{N} \frac{1}{\nu_i + 1}\right) + \cdots \\ & + \left(\prod_{i=1}^{N}((\nu_i + 1) - \frac{1}{\pi})\right) \log \pi(\nu_N + 1)\Bigg\} \\ & + \left(\prod_{i=1}^{N} \frac{p_{\nu_i}(\nu_i + 1)}{P_{\nu_i}}\right) \int_0^{\frac{1}{\nu_1+1}} \int_0^{\frac{1}{\nu_2+1}} \cdots \int_0^{\frac{1}{\nu_N+1}} \\ & |dg_{t_1}(t_1 t_{t_2} \cdots t_N)| + \cdots + \left(\prod_{i=1}^{N} \frac{p_{\nu_i}}{P_{\nu_i}}\right) \int_{\frac{1}{\nu_1+1}}^{\pi} \\ & \int_{\frac{1}{\nu_2+1}}^{\pi} \cdots \int_{\frac{1}{\nu_N+1}}^{\pi} \frac{|dg_{t_N}(t_1 t_{t_2} \cdots t_N)|}{t_1{}^2 t_2^2 \cdots t_N^2}\Bigg]. \end{aligned}$$

Similar to the two-dimensional case, we can define the lemmas for N-dimensional case.

Lemma 3. *Let $\{p_{\nu_1}\}, \cdots, \{p_{\nu_N}\}$ and are non-negative and non-decreasing sequences then*

$$\begin{aligned} |\mathcal{K}_{\nu_1}(t_1)| =& \mathcal{O}\left(\frac{p_{\nu_1}(\nu_1 + 1)}{P_{\nu_1}}\right) \text{ for } 0 < t_1 \le \frac{1}{\nu_1 + 1} \quad . \\ & \vdots \\ |\mathcal{K}_{\nu_N}(t_N)| =& \mathcal{O}\left(\frac{p_{\nu_N}(\nu_N + 1)}{P_{\nu_N}}\right) \text{ for } 0 < t_N \le \frac{1}{\nu_N + 1}. \end{aligned}$$

Lemma 4. *Let $\{p_{\nu_1}\}, \cdots, \{p_{\nu_N}\}$ be non-negative and non-decreasing sequences. Then*

$$|\mathcal{K}_{\nu_1}(t_1)| = \mathcal{O}\left(\frac{p_{\nu_1}}{t_1^2 P_{\nu_1}}\right) \text{ for } \frac{1}{\nu_1+1} < t_1 \leq \pi, \qquad .$$

$$\vdots$$

$$|\mathcal{K}_{\nu_N}(t_N)| = \mathcal{O}\left(\frac{p_{\nu_2}}{t_N^2 P_{\nu_N}}\right) \text{ for } \frac{1}{\nu_N+1} < t_N \leq \pi$$

Proof. Using Lemmas 3 and 4, we prove this result along the same lines of the proof of Theorem 2. ◄

5. Conclusion

In this paper, we obtained the degree of convergence of the function of double Fourier series in Sobolev norm using double Riesz means. We also obtained degree of convergence of a function of ℕ-dimensional variable of ℕ-multiple Fourier series in Sobolev spaces using ℕ-dimensional Riesz means which is an extension of two-dimesional case.

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