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On Sobolev-Poincare-Friedrichs Type Weight Inequalities

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Abstract. In this paper, we prove a Long-Nie type results on Sobolev-Poincare and Friedrichs inequalities

$$\Big(\int\limits_{\Omega}|f(x)|^q v(x)dx\Big)^{1/q} \leq C\Big(\int\limits_{\Omega}|\nabla f(x)|^p \omega dx\Big)^{1/p}, \quad q \geq p > 1,$$

where f is a locally Lipschitz function on Ω, the weights $v, \sigma = \omega^{-\frac{1}{p-1}} \in L^{1,loc}$ satisfy some cube conditions and Ω is a convex bounded domain in the case of Poincare's inequality. This result generalizes previously known weighted inequalities to more general class of weights.

Key Words and Phrases: weights, Sobolev inequality, Friedrich inequality, Poincare inequality, embedding, trace inequality.

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1. Introduction

In this paper, we prove new results on Poincare and Friedrichs type gradient inequalities. In the case of Sobolev's inequality, we get a new proof for the known R. Long and F. Nie's result [13]. A unique approach has been applied for proving the mentioned inequalities based not on the representation formula or inequalities (see (1) below).

There are a lot of papers dealing with the sufficiency conditions on weight functions for Poincare's type inequality to hold in regular domains (see, e.g. in [2, 4, 5, 7, 27]). We also prove the sufficiency conditions in this paper. Our class of weight functions (see (11) and (7)) for Poincare and Friedrichs inequalities is

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wider than in the known results. Note that a characterization of classical (p, q) -Poincare's inequality in terms of nonlinear capacity was given by V.G. Mazya in 1960s (see, e.g., [18, 19]).

The above mentioned inequalities are useful in analysis and in the theory of partial differential equations, especially in the study of Harnack's inequality and regularity of solutions of degenerated elliptic equations, continuation for the differential inequalities, absence of positive eigenvalues, estimation of negative eigenvalues and discreetness of spectrum of the Schrödinger operator (see [6, 8, 10, 12, 20, 25, 27]). As noted in [9], there exists a strict connection between Friedrichs inequality and Rellich's theorem concerning compact embedding. In [21, 5], such inequalities were applied to the error estimation of numerical solutions of degenerate elliptic equations.

One way for producing Sobolev's inequality (sometimes Poincare's inequality too) goes through the estimate

$$|f(x)| \leq C_n \, I_1\left(|\nabla f(x)|\right) \tag{1}$$

(see e.g. [26, 29]) that allows to apply weighted boundedness results for fractional integrals (those, for example, of [2, 11, 21, 22, 23, 27, 30]). By D.R. Adams'es general result on fractional integrals [1], it follows that the inequality (4) for the case $q > p$ and $v \in L^{1,loc}$, $\omega^{-\frac{1}{p-1}} \in A_\infty$ holds if

$$\sup_{x,r} \Big(\int_{|y-x|<r} v(y)dy\Big)^{\frac{1}{q}} \Big(\int_r^\infty t^{\frac{(1-np)}{p-1}} \big(\int_{|y-x|<t} \omega^{-\frac{1}{p-1}}(y)dy\big)dt\Big)^{\frac{p-1}{p}} < \infty$$

This assertion is not true for $q = p$. For the case $q = p$, according to Fefferman-Phong's [8, 22] result, the "r-bomp" condition for some $r > 1$:

$$|Q|^{1/n-1/p+1/q}\Big(\frac{1}{|Q|}\int\limits_Q v^r\,dx\Big)^{1/rq}\Big(\frac{1}{|Q|}\int\limits_Q \sigma^r\,dx\Big)^{1/rp'} \leq C$$

suffices for the validity of inequality (4) below. In [3] for $p = 2$ and in [2] for $p > 1$, a weaker sufficient condition for the validity of inequality (4) was found:

$$\int\limits_{\mathbb{R}^n} |f(x)|^p v dx \leq C \int\limits_{\mathbb{R}^n} |\nabla f(x)|^p dx, \; f \in Lip_0(\mathbb{R}^n). \tag{2}$$

Recall that result for $p = 2$. Let $\phi : [0, \infty) \to [1, \infty)$ be an increasing and doubling function such that $\int\limits_1^\infty \frac{1}{\phi(t)} \frac{dt}{t} < \infty$. Then for (2) to hold it is sufficient that

$$|Q|^{2/n-1} \int\limits_Q v(y)\phi\left(v(y)|Q|^{2/n}\right) dy \leq C. \tag{3}$$

An interesting corollary follows from (3) if we put $\phi(t) = \left(1 + \ln^+ t\right)^{1+\varepsilon}, \varepsilon > 0$. See also [28] for sufficiency conditions for the inequality (2). For other relevant results see also [15, 16, 17, 22].

This paper is focused on the weighted inequality

$$\Big(\int\limits_{\Omega} |f(x)|^q v dx\Big)^{1/q} \le C\Big(\int\limits_{\Omega} |\nabla f(x)|^p \omega dx\Big)^{1/p}, \quad q \ge p > 1 \tag{4}$$

in the class of functions $f \in Lip(\Omega)$ of Sobolev, Poincare, and Fredirichs types (convex bounded domains are considered in the case of Poincare's inequality). Using the equivalence of weak type inequalities to those of strong type for the Sobolev type inequalities proved by Sawyer-Wheeden [27] and based on other approaches, R. Long and F. Nie [13] proved (4) in $\mathbb{R}^n$ under the condition

$$\int\limits_{Q} \Big(\int\limits_{Q} \frac{v\, dx}{|x-y|^{n-1}}\Big)^{p'} \sigma(x)dx \le C_1\Big(\int\limits_{Q} vdx\Big)^{p'/q'} \tag{5}$$

for all cubes $Q \subset \mathbb{R}^n$ and weights $v, \sigma = \omega^{-\frac{1}{p-1}} \in L^{1,loc}$.

It is worth noting that the technique we will use for the proof of our results differs completely from the ones referred before. The one important detail in our result is that the integrations in condition (5) are taken not over the whole of the ball, but only over its intersection with the domain. Also we produce a simpler and clear proof for Friedrichs inequality when the integral $\int_{\mathbb{R}^n} vdx$ diverges and $f(x)$ is a Lipschitz continuous function. In Theorem 1, we derive new weight inequalities of Friedrichs and Hardy types (4) in whole $\mathbb{R}^n$. Note that there are many results on Friedrichs inequality in the bounded domains. To the best of our knowledge, those results for all space are new even for the unweighted cases. Concerning the Hardy type inequality, we have considered the class of functions f with finite $v(\Omega_\alpha)$ for a.e. $\alpha > 0$. This condition is satisfied, e.g., if $\int_{\mathbb{R}^n} |v(x)|^\beta dx < \infty$ for some $\beta > 0$.

The following main results are obtained in this paper.

Theorem 1. *Let $q \ge p > 1$, positive measurable functions $v, \sigma = \omega^{-1/(p-1)} \in L^{1,loc}$, and integral $\int\limits_{\mathbb{R}^n} vdx$ diverge. Let f be a Lipschitz continuous function in $\mathbb{R}^n$ satisfying one of the following conditions:*

1) There exist $r > 0$, $\delta \in (0,1)$ such that for any cube $Q(x,r)$

$$v\big(Q(x,r) \setminus supp\ f\big) > \delta v\big(Q(x,r)\big) \quad \textit{(Friedrich's type);}$$

2)

$$\lim_{x\to\infty} f(x) = 0 \quad \textit{(Sobolev's type);}$$

3) for a.e. fixed $\alpha > 0$

$$v\big(\{x : |f(x)| > \alpha\}\big) < \infty \quad \textit{(Hardy's type)}$$

.

Then the inequality

$$\Big(\int\limits_{\mathbb{R}^n} |f(x)|^q v(x)dx \Big)^{1/q} \le C_0\, A \Big(\int\limits_{\mathbb{R}^n} |\nabla f(x)|^p \omega dx \Big)^{1/p} \tag{6}$$

holds with positive constant C_0 *depending on* n, p, q

if

$$\int\limits_{Q\cap\Omega} \Big(\int\limits_{Q\cap\Omega} \frac{v(x)dx}{|x-y|^{n-1}} \Big)^{p'} \sigma(y)dy \le A^{p'} \Big(\int\limits_{Q\cap\Omega} v(x)dx \Big)^{p'/q'} \tag{7}$$

for all balls $\{ Q = Q(x,t) : \ x \in \Omega,\, 0 < t < r\}$; *we assume* $r := \infty$ *in cases 2) and 3) with* $\Omega :=$ *supp* $f = \{x \in \mathbb{R}^n : \ f(x) \ne 0\}$.

In the case of Poincare's inequality (4), we consider the functions $f \in Lip\,(\Omega)$ with

$$\int_\Omega f v dx = 0, \tag{8}$$

and the convex domain Ω. In this case we suppose that there exist $\varepsilon, \delta \in (0,1)$ such that

$$|Q \cap \Omega| \ge \delta\, |Q| \tag{9}$$

implies

$$v\,(Q \cap \Omega) \ge \varepsilon v\,(Q) \tag{10}$$

for a ball $Q = Q\,(x, \rho)$ with $x \in \Omega, \quad 0 < \rho < diam\ \Omega$.

Theorem 2. *Let* $1 < p \le q < \infty, \quad v, \omega^{l-p'} \in L^{1,loc}$. *Suppose* $\Omega \subset \mathbb{R}^n$ *is a bounded convex domain satisfying* (10) *together with the function* v. *Then for the inequality* (4) *to hold for the functions* $f \in Lip(\Omega)$ *satisfying* (8) *it is sufficient that condition*

$$\int\limits_{Q\cap\Omega} \sigma\,(x) \left(\int\limits_{Q\cap\Omega} \frac{v\,(y)\, dy}{|y-x|^{n-1}} \right)^{p'} dx \le A^{p'} \left(\int\limits_{Q\cap\Omega} v\,(x)\, dx \right)^{\frac{p'}{q'}} \tag{11}$$

hold for all balls $Q = Q\,(x, \rho)\,,\ x \in \Omega,\ \rho \in (0, d_\Omega)$ *and some* $A > 0$.

2. Notation

For a measurable set E, denote by $|E|$ the n-dimensional Lebesgue measure of E. For a measurable set E and a measurable function f, set $f(E) = \int_E f(x)dx$. Denote by Q a ball and $Q(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$. The diameter of domain Ω will be denoted by $d_\Omega = \sup\{|x-y| : x, y \in \Omega\}$.

$C, C_1, C_2, ...$ will be positive constants, which can be different in different places. $C = C(...)$ means that the constant C depends on the parameters in the parentheses. We will say that w is a weight function if it is a measurable function taking a.e. positive finite values. For a differentiable function f, denote by ∇f the gradient vector $\{\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_1}, ... \frac{\partial u}{\partial x_n}\}$ and $|\nabla f|^2 = (\partial f/\partial x_1)^2 + (\partial f/\partial x_2)^2 + ... + (\partial f/\partial x_n)^2$.

3. Proof of Theorem 1

Proof. Denote

$$\Omega_\alpha = \{x \in \mathbb{R}^n : |f(x)| > \alpha\}.$$

Let $\alpha > 0$ be such that $\Omega_{3\alpha} \neq \emptyset$. Fix a point $x \in \Omega_{3\alpha}$. In all three cases 1)-3), we will show that there exists a ball $Q = Q(x, \rho(x))$ such that

$$v(Q \setminus \Omega_\alpha) = \gamma\, v(Q), \tag{12}$$

where $\gamma \in (0, \delta)$ is a fixed constant to be specified later.

To show (12), consider the auxiliary function

$$F(t) = \frac{1}{\gamma} v\big(Q(x,t) \setminus \Omega_\alpha\big) - v(Q(x,t)), \quad t > 0. \tag{13}$$

Clearly, $F(t)$ is continuous and $\Omega_{3\alpha}$ is an open set since f is continuous.

In case 1), $Q(x,t) \subset \Omega_{3\alpha} \subset \Omega_\alpha$ for sufficiently small $t > 0$ and therefore $Q(x,t) \setminus \Omega_\alpha$ is empty for such t. Denote by t' the list $t' \geq 0$ such that $v\big(Q(x,t') \setminus \Omega_\alpha\big) = 0$. Evidently, $F(t') < 0$. On the other hand, for $t = r$ we have $v\big(Q(x,r) \setminus \Omega_\alpha\big) \geq v\big(Q(x,r) \setminus \Omega\big) > \delta v\big(Q(x,r)\big) > \gamma v\big(Q(x,r)\big)$. By the Cauchy theorem, there exists $t_0 \in [t', r]$ such that $F(t_0) = 0$; we put $\rho(x) = t_0$ and get (12).

In case 2), $\Omega_{3\alpha}$ is a bounded open set. Again there exists a small t' such that $F(t') < 0$. Due to the assumption $\lim_{x\to\infty} f(x) = 0$, the set $\Omega_{3\alpha}$ is contained in some large ball $Q = Q(0,R)$. Taking into the account the condition $v\big(\mathbb{R}^n\big) = \infty$, we have $v\big(Q(x,t) \setminus \Omega_{3\alpha}\big) > \gamma v\big(Q(x,t)\big)$ for sufficiently large $t > 0$. Therefore, there exists an $r = r(x) > 0$ such that $F(r) > 0$ and there exists $t_0 \in [t', r]$ such that $F(t_0) = 0$; we put $\rho(x) = t_0$ and get (12).

In case 3), again there exists a small t' such that $F(t') < 0$. By assumptions, $v\big(\mathbb{R}^n\big) = \infty$ and $v\big(\Omega_\alpha\big) < \infty$ for a.e. $\alpha > 0$. Therefore, there exists a large $r = r(x) > 0$ such that $v\big(Q(x,r) \setminus \Omega_{3\alpha}\big) > \gamma v\big(Q(x,r)\big)$. This means $F(r) > 0$ and there exists $t_0 \in [t', r]$ such that $F(t_0) = 0$; we put $\rho(x) = t_0$ and get (12).

Now, let $v\big(Q \cap \Omega_{3\alpha}\big) < \gamma v\big(Q\big)$. Then

$$\begin{aligned} v(Q) &= v\left(Q \cap \Omega_\alpha\right) + v\left(Q \setminus \Omega_\alpha\right) \\ &\le \gamma v(Q) + v\left(Q \cap \Omega_\alpha\right) \end{aligned} \tag{14}$$

or

$$v(Q) \le \frac{1}{1-\gamma} v\left(Q \cap \Omega_\alpha\right). \tag{15}$$

Therefore,

$$v\left(Q \cap \Omega_{3\alpha}\right) \le \frac{\gamma}{1-\gamma} v\left(Q \cap \Omega_\alpha\right). \tag{16}$$

Now, let

$$v\big(Q \cap \Omega_{3\alpha}\big) \ge \gamma v(Q). \tag{17}$$

There are two possibilities:

$$a) \quad \Big|Q \cap \Omega_{2\alpha}\Big| \ge \frac{1}{2}|Q|; \tag{18}$$

$$b) \quad \Big|Q \setminus \Omega_{2\alpha}\Big| \ge \frac{1}{2}|Q|. \tag{19}$$

If a) is satisfied, using (12) we have

$$\int\limits_{Q\cap\Omega_{2\alpha}} dy \int\limits_{Q\setminus\Omega_\alpha} v(x)dx \ge \frac{\gamma}{2}|Q|v(Q). \tag{20}$$

Fix the points $x \in Q \cap \Omega_{2\alpha}$ and $y \in Q \setminus \Omega_\alpha$. The line $\{x + t(y-x) : 0 < t < 1\}$ lies in Q. There are $t_1 < t_2$ depending on x, y such that $|f(x + t_1(y-x))| = 2\alpha$, $\quad |f(x+t_2(y-x))| = \alpha$. Note that $\{t_1 = t_1(x,y)\}$ and $\{t_2 = t_2(x,y)\}$ depend on x, y.

By (51),

$$\begin{aligned} 1 &\le \frac{2}{\alpha\gamma}\frac{1}{|Q|v(Q)} \int\limits_{Q\cap\Omega_{2\alpha}} \Big[\int\limits_{Q\setminus\Omega_\alpha} \Big||f\left(x+t_2(y-x)\right)| - |f\left(x+t_1(y-x)\right)|\Big| v(x)dx\Big] dy \\ &\le \frac{2}{\alpha\gamma}\frac{1}{|Q|v(Q)} \int\limits_{Q\cap\Omega_{2\alpha}} \Big[\int\limits_{Q\setminus\Omega_\alpha} \Big(\int\limits_{t_1(x,y)}^{t_2(x,y)} \Big|\frac{\partial}{\partial t}|f\left(x+t(y-x)\right)|\Big| dt\Big) v(x)dx\Big] dy \end{aligned}$$

$$\leq \frac{2}{\alpha\gamma}\frac{1}{|Q|v(Q)}\int\limits_{Q\cap\Omega_{2\alpha}}\Big[\int\limits_{Q\setminus\Omega_\alpha}\Big(\int\limits_{t_1(x,y)}^{t_2(x,y)}\Big|(x-y)\cdot\nabla|f\left(x+t(y-x)\right)|\Big|dt\Big)v(x)dx\Big]dy$$

$$\leq \frac{2}{\alpha\gamma}\frac{d(Q)}{|Q|v(Q)}\int\limits_{Q\cap\Omega_{2\alpha}}\Big[\int\limits_{Q\setminus\Omega_\alpha}\Big(\int\limits_{t_1(x,y)}^{t_2(x,y)}\big|\nabla f\left(x+t(y-x)\right)\big|dt\Big)v(x)dx\Big]dy$$

by Fubini's theorem,

$$\leq \frac{2}{\alpha\gamma c_n}\frac{d(Q)^{1-n}}{v(Q)}\int\limits_{Q\setminus\Omega_\alpha}\Big[\int\limits_0^1\Big(\int\limits_{\left\{\substack{y\in Q\cap\Omega_{2\alpha}:\\ x+t(y-x)\in Q\cap\Omega_\alpha\setminus\Omega_{2\alpha}}\right\}}\big|\nabla f\left(x+t(y-x)\right)\big|dy\Big)dt\Big]v(x)dx, \tag{21}$$

where c_n is a volume of a unit ball in $\mathbb{R}^n$. Making change of variables $z = x+t(y-x)$, $y \to z$, we see that the right hand side of (21) is exceeded by

$$\frac{2}{\alpha\gamma c_n}\frac{d(Q)^{1-n}}{v(Q)}\int\limits_{Q\setminus\Omega_\alpha}\Big[\int\limits_0^1\Big(\int\limits_{\left\{\substack{z\in Q\cap\Omega_\alpha\setminus\Omega_{2\alpha}:\\ x+\frac{z-x}{t}\in Q\cap\Omega_{2\alpha}}\right\}}\big|\nabla f(z)\big|dz\Big)t^{-n}dt\Big]v(x)dx$$

$$= \frac{2}{\alpha\gamma c_n}\frac{d(Q)^{1-n}}{v(Q)}\int\limits_{Q\setminus\Omega_\alpha}\Big[\int\limits_{Q\cap\Omega_\alpha\setminus\Omega_{2\alpha}}\Big(\int\limits_{\left\{\substack{0<t<1:\\ x+\frac{z-x}{t}\in Q\cap\Omega_{2\alpha}}\right\}}t^{-n}\,dt\Big)\big|\nabla f(z)\big|dz\Big]v(x)dx$$

Since $z = x+t(y-x)$, $t|y-x| = |z-x|$, we have $t \geq \frac{|z-x|}{d(Q)}$, hence

$$1 \leq \frac{2d(Q)^{1-n}}{\alpha\gamma c_n}\frac{1}{v(Q)}\int\limits_{Q\setminus\Omega_\alpha}\Big(\int\limits_{Q\cap\Omega_\alpha\setminus\Omega_{2\alpha}}\Big(\int_{\frac{|z-x|}{d(Q)}}^1 t^{-n}dt\Big)\big|\nabla f(z)\big|dz\Big)v(x)dx$$

$$\leq \frac{2}{\alpha\gamma c_n(n-1)}\frac{1}{v(Q)}\int\limits_{Q\setminus\Omega_\alpha}\Big(\int\limits_{Q\cap\Omega_\alpha\setminus\Omega_{2\alpha}}\frac{\big|\nabla f(z)\big|dz}{|z-x|^{n-1}}\Big)v(x)dx. \tag{22}$$

Therefore,

$$v\left(Q\cap\Omega_{3\alpha}\right) \leq \frac{2}{\alpha c_n\gamma(n-1)}\int\limits_{Q\setminus\Omega_\alpha}\Big(\int\limits_{Q\cap\Omega_\alpha\setminus\Omega_{2\alpha}}\frac{\big|\nabla f(z)\big|}{|z-x|^{n-1}}\,dz\Big)v(x)dx$$

$$= \frac{2}{\alpha c_n \gamma(n-1)} \int\limits_{Q\cap\Omega_\alpha\setminus\Omega_{2\alpha}} |\nabla f(z)| \Big(\int\limits_{Q\setminus\Omega_\alpha} \frac{v(x)dx}{|z-x|^{n-1}} \Big) dz$$

Applying Holder's inequality, we see that this is exceeded by

$$\frac{2}{\alpha c_n \gamma(n-1)} \Big(\int\limits_{Q\cap\Omega_\alpha\setminus\Omega_{2\alpha}} \omega(z) |\nabla f(z)| dz \Big)^{1/p} \times \tag{23}$$

$$\times \Big(\int\limits_{Q\cap\Omega_\alpha\setminus\Omega_{2\alpha}} \sigma(z) \Big(\int\limits_{Q\setminus\Omega_\alpha} \frac{v(x)dx}{|z-x|^{n-1}} \Big)^{p'} dz \Big)^{1/p'} \tag{24}$$

Using here the condition (7), we get

$$v(Q\cap\Omega_{3\alpha}) \le \frac{2A}{\alpha c_n \gamma(n-1)} \Big(\int\limits_{Q\cap\Omega_\alpha\setminus\Omega_{2\alpha}} \omega(z) |\nabla f(z)|\, dz \Big)^{1/p} \big(v(Q)\big)^{1/q'}$$

by (17) and $\Omega_{3\alpha} \subset \Omega_{2\alpha}$ we get

$$v(Q\cap\Omega_{3\alpha}) \le \frac{2A}{\alpha c_n \gamma^{1+\frac{1}{q'}}(n-1)} \Big(\int\limits_{Q\cap\Omega_\alpha\setminus\Omega_{3\alpha}} \omega(z) |\nabla f(z)|\, dz \Big)^{1/p} \Big(v(Q\cap\Omega_{3\alpha}) \Big)^{1/q'}. \tag{25}$$

The same inequality holds if the case b) is considered. Indeed, we come to the inequality

$$\int\limits_{Q\setminus\Omega_{2\alpha}} \Big(\int\limits_{Q\cap\Omega_{3\alpha}} v(x)dx \Big) dy \ge \frac{\gamma}{2} |Q| v(Q) \tag{26}$$

instead of (51). Further, repeating the above argument, we come to

$$\frac{2}{\alpha c_n \gamma(n-1)} \Big(\int\limits_{Q\cap\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \omega(z) |\nabla f(z)| dz \Big)^{1/p} \times$$

$$\times \Big(\int\limits_{Q\cap\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \sigma(z) \Big(\int\limits_{Q\cap\Omega_{3\alpha}} \frac{v(x)dx}{|z-x|^{n-1}} \Big)^{p'} dz \Big)^{1/p'}$$

instead of (24). Since $\Omega_{2\alpha} \setminus \Omega_{3\alpha} \subset \Omega_\alpha \setminus \Omega_{3\alpha}$, the same inequality (25) holds in this case.

Now, combining (48) and (25) we get

$$v(Q\cap\Omega_{3\alpha}) \le \frac{\gamma}{1-\gamma} v(Q\cap\Omega_\alpha)$$

$$+\frac{2A}{\alpha c_n \gamma^{1+\frac{1}{q'}}(n-1)}\Big(v(Q\cap\Omega_{3\alpha})\Big)^{1/q'}\Big(\int\limits_{Q\cap\Omega_\alpha\setminus\Omega_{3\alpha}}\omega(z)\big|\nabla f(z)\big|\,dz\Big)^{1/p}. \qquad (27)$$

We have a ball $Q(x,\rho(x))$ for every fixed $x\in\Omega_{3\alpha}$ with property (27). On the other hand, by constructions above, we have $\rho(x)_{x\in\Omega_{3\alpha}}<\infty$. Due to the Besicovitch covering theorem (see, [24]), we can select a countable subcover $\{Q^i\}$ from $\{Q(x,\rho(x)) :\, x\in\Omega_{3\alpha}\}$ with finite multiplicity, i.e.

$$\sum_{i=1}^{\infty}\chi_{Q^i}(x)\le\kappa_n. \qquad (28)$$

From (27) it follows that

$$v\left(Q^i\cap\Omega_{3\alpha}\right)\le\frac{\gamma}{1-\gamma}v\left(Q^i\cap\Omega_\alpha\right)$$
$$+\frac{2A}{\alpha c_n \gamma^{1+\frac{1}{q'}}(n-1)}\Big(v(Q^i\cap\Omega_{3\alpha})\Big)^{1/q'}\Big(\int\limits_{Q^i\cap\Omega_\alpha\setminus\Omega_{3\alpha}}\omega(z)\big|\nabla f(z)\big|\,dz\Big)^{1/p}. \qquad (29)$$

Summing (29) by $i\in\mathbb{N}$, using (28) and Holder's inequality, we get

$$v\left(\Omega_{3\alpha}\right)\le\frac{\kappa_n\gamma}{1-\gamma}v\left(\Omega_\alpha\right)$$
$$+\frac{2A}{\alpha c_n \gamma^{1+\frac{1}{q'}}(n-1)}\Big(\sum_i\left(v(Q^i\cap\Omega_{3\alpha})\right)^{p'/q'}\Big)^{1/p'}\Big(\sum_i\int\limits_{Q^i\cap\Omega_\alpha\setminus\Omega_{3\alpha}}\omega(z)\big|\nabla f(z)\big|\,dz\Big)^{1/p}.$$

Using $p'/q'\ge 1$, we obtain

$$v\left(\Omega_{3\alpha}\right)\le\frac{\kappa_n\gamma}{1-\gamma}v\left(\Omega_\alpha\right)$$
$$+\frac{2\kappa_n^{\frac{1}{q'}+\frac{1}{p}}A}{\alpha c_n \gamma^{1+\frac{1}{q'}}(n-1)}\,v(\Omega_{3\alpha})^{1/q'}\Big(\int\limits_{\Omega_\alpha\setminus\Omega_{3\alpha}}\omega(z)\big|\nabla f(z)\big|\,dz\Big)^{1/p}. \qquad (30)$$

Now, it remains to integrate (30) to have the inequality (6):

$$\int\limits_0^\infty v\left(\Omega_{3\alpha}\right)d\alpha^q\le\frac{\gamma\kappa_n}{1-\gamma}\int\limits_0^\infty v\left(\Omega_\alpha\right)d\alpha^q$$
$$+\frac{2q\kappa_n^{\frac{1}{q'}+\frac{1}{p}}A}{c_n \gamma^{1+\frac{1}{q'}}(n-1)}\int\limits_0^\infty\alpha^{q-1}\,v(\Omega_{3\alpha})^{1/q'}\Big(\int\limits_{\Omega_\alpha\setminus\Omega_{3\alpha}}\omega(z)\big|\nabla f(z)\big|\,dz\Big)^{1/p}\frac{d\alpha}{\alpha}. \qquad (31)$$

Due to $\int\limits_0^\infty v\left(\Omega_{3\alpha}\right) d\alpha^q = \frac{1}{3^q}\int\limits_{\mathbb{R}^n} |f(x)|^q v(x)dx$ and applying Holder's inequality, we get

$$\frac{1}{3^q}\int\limits_{\mathbb{R}^n} |f(x)|^q v(x)dx \le \frac{\gamma\kappa_n}{(1-\gamma)}\int\limits_{\mathbb{R}^n} |f(x)|^q v(x)dx$$
$$+\frac{2\kappa_n^{\frac{1}{q'}+\frac{1}{p}}A}{c_n\gamma^{1+\frac{1}{q'}}(n-1)}\Big(\int\limits_0^\infty \alpha^{(q-1)p'}v(\Omega_{3\alpha})^{p'/q'}\frac{d\alpha}{\alpha}\Big)^{1/p'}\times$$
$$\times\Big(\int\limits_0^\infty\Big(\int\limits_{Q^i\cap\Omega_\alpha\setminus\Omega_{3\alpha}}\omega(z)\big|\nabla f(z)\big|dz\Big)\frac{d\alpha}{\alpha}\Big)^{1/p}. \tag{32}$$

Using Minkowski inequality for $p'/q'\ge 1$, we have

$$\Big(\int\limits_0^\infty \alpha^{(q-1)p'}v(\Omega_{3\alpha})^{p'/q'}\frac{d\alpha}{\alpha}\Big)^{1/p'} \le \Big(\int\limits_0^\infty\left(\int\limits_0^{\frac{1}{3}|f(x)|}\alpha^{(q-1)p'}\frac{d\alpha}{\alpha}\right)^{q'/p'} v(x)dx\Big)^{1/q'}$$
$$=\left(\frac{1}{(q-1)p'}\right)^{\frac{1}{p'}}\frac{1}{3^{q-1}}\Big(\int\limits_{\mathbb{R}^n}|f(x)|^q v(x)dx\Big)^{1/q'}. \tag{33}$$

Choosing $\gamma\in(0,\delta)$ such that $\frac{\gamma\kappa_n}{1-\gamma}<\frac{1}{3^q}$, from the inequality (34) we get

$$\Big(\int\limits_{\mathbb{R}^n}|f(x)|^q v(x)dx\Big)^{1/q}\le C_0 A\Big(\int\limits_{\mathbb{R}^n}\omega(z)\big|\nabla f(z)\big|dz\Big)^{1/p}, \tag{34}$$

where $C_0=\left(\frac{1}{3^q}-\frac{\gamma\kappa_n}{(1-\gamma)}\right)^{-1}\left(\frac{1}{(q-1)p'}\right)^{\frac{1}{p'}}\frac{(\ln 3)^{\frac{1}{p}}}{3^{q-1}}\frac{2q\kappa_n^{\frac{1}{q'}+\frac{1}{p}}}{c_n\gamma^{1+\frac{1}{q'}}(n-1)}$.

Theorem 1 is proved. ◄

Proof of Theorem 2. The idea of the proof of Theorem 2 is close to [14]. To save the completeness of contents, we present it below. Take $a\in R$ such that

$$v\left(\{u\ge a\}\cap\Omega\right)=v\left(\{u<a\}\cap\Omega\right)=\frac{1}{2}v\left(\Omega\right). \tag{35}$$

Denote $\Omega_\alpha=\{x\in\Omega:\ f(x)>\alpha+a\}$ for $\alpha>0$.

Let γ be a sufficiently little positive number that will be specified later. For any fixed point $x \in \Omega_{3\alpha}$ there exists a ball $Q = Q(x, \rho(x))$ such that

$$v(Q \cap \Omega \setminus \Omega_\alpha) = \gamma v(Q). \tag{36}$$

Indeed, the continuous function

$$F(t) = \frac{1}{\gamma} v\Big(\Omega \cap Q_t^x \setminus \Omega_\alpha\Big) - v(Q_t^x)$$

is negative for sufficiently small $t > 0$ since x is an interior point of $\Omega_{3\alpha}$. Also, $F(t)$ is positive for $t \to d_\Omega$:

$$F(d_\Omega) \geq \frac{1}{2} v(\Omega) - \gamma v\left(Q_{d_\Omega}^x\right) = \frac{1}{2} v\left(\Omega \cap Q_{d_\Omega}^x\right) - \gamma v\left(Q_{d_\Omega}^x\right) \geq \left(\frac{\varepsilon}{2} - \gamma\right) v\left(Q_{d_\Omega}^x\right)$$

since (10) is satisfied, where $0 < \gamma < \frac{\varepsilon}{2}$ is a number. Applying Cauchy's theorem, we find

$$F(t_1) = 0 \quad \text{by some } t_1 \in (0, d_\Omega).$$

Put $\rho(x) = t_1$ and get (36).

If 1)

$$v\left(Q^i \cap \Omega_{3\alpha}\right) \leq \gamma v\left(Q^i\right), \tag{37}$$

using the condition (10) we have

$$\varepsilon v\left(Q^i\right) \leq v\left(\Omega \cap Q^i\right) \tag{38}$$

and

$$v\left(Q^i \cap \Omega_{3\alpha}\right) \leq \frac{\gamma}{\varepsilon} v\left(Q^i \cap \Omega\right). \tag{39}$$

Now, by (36), we get

$$v\left(Q^i \cap \Omega\right) = v\left(\Omega_\alpha \cap Q^i\right) + v\left(\Omega \cap Q^i \setminus \Omega_\alpha\right) \leq$$

$$\leq v\left(\Omega_\alpha \cap Q^i\right) + \frac{\gamma}{\varepsilon} v\left(Q^i \cap \Omega\right).$$

Therefore, from (39) we obtain

$$v\left(\Omega \cap Q^i\right) \leq \left(1 - \frac{\gamma}{\varepsilon}\right)^{-1} v\left(\Omega_\alpha \cap Q^i\right) \tag{40}$$

and

$$v\left(\Omega_{3\alpha} \cap Q^i\right) \leq \frac{\gamma}{\varepsilon} \frac{1}{1 - \frac{\gamma}{\varepsilon}} v\left(\Omega \cap Q^i\right) = \frac{\gamma}{\varepsilon - \gamma} v\left(Q^i \cap \Omega_\alpha\right). \tag{41}$$

Now, let

$$v\left(Q^i\cap\Omega_{3\alpha}\right)\geq\gamma v\left(Q^i\right). \tag{42}$$

Then we have at least one of two variants:

$$a)\quad \left|Q\cap\Omega_{2\alpha}\right|\geq\frac{\delta}{2}\left|Q\right| \tag{43}$$

or

$$b)\quad \left|Q\setminus\Omega_{2\alpha}\right|\geq\frac{\delta}{2}\left|Q\right|, \tag{44}$$

since (9) is satisfied for some $\delta\in(0,1)$ and a convex domain.

Let the condition a) and (42) be satisfied. For any fixed pair of points $x\in Q\cap\Omega\setminus\Omega_\alpha,\ y\in Q\cap\Omega_{2\alpha}$ the line $\{x+t\,(y-x):0<t<1\}$ is contained in Ω, since Ω is a convex domain. Then

$$\int_{Q\cap\Omega_{2\alpha}}dy\int_{Q\cap\Omega\setminus\Omega_\alpha}v(x)dx\geq\frac{\gamma\delta}{2}\left|Q\right|v\left(Q\right). \tag{45}$$

Let $t_1\left(x,y\right),t_2\left(x,y\right)$ be the points of the line $x+t\left(y-x\right)$ where it intersects the level sets $\partial\Omega_\alpha$ and $\partial\Omega_{3\alpha}$.

As

$$1\leq\frac{2}{\alpha\gamma\delta}\frac{1}{|Q|\,v(Q)}\int_{Q\cap\Omega_{2\alpha}}dy\int_{Q\cap\Omega\setminus\Omega_\alpha}v(x)dx\times$$
$$\times\Big(\int_{t_1(x,y)}^{t_2(x,y)}\frac{\partial}{\partial t}\,f\left(x+t(y-x)\right)dt\Big),$$

we have

$$1\leq\frac{2}{\alpha\gamma\delta}\frac{1}{|Q|\,v\left(Q\right)}\int_{Q\cap\Omega_{2\alpha}}dy\int_{Q\cap\Omega\setminus\Omega_\alpha}v\left(x\right)dx\times$$
$$\times\int_{t_1(x,y)}^{t_2(x,y)}\left|\left(x-y\right)\nabla f\left(x+t\left(y-x\right)\right)\right|dt.$$

By using Fubini's theorem and the fact that $|x-y|<d_\Omega$ for $x\in Q\cap\Omega\setminus\Omega_\alpha,\ y\in Q\cap\Omega_{2\alpha}$, we have

$$1\leq\frac{2}{\alpha\gamma\delta}\frac{d_Q}{|Q|\,v\left(Q\right)}\int_{Q\cap\Omega_{2\alpha}}dy\int_{Q\cap\Omega\setminus\Omega_\alpha}v\left(x\right)dx\int_{t_1(x,y)}^{t_2(x,y)}\left|\nabla f\left(x+t\left(y-x\right)\right)\right|dt$$

$$\leq\frac{2}{c_0\alpha\gamma\delta}\frac{d_Q^{1-n}}{v\left(Q\right)}\int_{Q\cap\Omega\setminus\Omega_{2\alpha}}v\left(x\right)dx\int_0^1dt\int_{Q\cap\Omega_{2\alpha}}\left|\nabla f\left(x+t\left(y-x\right)\right)\right|dy,$$

where c_0 is a volume of a unit ball. Change variables in the interior integral $z = x + t\,(y - x)$ passing from y to z. Then we have

$$1 \le \frac{2}{c_0\alpha\delta\gamma}\frac{d_Q^{1-n}}{v\,(Q)}\int_{Q\cap\Omega\setminus\Omega_\alpha} v\,(x)\,dx\int_0^1\frac{dt}{t^n}\times$$

$$\times\Bigl(\int\limits_{\left\{\begin{array}{l}\frac{z-x}{t}+x\in Q\cap\Omega_{2\alpha}\\ z\in Q\cap\Omega_\alpha\setminus\Omega_{2\alpha}\end{array}\right\}}|\nabla f\,(z)|\,dz\Bigr).$$

Applying again the Fubini's theorem to the interior integrals, we get the right hand side equal to

$$\frac{2}{c_0\alpha\gamma\delta}\frac{d_Q^{1-n}}{v\,(Q)}\int_{Q\cap\Omega\setminus\Omega_\alpha} v(x)dx\int_{Q\cap\Omega_\alpha\setminus\Omega_{2\alpha}}|\nabla f\,(z)|\,dx\int\limits_{R(x,z)} t^{-n}dt,$$

where $R(x,z)$ is the set $\left\{t\in(0,1) : \frac{z-x}{t}+x\in Q\cap\Omega_{2\alpha}\right\}$. Now, since $|z-x| < t\,|y-x|$, we have $t > \frac{|z-x|}{d_Q}$. Hence, calculating the interior integral, we get

$$1 \le \frac{2}{c_0(n-1)\alpha\gamma\delta}\frac{1}{v\,(Q)}\int_{Q\cap\Omega\setminus\Omega_\alpha} v\,(x)\,dx\int_{Q\cap\Omega_\alpha\setminus\Omega_{2\alpha}}\frac{|\nabla f\,(z)|\,dz}{|z-x|^{n-1}}. \tag{46}$$

Therefore, applying Fubini's theorem, we obtain

$$\begin{aligned} &v\,(Q\cap\Omega_{3\alpha})\\ &\le v(Q)\Bigl[\frac{2}{c_0(n-1)\alpha\gamma\delta}\frac{1}{v\,(Q)}\int_{Q\bigcap\Omega_\alpha\setminus\Omega_{2\alpha}}|\nabla f(z)|\,dz\times\\ &\times\left(\int_{Q\bigcap\Omega\setminus\Omega_\alpha}\frac{v(x)dx}{|x-z|^{n-1}}\right)\Bigr]^q \end{aligned} \tag{47}$$

Applying Holder's inequality, we have

$$v\,(Q\cap\Omega_{3\alpha}) \le v\,(Q)\left(\frac{2}{c_0(n-1)\gamma\delta\alpha}\frac{1}{v\,(Q)}\right)^q\left(\int_{Q\cap\Omega_\alpha\setminus\Omega_{2\alpha}}|\nabla f\,(z)|^p\,\omega dz\right)^{\frac{q}{p}}\times$$

$$\times\left(\int_{Q\cap\Omega_\alpha\setminus\Omega_{2\alpha}}\sigma\,(z)\left(\int_{Q\bigcap\Omega\setminus\Omega_\alpha}\frac{v\,(x)\,dx}{|z-x|^{n-1}}\right)^{p'}\right)^{\frac{q}{p'}}. \tag{48}$$

This inequality and the condition (11) give

$$v\left(Q\cap\Omega_{3\alpha}\right)\leq\frac{A^{q}}{\alpha^{q}}\left(\frac{2}{\gamma\delta}\right)^{q}\left(\int_{Q\cap\Omega_{\alpha}\backslash\Omega_{2\alpha}}\left|\nabla f\left(z\right)\right|\omega dz\right)^{\frac{q}{p}} \tag{49}$$

The same inequality holds if the condition b) and (42) hold. Indeed, we have the inequality

$$\int_{Q\cap\Omega\backslash\Omega_{2\alpha}}dy\int_{Q\cap\Omega_{3\alpha}}v\left(x\right)dx\geq\frac{\gamma\delta}{2}\left|Q\right|v\left(Q\right) \tag{50}$$

instead of (45). Repeating the above considerations, we come to the inequality

$$v\left(Q\cap\Omega_{3\alpha}\right)\leq v\left(Q\right)\left(\frac{2}{\gamma\delta\alpha}\frac{1}{v\left(Q\right)}\right)^{q}\left(\int_{Q\bigcap\Omega_{2\alpha}\backslash\Omega_{3\alpha}}\left|\nabla f\left(z\right)\right|^{p}\omega dz\right)^{\frac{q}{p}}\times$$

$$\times\left(\int_{Q\bigcap\Omega_{2\alpha}\backslash\Omega_{3\alpha}}\sigma\left(z\right)\left(\int_{Q\bigcap\Omega_{3\alpha}}\frac{v\left(x\right)dx}{\left|x-z\right|^{n-1}}\right)^{p'}dz\right)^{\frac{q}{p'}} \tag{51}$$

instead of (48).

So the inequality (49) is also true for the case b) if the condition (40) holds. Combining the cases a), b) and (41), (42), we get the inequality

$$v\left(Q\cap e_{3\alpha}\right)\leq\frac{\gamma}{\varepsilon-\gamma}v\left(Q\cap\Omega_{\alpha}\right)+\left(\frac{2}{\gamma\delta}\right)^{q}\left(\frac{A}{\alpha}\right)^{q}\left(\int_{Q\cap\Omega_{\alpha}\backslash\Omega_{3\alpha}}\left|\nabla f\left(z\right)\right|^{p}\omega dz\right)^{\frac{q}{p}}. \tag{52}$$

Now all that's left is to repeat all considerations of Theorem 1

Theorem 2 is proved. ◄

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