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Fractional Fourier Transform to Stability Analysis of Fractional Differential Equations with Prabhakar Derivatives

S. Deepa, A. Ganesh, V. Ibrahimov, S.S. Santra, V. Govindan,
K.M. Khedher, S. Noeiaghdam*

Abstract. In this paper, the authors introduce the Prabhakar derivative associated with the generalised Mittag-Leffler function. Some properties of the Prabhakar integrals, Prabhakar derivatives and some of their extensions, like fractional Fourier transform of Prabhakar integrals and fractional Fourier transform of Prabhakar derivatives are introduced. This note aims to study the Mittag-Leffler-Hyers-Ulam stability of the linear and nonlinear fractional differential equations with the Prabhakar derivative. Furthermore, we give a brief definition of the Mittag-Leffler-Hyers-Ulam problem and a method for solving fractional differential equations using the fractional Fourier transform. We show that the fractional differential equations are Mittag-Leffler-Hyers-Ulam stable in the sense of Prabhakar derivatives.

Key Words and Phrases: fractional differential equation (FDE), Prabhakar fractional integral, Prabhakar fractional derivative, Mittag-Leffler function (MLF), fractional Fourier transform (FrFT), Hyers-Ulam stability (HUS).

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1. Introduction

Fractional calculus is a generalisation of ordinary calculus to non-integer differentiation and integration. There have been a lot of contributions to the theory of fractional calculus by famous mathematicians, such as Laplace, Fourier, Abel, Leoville Letnikov, Heaviside, Weyl, Erdelyi, Go Renflo, Mainardi et al. FDEs have recently become a very strong tool in many fields, such as physics, thermodynamics, electrical circuit theory and seepage flow in porous media, etc. The mathematicians [1, 2, 3, 4, 5] discussed FDEs and their applications up to date

*Corresponding author.

in their books. Also, there are many applications of fractional order models in bio-mathematics and engineering. In [6, 7], the wavelet method was applied for solving fractional order model of COVID-19. The fractional models have been presented for malaria infection in [8, 9]. The fractional model of energy supply-demand system and its stability analysis can be found in [10, 11].

The most significant benefit of using an FDE is its non-local property, which means that the next state of a system depends not only on its current state but also on all of its historical states. Despite describing differential, fractional, and functional equations, finding exact solutions to physical and practical problems is improbably difficult. If there are exact solutions, they are often so complicated that it is not convenient to have numerical solutions. The conduct of the analytical solutions of the fractional differential equation represented by the fractional-order derivative operators is the fundamental principle in numerous stability issues. Motivated by the usage of the Mittag-Leffler functions in many areas of science and design, we present this paper.

Stanislaw Marcin Ulam introduced the HUS problem in 1940. Hyers [12] provided a brilliant answer to Ulam's question about the stability of functional equations for the case of approximately additive mappings, where $\mathcal{G}_1$ and $\mathcal{G}_2$ were assumed to be Banach spaces, in 1941. Rassias proved the HUS for the additive Cauchy equation; later it was generalised by Aoki to additive mappings [13, 14]. Obbloza [15] was one of the first authors who investigated the HUS of differential equations. Wang [16] demonstrated the stability of FDEs with fractional integrals and contributed some new concepts in FDE stability. Liu et al. [17], established the HUS of linear Caupto-Fabinizio FDEs using the Laplace transform method, and [18], discussed the Mittag-Leffler-Ulam stabilities of fractional evolution equations. In 1948, Magnus Gosta Mittag-Leffler defined a new special function for divergent series that is known as MLF. Pollard [19] derived one parameter MLF. P. Humbert et al. [20, 21, 22] derived the main features of MLF and generalised two-parameter MLF. Extension of MLF and the solution of differential equations of non-integer order are discussed in [23, 24].

The three-parameter extension of MLF was introduced by Prabhakar [25] in 1971. In 2002, Saxena [26] developed Volterra operations involving the Prabhakar function. D'Ovidio [27] provided a regularisation of the fractional derivative and named it after [28], where the features of the Prabhakar derivatives and their applications were discussed. Beghin and Orisingher's [29] studied the relationship between Prabhakar and Wright functions. In [30], the stability region of fractional differential systems was studied using Prabhakar derivative analyses with Caputo and Riemann derivatives. Citation [31] discusses the Hilfiger-Prabhakar derivatives and their applications in time delay, the time-fractional Poisson process, and its renewal structure. See the discussion of Lyapunov type inequality for a hybrid fractional differential equation with Prabhakar derivative in [32]. A class of nonlinear variable order FDE's in the Caputo-Prabhakar sense was solved by using Bernteon polynomials in [33]. Recently, HUS of linear differential equations using FT was discussed in [34, 35, 36, 37, 38].

We establish the fractional Fourier transform and present it in integral form. Furthermore, using the convolution concept and properties of the fractional Fourier transform, the solution of the Mittag-Leffler-Hyers-Ulam stability conditions concerning the fractional differential equation is established. In particular, we prove the Mittag-Leffler-Hyers-Ulam stability of the following FDEs using the FrFT:

$$\begin{cases} \mathcal{D}^{\emptyset}_{\varrho,\psi,\sigma} u(e) + du(e) = q(e), \quad e \in (0, \mathcal{Q}], \ d \in \mathbb{R} \\ e^{\psi-1} u(e)|_{e=0}, \end{cases}$$

$$\begin{cases} \mathcal{D}^{\emptyset}_{\varrho,\psi,\sigma} u(e) + du(e) = \mathcal{G}(e, u(e)), \quad e \in (0, \mathcal{Q}], \ d \in \mathbb{R} \\ e^{1-\psi} u(e)|_{e=0} \end{cases}$$

and

$$\begin{cases} \mathcal{D}^{\emptyset}_{\varrho,\psi,\sigma} u(e) + du(e) - q(e) = F(e), \quad e \in (0, \mathcal{Q}], \ d \in \mathbb{R} \\ e^{\psi-1} u(e)|_{e=0}, \end{cases}$$

$$\begin{cases} \mathcal{D}^{\emptyset}_{\varrho,\psi,\sigma} u(e) + du(e) - \mathcal{G}(e, u(e)) = F(e), \quad e \in (0, \mathcal{Q}], \ d \in \mathbb{R} \\ e^{1-\psi} u(e)|_{e=0}. \end{cases}$$

The paper is organised as follows: In Section 2, basic definitions, theorems, and lemmas related to fractional derivatives, MLF, Prabhakar derivatives, and FrFT are given. In Sections 3 and 4, Mittag-Leffler-HUS of linear and nonlinear FDEs is proved by using the FrFT. In Sections 5 and 6, examples and conclusions are given.

2. Preliminaries

The Fourier transform (FT) was defined by the French mathematician Joseph Fourier. The term fraction power for the Fourier operators appeared in 1929. FrFTs represent the generalisation of the conventional FT. Victor Nami first introduced the FrFT in 1980, which, due to its time-frequency characteristics, produces better results for non-stationary signals than the FT. FrFT is used in several scientific fields, such as optics, time-frequency distribution, image processing, satellite image compression, signal image recovery and noise removal, image smoothing, encryption and decryption. V. Namias as a way to solve FrFT plays a very important role in solving ordinary and partial equations. The analytical solutions of the FDE described by the fractional-order derivative operators play the main role in many stability problems. Motivated by the success of the application of the MLFs in many areas of fractional calculus, we present this paper.

In this section, we state some essential definitions and lemmas used in this study.

Definition 1. *The Riemann–Liouville derivative operator of order θ is defined by*

$$D^{\theta} f(e) = \begin{cases} \frac{1}{\Gamma(n-\theta)} \frac{d^n}{de^n} \int_a^e \frac{f(\mathcal{Q})}{(e-\mathcal{Q})^{\theta+1-n}}, & n-1 < \theta < n, n \in \mathbb{N} \\ \frac{d^n}{de^n} f(e), & \theta = n \in \mathbb{N}, \end{cases}$$

where $\theta > 0, \mathcal{Q} > a, \theta, a, e \in \mathbb{R}$.

Definition 2. *The Mittag–Leffler function can be defined in terms of a power series as*

$$\mathcal{E}_{\theta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\theta k + 1)}, \quad \theta > 0, \quad (\textit{one parameter}) \tag{1}$$

$$\mathcal{E}_{\theta,\tau}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\tau + \theta k)}, \quad \tau > 0, \theta > 0 \quad (\textit{two parameters}). \tag{2}$$

Definition 3. *The generalized MLF with three-parameters was introduced by Prabhakar (known as the Prabhakar function) as*

$$\mathcal{E}^{\o}_{\varsigma,\tau}(z) = \sum_{k=0}^{\infty} \frac{(\o)_k z^k}{\Gamma(\varsigma k + \tau)}$$

or

$$\mathcal{E}^{\o}_{\varsigma,\tau}(z) = \frac{1}{\Gamma(\o)} \sum_{k=0}^{\infty} \frac{\Gamma(\o + k)}{k! \Gamma(\varsigma k + \tau)} z^k,$$

where $z, \varsigma, \tau, \o \in \mathcal{C}, Re(\varsigma) > 0, Re(\tau) > 0, Re(\o) > 0$. Here

$$(\o)_k = \o(\o+1)(\o+2)......(\o+k-1) = \frac{\Gamma(\o+k)}{\Gamma(\o)}, (\o)_0 = 1, \o \neq 0$$

is a Pochhammer symbol.

Definition 4. *The Prabhakar integral operator and Prabhakar derivative including the generalized MLF are defined as follows:*

$$\mathcal{E}^{\o}_{\varrho,\psi,w.0+} f(e) = \int_0^e (e-u)^{u-1} \mathcal{E}^{\o}_{\varrho,\psi}(w(e-u)^{\varrho}) f(u) du$$

and

$$\mathcal{D}^{\o}_{\varrho,\psi,w.0+}f(e) = \frac{d^m}{de^m}\mathcal{E}^{-\o}_{\varrho,m-\psi,w.0+}f(e),$$

where $0 < e < b \leq \infty; \varrho, \psi, w, \o \in \mathcal{C}, R(\varrho), R(\psi) > 0.$

Lemma 1. *Let* $\o, \varrho, \psi, \sigma \in \mathcal{C}$ *with* $\mathbb{R}(\psi) > 0$. *Then the differentiation of the generalized MLF is given by*

$$\left(\frac{d}{dx}\right)^n \left[x^{\psi-1}\mathcal{E}^{\o}_{\varrho,\psi}\left(\sigma x^p\right)\right] = x^{\psi-n-1}\mathcal{E}^{\o}_{\varrho,\psi-n}(\sigma x^p), \quad for\ any\ n \in \mathbb{N}.$$

Lemma 2. *The generalized MLF*

$$S = \sum_{n=0}^{\infty} e^{\psi n}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma e^{\varrho}), \quad e \in \mathcal{C}, w, \varrho \in \mathbb{R}$$

is absolutely convergent, where $\mathbb{R}(\varrho), \mathbb{R}(\psi) > 0.$

Lemma 3. *Let*

$$\mathcal{C}_{1-\psi}[a, \mathcal{Q}] = \left\{u(e) \in \mathcal{C}[a, \mathcal{Q}]; e^{1-\psi}u(e) \in \mathcal{C}[a, \mathcal{Q}]\right\}, 0 < \psi < 1$$

and

$$\lim_{e \to a^+}[(t-a)^{1-\psi}u(e)] = c,\ c \in \mathcal{C}.$$

Then

$$\left(\mathcal{E}^{-\o}_{\varrho 1-,\psi,\sigma,a^+}u\right)(a^+) = c\Gamma(\psi)$$

holds. Also, let

$$\left(\mathcal{E}^{-\o}_{\varrho 1-,\psi,\sigma,a^+}u\right)(a^+) = \chi, \quad \chi \in \mathcal{C}$$

and $\lim_{e \to a^+}[(t-a)^{1-\psi}u(e)]$. *Then*

$$\lim_{e \to a^+}[(t-a)^{1-\psi}u(e)] = \frac{b}{\Gamma(\psi)}.$$

Definition 5. *For a function* $u \in \phi \in (R)$, *the FrFT of the order* $\varsigma(0 < \varsigma \leq 1), \widehat{u_\varsigma}$ *is defined as*

$$\widehat{u_\varsigma} = (\mathcal{F}_\varsigma u)(\Omega) = \int_{-\infty}^{\infty} u(e)\mathcal{E}_\varsigma(\Omega, e)de, \quad \Omega \in \mathbb{R},$$

where

$$\mathcal{E}_\varsigma(\Omega, e) = \begin{cases} e^{-i|\Omega|^{\frac{1}{\varsigma}}e}, & \Omega \leq 0 \\ e^{i|\Omega|^{\frac{1}{\varsigma}}e}, & \Omega \geq 0. \end{cases}$$

If $\varsigma = 1$, the kernel $\mathcal{E}_\varsigma$ is defined by the kernel of the conventional FT

$$\mathcal{E}_1(\Omega, e) = \begin{cases} e^{-i|\Omega|e}, & \Omega \leq 0 \\ e^{i|\Omega|e}, & \Omega \geq 0, \end{cases} = e^{i\Omega t}, \ \Omega, e \in \mathbb{R}.$$

This means that the FrFT of the order 1 is a conventional FT.

Definition 6. *Let $f, g \in \mathbb{R} \to F$ be continuous and $f^*.g^*$ be piece-wise continuous. If $F(f(x)) = F(g(x))$, then $f(x) = g(x)$ for every x.*

Definition 7. *The FrFT of the convolution of $f(x)$ and $g(x)$ is a product of the FrFT of $f(x)$ and $g(x)$. Let $k, u \in \phi(R)$ and $\varsigma > 0$. Then for any $\Omega \in \mathbb{R}$,*

$$(\mathcal{F}_\varsigma(k * u)(\Omega)) = (\mathcal{F}_\varsigma k)(\Omega)(\mathcal{F}_\varsigma u).$$

In particular

$$(\mathcal{F}(k * u))(\Omega) = (\mathcal{F}k)(\Omega)(\mathcal{F}u)(\Omega), (\Omega \in \mathbb{R}).$$

The convolution in the time domain is equivalent to the multiplication in the frequency domain.

Remark 1. *Let $\tau \in \mathcal{C}, \mathbb{R}(\tau) > 0, \mathcal{Q} > 0$. Then*

$$\mathcal{F}_\varsigma(e^\tau) = \frac{\Gamma(\tau + 1)}{\left(-i\Omega^{1/\varsigma}\right)^{\tau+1}}.$$

Theorem 1. *Let $\o, \tau C, \mathbb{R}(\o) > 0, \mathbb{R}(\tau) > 0, \sigma \in \mathbb{R}$. Then*

$$\mathcal{F}_\varsigma\left(e^{\o m+\tau-1}\mathcal{E}^{(m)}_{\o.\tau}(\sigma e^{\o})\right) = \frac{\left(-i\Omega^{1/\varsigma}\right)^{\o-\tau} m!}{\left[\left(-i\Omega^{1/\varsigma}\right)^{\o} - \sigma\right]^{m+1}}$$

holds.

Lemma 4. *The FrFT of Prabhakar integral is given by*

$$F_\varsigma\left[\mathcal{E}^{\o}_{\varrho,\psi,\sigma.0+}f(e)\right] = \left(-i\Omega^{1/\varsigma}\right)^{-\psi}\left[1 - \sigma(-i\Omega^{1/\varsigma})^{-\o}F_\varsigma(-i\Omega^{1/\varsigma})\right].$$

Proof. We know that the Prabhakar integral operator including the generalized MLF is

$$\mathcal{E}^{\o}_{\varrho,\psi,\sigma.0+}f(e) = \int_0^e (e-u)^{u-1}\mathcal{E}^{\o}_{\varrho,\psi}(\sigma(e-u)^{\varrho})f(u)du, \ e > 0.$$

Taking FrFT on both sides, we have

$$
\begin{aligned}
F_\varsigma\left[\mathcal{E}^{\o}_{\varrho,\psi,\sigma.0+}f(e)\right] &= F_\varsigma\left[\int_0^e (e-u)^{\psi-1}\mathcal{E}^{\o}_{\varrho,\psi}[\sigma(e-u)^{\varrho}]f(u)du\right] \\
&= F_\varsigma\left[e^{\psi-1}\mathcal{E}^{\o}_{\varrho,\psi}(\sigma e^{\varrho})\right]F_\varsigma[f(e)] \\
&= F_\varsigma\left[e^{\psi-1}\sum_{k=0}^{\infty}\frac{(\o_k)}{k!}\frac{(\sigma e^{\varrho})^k}{\Gamma(\varrho k+\psi)}\right]F_\varsigma[f(e)] \\
&= \sum_{k=0}^{\infty}\frac{(\o_k)}{k!}\sigma^k\frac{F_\varsigma[e^{\psi+\varrho k-1}]}{\Gamma(\varrho k+\psi)}F_\varsigma[f(e)] \\
&= \sum_{k=0}^{\infty}\frac{(\o_k)}{k!}\frac{\sigma^k}{\Gamma(\varrho k+\psi)}\frac{\Gamma(\varrho k+\psi)}{\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\varrho k+\psi}}F_\varsigma[f(e)] \\
&= \sum_{k=0}^{\infty}\frac{(\o_k)}{k!}\sigma^k\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho k}\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\psi}F_\varsigma[f(e)] \\
&= \left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\psi}\left[\sum_{k=0}^{\infty}\frac{(\o_k)}{k!}\sigma^k\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho k}\right]F_\varsigma[f(e)].
\end{aligned}
$$

Expanding the summation, we have

$$
F_\varsigma\left[\mathcal{E}^{\o}_{\varrho,\psi,\sigma.0+}f(e)\right] = \left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\psi}\left[1-\sigma\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{-\o}F_\varsigma[f(e)].
$$

◀

Lemma 5. *The FrFT of Prabhakar derivative has the form*

$$
F_\varsigma\left[\mathcal{D}^{\o}_{\varrho,\psi,\sigma.0+}f(e)\right] = \left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\psi}\left[1-\sigma\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o}F_\varsigma[f(e)], \quad m-1\le\psi\le m.
$$

Proof. We know that, the Prabhakar derivative is

$$
\mathcal{D}^{\o}_{\varrho,\psi,\sigma.0+}f(e) = \frac{d^m}{de^m}\mathcal{E}^{-\o}_{\varrho,m-\psi,\sigma.0+}f(e).
$$

Taking FrFT on both sides, we have

$$
\begin{aligned}
F_\varsigma\left[\mathcal{D}^{\o}_{\varrho,\psi,\sigma.0+}f(e)\right] &= F_\varsigma\left[\frac{d^m}{de^m}\mathcal{E}^{-\o}_{\varrho,m-\psi,\sigma.0+}f(e)\right] \\
&= F_\varsigma\left[\frac{d^m}{de^m}\int_0^e (e-u)^{m-\psi-1}\mathcal{E}^{-\o}_{\varrho,m-\psi,\sigma.0+}\sigma(e-\psi)^{\varrho}f(u)du\right]
\end{aligned}
$$

$$
\begin{aligned}
&= F_{\varsigma}\left[e^{m-\psi-1-m}\mathcal{E}^{-\emptyset}_{\varrho,m-\psi-m.,0+}\sigma e^{\varrho}\right]F_{\varsigma}[f(u)] \\
&= F_{\varsigma}\left[e^{\psi-1}\mathcal{E}^{-\emptyset}_{\varrho,-\psi}\sigma e^{\varrho}\right]F_{\varsigma}[f(u)] \\
&= F_{\varsigma}\left[e^{-\psi-1}\sum_{k=0}^{\infty}\frac{(-\emptyset)_k}{k!}\frac{(\sigma e^{\varrho})^k}{\Gamma(\varrho k+\psi)}\right]F_{\varsigma}[f(e)] \\
&= \sum_{k=0}^{\infty}\frac{(-\emptyset)_k}{k!}\sigma^k\frac{F_{\varsigma}[e^{-\psi+\varrho k-1}]}{\Gamma(\varrho k+(-\psi))}F_{\varsigma}[f(e)] \\
&= \sum_{k=0}^{\infty}\frac{(-\emptyset)_k}{k!}\frac{\sigma^k}{\Gamma(\varrho k+(-\psi))}\frac{\Gamma(\varrho k+(-\psi))}{\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\varrho k-\psi}}F_{\varsigma}[f(e)] \\
&= \sum_{k=0}^{\infty}\frac{(-\emptyset)_k}{k!}\sigma^k\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho k}\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\psi}F_{\varsigma}[f(e)] \\
&= \left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\psi}\left[\sum_{k=0}^{\infty}\frac{(-\emptyset)_k}{k!}\sigma^k\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho k}\right]F_{\varsigma}[f(e)].
\end{aligned}
$$

Expanding the summation, we have

$$
F_{\varsigma}\left[\mathcal{D}^{\emptyset}_{\varrho,\psi,\sigma.0+}f(e)\right] = \left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\psi}\left[1-\sigma\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\emptyset}F_{\varsigma}[f(e)].
$$

◀

3. Mittag-Leffler-Hyers–Ulam Stability of fractional differential equations using fractional Fourier transform

In this section, we are going to prove Mittag-Leffler HUS for linear and non-linear FDEs

$$
\begin{cases}\mathcal{D}^{\emptyset}_{\varrho,\psi,\sigma}u(e)+du(e)=q(e), \quad e\in(0,\mathcal{Q}],\ d\in\mathbb{R}\\ e^{\psi-1}u(e)|_{e=0},\end{cases}
$$

$$
\begin{cases}\mathcal{D}^{\emptyset}_{\varrho,\psi,\sigma}u(e)+du(e)=\mathcal{G}(t,u(e)), \quad e\in(0,\mathcal{Q}],\ d\in\mathbb{R}\\ e^{1-\psi}u(e)|_{e=0}\end{cases}
$$

with Prabhakar derivatives using FrFT.

3.1. Stability of linear FDE $\mathcal{D}^{\o}_{\varrho,\psi,\sigma}u(e)+du(e)=q(e), e\in(0,\mathcal{Q}],\ d\in\mathbb{R}$ and $e^{\psi-1}u(e)|_{e=0}$

Definition 8. *The FDE*

$$\varphi\left\{q(e),u(e),\mathcal{D}^{\o_1}_{\varrho_1,\psi_1,\sigma}u,...,\mathcal{D}^{\o_n}_{\varrho_n,\psi_n,\sigma.0+}u\right\}=0,\ \o_i\geq 0, i=1\ to\ n$$

has HUS if

$$\left|\varphi\left\{q(e),u(e),\mathcal{D}^{\o_1}_{\varrho_1,\psi_1,\sigma}u,...,\mathcal{D}^{\o_n}_{\varrho_n,\psi_n,\sigma.0+}u\right\}\right|\leq\varepsilon,\quad for\ any\ \varepsilon>0.$$

Then there exists a solution u_a of the FDE such that

$$|u(e)-u_a(e)|\leq K(\varepsilon)$$

and

$$lim_{\varepsilon\to 0}K(\varepsilon)=0.$$

Theorem 2. *Let $u(e)\in\mathcal{C}_{1-\psi}[0,\mathcal{Q}]$ and $u(e)$ satisfy*

$$|\mathcal{D}^{\o_1}_{\varrho,\psi,w}u+du(e)-q(e)|\leq\varepsilon,\ 0<\psi<1,\ d\in\mathbb{R}. \tag{3}$$

Then there exists a solution $u_a(e):(0,\mathcal{Q}]\to\mathcal{C}$ of FDE

$$\mathcal{D}^{\o_1}_{\varrho,\psi,w}u+du(e)=q(e) \tag{4}$$

such that

$$|u(e)-u_a(e)|\leq\frac{\varepsilon e^{\psi}}{|d|}\sum_{n=0}^{\infty}\frac{e^{\psi n}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi+1}(|\sigma|e^{\varrho})}{|d|^n},$$

where $\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi+1}(|\sigma|e^{\varrho})$ is the generalized MLF.

Proof. For all $e\in(0,\mathcal{Q}]$, we consider

$$y(e)=\mathcal{D}^{\o_1}_{\varrho,\psi,w}u+du(e)-q(e).$$

Taking FrFT on both sides, we get

$$\begin{aligned}F_{\varsigma}[y(e)]&=F_{\varsigma}[\mathcal{D}^{\o_1}_{\varrho,\psi,w}u]+F_{\varsigma}[du(e)]-F_{\varsigma}[q(e)]\\&=\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\psi}\left[1-\sigma\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o}F_{\varsigma}[u(e)]\\&-\left(\mathcal{E}^{-\o}_{\varrho,1-u,\sigma,0+}u\right)(0)+dF_{\varsigma}[u(e)]-F_{\varsigma}[q(e)].\end{aligned}$$

By using Lemma 3 and considering $b = du_0\Gamma(\psi)$ and $c = \frac{b}{d\Gamma(\psi)}$, we get

$$F_\varsigma[y(e)] = \left(-i\Omega^{\frac{1}{\varsigma}}\right)^\psi \left[1 - \sigma\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o} F_\varsigma[u(e)] - du_0\Gamma(u) + dF_\varsigma[u(e)] - F_\varsigma[q(e)]$$

$$F_\varsigma[y(e)] = \left\{\left(-i\Omega^{\frac{1}{\varsigma}}\right)^\psi \left[1 - \sigma\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o} + d\right\} F_\varsigma[u(e)] - du_0\Gamma(u) - F_\varsigma[q(e)]$$

$$\begin{aligned} F_\varsigma[u(e)] = &\frac{F_\varsigma[y(e)]}{\left\{\left(-i\Omega^{\frac{1}{\varsigma}}\right)^\psi \left[1 - \sigma\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o} + d\right\}} \\ &+ \frac{du_0\Gamma(u) + F_\varsigma[q(e)]}{\left\{\left(-i\Omega^{\frac{1}{\varsigma}}\right)^\psi \left[1 - \sigma\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o} + d\right\}}. \end{aligned} \tag{5}$$

At this point, by setting

$$\begin{aligned} u_a(e) = &\frac{du_0\Gamma(u)\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n e^{\psi n+\psi-1}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma e^\varrho)}{d} \\ &+ \frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n \int_0^e q(x)(e-x)^{\psi n+\psi-1}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma(e-x)^\varrho)dx}{d}, \end{aligned} \tag{6}$$

we have

$$\begin{aligned} e^{1-\psi}u_a(e) = &\frac{du_0\Gamma(u)\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n e^{\psi n}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma e^\varrho)}{d} \\ &+ \frac{e^{1-\psi}\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n \int_0^e q(x)(e-x)^{\psi n+\psi-1}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma(e-x)^\varrho)dx}{d}. \end{aligned}$$

This implies that $u_a(e)$ satisfies

$$e^{1-\psi}u_a(e)|_{e=0}.$$

Now, applying FrFT and using the convolution property, we have

$$\begin{aligned} F_\varsigma[u_a(e)] = &F_\varsigma\left[\frac{du_0\Gamma(u)\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n e^{\psi n+\psi-1}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma e^\varrho)}{d}\right] \\ &+ F_\varsigma\left[\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n \int_0^e q(x)(e-x)^{\psi n+\psi-1}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma(e-x)^\varrho)dx}{d}\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{du_0\Gamma(u)\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n}{d} F_\varsigma\left[e^{\psi n+\psi-1}\mathcal{E}_{\varrho,\psi n+\psi}^{\o n+\o}(\sigma e^{\varrho})\right] \\
&\quad + \frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n}{d} F_\varsigma\left[\int_0^e q(x)(e-x)^{\psi n+\psi-1}\mathcal{E}_{\varrho,\psi n+\psi}^{\o n+\o}(\sigma(e-x)^{\varrho})dx\right] \\
&= \frac{du_0\Gamma(u)\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n}{d} F_\varsigma\left[e^{\psi n+\psi-1}\mathcal{E}_{\varrho,\psi n+\psi}^{\o n+\o}(\sigma e^{\varrho})\right] \\
&\quad + \frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n}{d} F_\varsigma\left[e^{\psi n+\psi-1}\mathcal{E}_{\varrho,\psi n+\psi}^{\o n+\o}(\sigma e^{\varrho})\right] F_\varsigma[q(e)]
\end{aligned}
$$

$$
\begin{aligned}
&= du_0\Gamma(u) + F_\varsigma[q(e)]\left[\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n F_\varsigma\left[e^{\psi n+\psi-1}\mathcal{E}_{\varrho,\psi n+\psi}^{\o n+\o}(\sigma e^{\varrho})\right]}{d}\right] \\
&= du_0\Gamma(u) + F_\varsigma[q(e)]\left[\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\varrho(\o n+\o)-(\psi n+\psi)}}{\left[\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\varrho}-\sigma\right]^{\o n+\o}}\right] \\
&= du_0\Gamma(u) + F_\varsigma[q(e)]\left[\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\varrho(\o n+\o)-(\psi n+\psi)}}{\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\varrho}\left[1-\sigma\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o n+\o}}\right].
\end{aligned}
$$

Expanding the summation, we get

$$
F_\varsigma\left[u_a(e)\right] = \frac{du_0\Gamma(u) + F_\varsigma[q(e)]}{\left\{\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\psi}\left[1-\sigma\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o} + d\right\}}. \tag{7}
$$

Taking (7) into account, we deduce that $u_a(e)$ is a solution of (4).

By subtracting (5) and (7) from each other, we have

$$
\begin{aligned}
F_\varsigma\left[u(e)\right] + F_\varsigma\left[u_a(e)\right] &= \frac{F_\varsigma[y(e)] + du_0\Gamma(u) + F_\varsigma[q(e)] - [du_0\Gamma(u) + F_\varsigma[q(e)]]}{\left\{\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\psi}\left[1-\sigma\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o} + d\right\}} \\
F_\varsigma\left[u(e)\right] + F_\varsigma\left[u_a(e)\right] &= \frac{F_\varsigma[y(e)]}{\left\{\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\psi}\left[1-\sigma\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o} + d\right\}}. \qquad (8)
\end{aligned}
$$

By Lemma 4, the LHS of (8) implies

$$F_{\varsigma}\left[\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^{n} e^{\psi n}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma e^{\varrho})}{d} * y(e)\right]$$

$$= F_{\varsigma}\left[\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^{n} e^{\psi n}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma e^{\varrho})}{d}\right] F_{\varsigma}\left[y(e)\right]$$

$$= \frac{F_{\varsigma}[y(e)]}{\left\{\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\psi}\left[1-\sigma\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o} + d\right\}}.$$

Hence, (8) can be written as

$$F_{\varsigma}\left[u(e)\right] + F_{\varsigma}\left[u_{a}(e)\right] = F_{\varsigma}\left[\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^{n} e^{\psi n}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma e^{\varrho})}{d} * y(e)\right].$$

Finally, (3) implies

$$|u(e)+u_{a}(e)| = \left|\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^{n} e^{\psi n}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma e^{\varrho})}{d} * y(e)\right|$$
$$= \frac{\varepsilon e^{\psi}}{|d|}\sum_{n=0}^{\infty}\frac{e^{\psi n}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi+1}(|\sigma| e^{\varrho})}{|d|^{n}},$$

where $\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma e^{\varrho})$ is the generalized MLF.

By the definition of the HUS theorem, the FDE (3) has Mittag-Leffler-Hyers–Ulam Stability. ◀

3.2. Stability of Non-linear FDE $\mathcal{D}^{\o}_{\varrho,\psi,\sigma}u(e) + du(e) = \mathcal{G}(t,u(e)), e \in (0,\mathcal{Q}],\ d \in \mathbb{R}$ and $e^{1-\psi}u(e)|_{e=0}$

Definition 9. *For $\mathcal{G} : [0,\mathcal{Q}] \times \mathbb{R} \to \mathbb{R}$, the FDE*

$$\varphi\left\{\mathcal{G}(t,u(e)), u(e), \mathcal{D}^{\o_1}_{\varrho_1,\psi_1,\sigma}u, ..., \mathcal{D}^{\o_n}_{\varrho_n,\psi_n,\sigma.0+}u\right\} = 0,\ \o_i \geq 0, i = 1\ to\ n$$

has HUS if

$$\left|\varphi\left\{\mathcal{G}(t,u(e)), u(e), \mathcal{D}^{\o_1}_{\varrho_1,\psi_1,\sigma}u, ..., \mathcal{D}^{\o_n}_{\varrho_n,\psi_n,\sigma.0+}u\right\}\right| \leq \varepsilon, \quad for\ any \varepsilon > 0.$$

Then there exists a solution u_a of the FDE such that

$$|u(e) - u_a(e)| \leq K(\varepsilon)$$

and

$$lim_{\varepsilon \to 0} K(\varepsilon) = 0.$$

Theorem 3. *Let $\mathcal{G} : [0, \mathcal{Q}] \times \mathbb{R} \to \mathbb{R}, u(e) \in \mathcal{C}_{1-\psi}[0, \mathcal{Q}]$ and $u(e)$ satisfy*

$$|\mathcal{D}^{\o}_{\varrho,\psi,w} u + du(e) - \mathcal{G}(t, u(e))| \leq \varepsilon, \ \ 0 < \psi < 1, \ \ d \in \mathbb{R}.$$

Then there exists a solution $u_a(e) : (0, \mathcal{Q}] \to \mathcal{C}$ of FDE such that

$$\mathcal{D}^{\o}_{\varrho,\psi,w} u + du(e) = \mathcal{G}(t, u(e))$$

and

$$|u(e) - u_a(e)| \leq \frac{\varepsilon e^{\psi}}{|d|} \sum_{n=0}^{\infty} \frac{e^{\psi n} \mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi+1}(|\sigma| e^{\varrho})}{|d|^n},$$

where $\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi+1}(|\sigma| e^{\varrho})$ is the generalized MLF.

Proof. The proof is similar to that of Theorem 2. ◄

4. Mittag-Leffler-Hyers-Ulam stability of fractional differential equations

In this section, we are going to prove Mittag-Leffler-HUS for linear and non-linear FDEs

$$\begin{cases} \mathcal{D}^{\o}_{\varrho,\psi,\sigma} u(e) + du(e) - q(e) = F(e), \quad e \in (0, \mathcal{Q}], \ d \in \mathbb{R} \\ e^{\psi-1} u(e)|_{e=0}, \end{cases}$$

$$\begin{cases} \mathcal{D}^{\o}_{\varrho,\psi,\sigma} u(e) + du(e) - \mathcal{G}(t, u(e)) = F(e), \quad e \in (0, \mathcal{Q}], \ d \in \mathbb{R} \\ e^{1-\psi} u(e)|_{e=0} \end{cases}$$

with Prabhakar derivatives using FrFT.

4.1. Stability of linear FDE $\mathcal{D}^{\o}_{\varrho,\psi,\sigma} u(e) + du(e) - q(e) = F(e), e \in (0, \mathcal{Q}], \ d \in \mathbb{R}$ and $e^{\psi-1} u(e)|_{e=0}$

Theorem 4. *If a function $u(e)$ satisfies*

$$|\mathcal{D}^{\o_1}_{\varrho,\psi,w} u + du(e) - q(e)| \leq F(e), \ 0 < \psi < 1, \ d \in \mathbb{R}, \ \ \forall \ e \in \mathcal{C}_{1-\psi}[0, \mathcal{Q}], \ F(e) > 0, \tag{9}$$

then there exists a solution $u_a(e) : (0, \mathcal{Q}] \to \mathcal{C}$ of FDE such that

$$\mathcal{D}^{\o_1}_{\varrho,\psi,w} u + du(e) - q(e) = F(e) \tag{10}$$

and

$$|u(e) - u_a(e)| \leq \frac{F(e)}{|d|} \sum_{n=0}^{\infty} \frac{e^{\psi n} \mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi+1}(|\sigma| e^{\varrho})}{|d|^n}.$$

Proof. Consider

$$y(e) = \mathcal{D}^{\o_1}_{\varrho,\psi,w} u + du(e) - q(e) - F(e), \quad e \in (0, \mathcal{Q}].$$

Taking FrFT on both sides, we get

$$\begin{aligned} F_\varsigma[y(e)] &= F_\varsigma[\mathcal{D}^{\o_1}_{\varrho,\psi,w} u] + F_\varsigma[du(e)] - F_\varsigma[q(e)] - F_\varsigma[F(e)] \\ &= \left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\psi} \left[1 - \sigma \left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o} F_\varsigma[u(e)] \\ &- \left(\mathcal{E}^{-\o}_{\varrho,1-u,\sigma,0+} u\right)(0) + dF_\varsigma[u(e)] - F_\varsigma[q(e)] - F_\varsigma[F(e)]. \end{aligned}$$

By using Lemma 3 and considering $b = du_0\Gamma(\psi)$ and $c = \frac{b}{d\Gamma(\psi)} t$, we get

$$\begin{aligned} F_\varsigma[y(e)] &= \left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\psi} \left[1 - \sigma \left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o} F_\varsigma[u(e)] \\ &\quad - du_0\Gamma(u) + dF_\varsigma[u(e)] - F_\varsigma[q(e)] - F_\varsigma[F(e)] \\ F_\varsigma[y(e)] &= \left\{\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\psi} \left[1 - \sigma \left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o} + d\right\} F_\varsigma[u(e)] \\ &\quad - du_0\Gamma(u) - F_\varsigma[q(e)] - F_\varsigma[F(e)] \\ F_\varsigma[u(e)] &= \frac{F_\varsigma[y(e)]}{\left\{\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\psi} \left[1 - \sigma \left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o} + d\right\}} \\ &\quad + \frac{du_0\Gamma(u) + F_\varsigma[q(e)] + F_\varsigma[F(e)]}{\left\{\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{\psi} \left[1 - \sigma \left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o} + d\right\}}. \end{aligned} \tag{11}$$

Let $u_a(e)$ be the solution of FDE. Then

$$\mathcal{C}_{1-\psi}[a, \mathcal{Q}] = \left\{u(e) \in \mathcal{C}[a, \mathcal{Q}]; e^{1-\psi} u_a(e) \in \mathcal{C}[a, \mathcal{Q}]\right\}, 0 < \psi < 1$$

and

$$\lim_{e\to a^+}[(t-a)^{1-\psi}u_a(e)] = c,\ \ c\in\mathcal{C}.$$

Consequently

$$\left(\mathcal{E}^{-\o}_{\varrho 1-,\psi,\sigma,a^+}u\right)(a^+) = c\Gamma(\psi)$$

holds. Therefore,

$$\begin{aligned}u_a(e) = {}&\frac{du_0\Gamma(u)\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n e^{\psi n+\psi-1}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma e^{\varrho})}{d}\\ &+\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n\int_0^e q(x)(e-x)^{\psi n+\psi-1}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma(e-x)^{\varrho})dx}{d}\\ &+\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n\int_0^e F(x)(e-x)^{\psi n+\psi-1}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma(e-x)^{\varrho})dx}{d}.\end{aligned}\tag{12}$$

We have

$$\begin{aligned}e^{1-\psi}u_a(e) = {}&\frac{du_0\Gamma(u)\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n e^{\psi n}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma e^{\varrho})}{d}\\ &+\frac{e^{1-\psi}\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n\int_0^e q(x)(e-x)^{\psi n+\psi-1}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma(e-x)^{\varrho})dx}{d}\\ &+\frac{e^{1-\psi}\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n\int_0^e F(x)(e-x)^{\psi n+\psi-1}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma(e-x)^{\varrho})dx}{d}.\end{aligned}$$

This implies that $u_a(e)$ satisfies $e^{1-\psi}u_a(e)|_{e=0}$.

Now, applying FrFT and using the convolution property, we have

$$\begin{aligned}F_{\varsigma}\left[u_a(e)\right] = {}&F_{\varsigma}\left[\frac{du_0\Gamma(u)\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n e^{\psi n+\psi-1}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma e^{\varrho})}{d}\right]\\ &+F_{\varsigma}\left[\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n\int_0^e q(x)(e-x)^{\psi n+\psi-1}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma(e-x)^{\varrho})dx}{d}\right]\\ &+F_{\varsigma}\left[\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n\int_0^e F(x)(e-x)^{\psi n+\psi-1}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma(e-x)^{\varrho})dx}{d}\right]\\ = {}&\frac{du_0\Gamma(u)\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n}{d}F_{\varsigma}\left[e^{\psi n+\psi-1}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma e^{\varrho})\right]\\ &+\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n}{d}F_{\varsigma}\left[\int_0^e q(x)(e-x)^{\psi n+\psi-1}\mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi}(\sigma(e-x)^{\varrho})dx\right]\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^{n}}{d} F_{\varsigma}\left[\int_{0}^{e} F(x)(e-x)^{\psi n+\psi-1} \mathcal{E}_{\varrho, \psi n+\psi}^{\o n+\o}\left(\sigma(e-x)^{\varrho}\right) d x\right] \\
=&\frac{d u_{0} \Gamma(u) \sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^{n}}{d} F_{\varsigma}\left[e^{\psi n+\psi-1} \mathcal{E}_{\varrho, \psi n+\psi}^{\o n+\o}\left(\sigma e^{\varrho}\right)\right] \\
&+\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^{n}}{d} F_{\varsigma}\left[e^{\psi n+\psi-1} \mathcal{E}_{\varrho, \psi n+\psi}^{\o n+\o}\left(\sigma e^{\varrho}\right)\right] F_{\varsigma}[q(e)] \\
&+\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^{n}}{d} F_{\varsigma}\left[e^{\psi n+\psi-1} \mathcal{E}_{\varrho, \psi n+\psi}^{\o n+\o}\left(\sigma e^{\varrho}\right)\right] F_{\varsigma}[F(e)]
\end{aligned}
$$

$$
\begin{aligned}
&F_{\varsigma}\left[u_{a}(e)\right]= \\
&=\left\{d u_{0} \Gamma(u)+F_{\varsigma}[q(e)]+F_{\varsigma}[F(e)]\right\}\left[\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^{n} F_{\varsigma}\left[e^{\psi n+\psi-1} \mathcal{E}_{\varrho, \psi n+\psi}^{\o n+\o}\left(\sigma e^{\varrho}\right)\right]}{d}\right] \\
&=\left\{d u_{0} \Gamma(u)+F_{\varsigma}[q(e)]+F_{\varsigma}[F(e)]\right\}\left[\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^{n}\left(-i \Omega^{\frac{1}{\varsigma}}\right)^{\varrho(\o n+\o)-(\psi n+\psi)}}{\left[\left(-i \Omega^{\frac{1}{\varsigma}}\right)^{\varrho}-\sigma\right]^{\o n+\o}}\right] \\
&=\left\{d u_{0} \Gamma(u)+F_{\varsigma}[q(e)]+F_{\varsigma}[F(e)]\right\}\left[\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^{n}\left(-i \Omega^{\frac{1}{\varsigma}}\right)^{\varrho(\o n+\o)-(\psi n+\psi)}}{\left(-i \Omega^{\frac{1}{\varsigma}}\right)^{\varrho}\left[1-\sigma\left(-i \Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o n+\o}}\right].
\end{aligned}
$$

Expanding the summation, we get

$$
F_{\varsigma}\left[u_{a}(e)\right]=\frac{d u_{0} \Gamma(u)+F_{\varsigma}[q(e)]+F_{\varsigma}[F(e)]}{\left\{\left(-i \Omega^{\frac{1}{\varsigma}}\right)^{\psi}\left\{1-\sigma\left(-i \Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right\}^{\o}+d\right\}}. \tag{13}
$$

Taking (13) into account, we deduce that $u_a(e)$ is a solution of (10).

Subtracting (11) and (13) from each other, we get

$$
F_{\varsigma}[u(e)]+F_{\varsigma}\left[u_{a}(e)\right]=
$$

$$
\frac{F_{\varsigma}[y(e)]+d u_{0} \Gamma(u)+F_{\varsigma}[q(e)]+F_{\varsigma}[F(e)]-\left[d u_{0} \Gamma(u)+F_{\varsigma}[q(e)]+F_{\varsigma}[F(e)]\right]}{\left\{\left(-i \Omega^{\frac{1}{\varsigma}}\right)^{\psi}\left[1-\sigma\left(-i \Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^{\o}+d\right\}} \tag{14}
$$

$$F_\varsigma\left[u(e)\right] + F_\varsigma\left[u_a(e)\right] = \frac{F_\varsigma[y(e)]}{\left\{\left(-i\Omega^{\frac{1}{\varsigma}}\right)^\psi \left[1-\sigma\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^\o + d\right\}}.$$

By Lemma 4, the RHS of (14) implies

$$F_\varsigma\left[\frac{\sum_{n=0}^\infty \left(\frac{-1}{d}\right)^n e^{\psi n}\mathcal{E}_{\varrho,\psi n+\psi}^{\o n+\o}(\sigma e^\varrho)}{d} * y(e)\right]$$

$$= F_\varsigma\left[\frac{\sum_{n=0}^\infty \left(\frac{-1}{d}\right)^n e^{\psi n}\mathcal{E}_{\varrho,\psi n+\psi}^{\o n+\o}(\sigma e^\varrho)}{d}\right] F_\varsigma\left[y(e)\right]$$

$$= \frac{F_\varsigma[y(e)]}{\left\{\left(-i\Omega^{\frac{1}{\varsigma}}\right)^\psi \left[1-\sigma\left(-i\Omega^{\frac{1}{\varsigma}}\right)^{-\varrho}\right]^\o + d\right\}}.$$

Hence, (14) can be written as

$$F_\varsigma\left[u(e)\right] + F_\varsigma\left[u_a(e)\right] = F_\varsigma\left[\frac{\sum_{n=0}^\infty \left(\frac{-1}{d}\right)^n e^{\psi n}\mathcal{E}_{\varrho,\psi n+\psi}^{\o n+\o}(\sigma e^\varrho)}{d} * y(e)\right].$$

Finally, from (9) we get

$$|u(e)+u_a(e)| = \left|\frac{\sum_{n=0}^\infty \left(\frac{-1}{d}\right)^n e^{\psi n}\mathcal{E}_{\varrho,\psi n+\psi}^{\o n+\o}(\sigma e^\varrho)}{d} * y(e)\right|$$
$$= \frac{F(e)e^\psi}{|d|}\sum_{n=0}^\infty \frac{e^{\psi n}\mathcal{E}_{\varrho,\psi n+\psi+1}^{\o n+\o}(|\sigma| e^\varrho)}{|d|^n},$$

where $\mathcal{E}_{\varrho,\psi n+\psi}^{\o n+\o}(\sigma e^\varrho)$ is the generalized MLF.

By the definition of the HUS theorem, the FDE (3) has Mittag-Leffler-Hyers–Ulam Stability. ◀

4.2. Stability of Non-linear FDE $\mathcal{D}_{\varrho,\psi,\sigma}^{\o}u(e) + du(e) - \mathcal{G}(t, u(e)) = F(e), e \in (0, \mathcal{Q}],\ d \in \mathbb{R}$ and $e^{\psi-1}u(e)|_{e=0}$

Theorem 5. *Let $\mathcal{G} : [0, \mathcal{Q}] \times \mathbb{R} \to \mathbb{R}, u(e) \in \mathcal{C}_{1-\psi}[0, \mathcal{Q}]$ and $u(e)$ satisfy*

$$|\mathcal{D}_{\varrho,\psi,w}^{\o_1}u + du(e) - \mathcal{G}(t, u(e))| \le F(e),\ 0 < \psi < 1,\ d \in \mathbb{R}. \tag{15}$$

Then there exists a solution $u_a(e) : (0, \mathcal{Q}] \to \mathcal{C}$ of FDE such that

$$\mathcal{D}^{\o_1}_{\varrho,\psi,w} u + du(e) - \mathcal{G}(t, u(e)) = F(e) \tag{16}$$

and

$$|u(e) - u_a(e)| \leq \frac{F(e)e^{\psi}}{|d|} \sum_{n=0}^{\infty} \frac{e^{\psi n} \mathcal{E}^{\o n+\o}_{\varrho,\psi n+\psi+1}(|\sigma| e^{\varrho})}{|d|^n}.$$

Proof. The proof is similar to that of Theorem 4. ◄

5. Example

Let us consider the FDE

$$\mathcal{D}^{1}_{1,\frac{1}{2},1} u(e) + 6u(e) = \frac{5\sqrt{e^t}}{2} + 6e^{\frac{1}{2}} + \frac{1}{16}, \tag{17}$$

where $\psi = 1, \varrho = 1, \delta = 1$ and $q(e) = \frac{5\sqrt{e^t}}{2} + 6e^{\frac{1}{2}} + \frac{1}{16}$.
For $\varepsilon = \frac{1}{8}$, the function $u(e)$ satisfies

$$\left| \mathcal{D}^{1}_{1,\frac{1}{2},1} u(e) + 6u(e) - \frac{5\sqrt{e^t}}{2} + 6e^{\frac{1}{2}} + \frac{1}{16} \right| \leq \frac{1}{8}$$

and the initial condition is $e^{1/2}u(e)|_{e=0}$. The exact solution of (17) is

$$u_\varsigma(e) = \frac{1}{6} \sum_{n=0}^{\infty} \left(\frac{-1}{7}\right)^n \int_0^e \left(\frac{5\sqrt{e^t}}{2} + 6e^{\frac{1}{2}} + \frac{1}{16}\right) (e-x)^{\frac{1}{2}n-\frac{1}{2}} \mathcal{E}^{n+1}_{1,\frac{1}{2}n+\frac{3}{2}}(e-x)dx$$

and the approximate solution $u_1(e)$ is

$$|u_1(e) - u_\varsigma(e)| < \frac{\varepsilon e^{\frac{1}{2}}}{|d|} \sum_{n=0}^{\infty} \frac{e^{\frac{1}{2}n} \mathcal{E}^{n+1}_{1,\frac{1}{2}n+\frac{3}{2}}(e)}{|6|^n}.$$

So

$$\frac{e^{\frac{1}{2}}}{48} \sum_{n=0}^{\infty} \frac{e^{\frac{1}{2}n} \mathcal{E}^{n+1}_{1,\frac{1}{2}n+\frac{3}{2}}(e)}{|6|^n}$$

is the control function of $u_1(e)$.

6. Conclusion

In this paper, we investigate Prabhakar derivatives in the sense of fractional calculus to find their generalized transforms. These derivatives are further generalization of fractional derivatives and effectively applicable for various applications like Cauchy problems, heat transfer problem. In order to explain the obtained results, some examples were given. Also, we introduced some standard approaches to the definition of Prabhakar derivatives, fractional differential equations, the Riemann-Liouville fractional differential operator, and the Mittag-Leffler function and studied their basic properties. In particular, we formulate the theorem describing the structure of the Mittag-Leffler-Hyers-Ulam problem for linear and nonlinear fractional differential equations associated with the Prabhakar derivatives and derive the Prabhakar derivative step response functions of those generalised systems. We discussed the basic properties of derivatives, including the rules for their properties and the conditions for the equivalence of various definitions. Finally, we proved the standard approaches to the Mittag-Leffler-Hyers-Ulam problem of the linear and nonlinear fractional differential equations with Prabhakar derivatives using a fractional Fourier transform.

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Swaminathan Deepa
Department of Mathematics, Adhiyamaan College of Engineering, Hosur-635109, India.
E-mail: febdeepa@gmail.com

Anumanthappa Ganesh
Department of Mathematics, Government Arts and Science College, Hosur-635 110, India.
E-mail: dr.aganesh14@gmail.com

Vagif R. Ibrahimov
Institute of Control Systems named after Academician A.Huseynov, Baku AZ1141, Azerbaijan.
E-mail: ibvag47@mail.ru

Shyam Sundar Santra
Department of Mathematics, JIS College of Engineering, Kalyani 741235, India.
E-mail: shyam01.math@gmail.com; shyamsundar.santra@jiscollege.ac.in

Vediyappan Govindan
Department of mathematics, Dmi St John the Baptist University, Malawi, Central Africa.
E-mail: govindoviya@gmail.com

Khaled Mohamed Khedher
Department of Civil Engineering, College of Engineering, King Khalid University, Abha 61421, Saudi Arabia.
Department of Civil Engineering, High Institute of Technological Studies, Mrezgua University Campus, Nabeul 8000, Tunisia.
E-mail: kkhedher@kku.edu.sa

Samad Noeiaghdam
Industrial Mathematics Laboratory, Baikal School of BRICS, Irkutsk National Research Technical University, Irkutsk, 664074, Russia.
Department of Applied Mathematics and Programming, South Ural State University, Lenin prospect 76, Chelyabinsk, 454080, Russia.
E-mail: snoei@istu.edu; noiagdams@susu.ru

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