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On a Boundary Value Problem For a Fifth Order Partial Integro-Differential Equation

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Abstract. The problems of the unique classical solvability and the construction of the solution of a multidimensional boundary value problem for a homogeneous fifth order partial integro-differential equations with a degenerate kernel are studied. The multidimensional Fourier series method, based on the separation of many variables, is used. A system of countable systems of integral equations is derived. Iteration process of solving the problem is constructed. Sufficient coefficient conditions for the unique classical solvability of the boundary value problem are established.

Key Words and Phrases: nonlocal boundary value problem, fifth order integro-differential equation, degenerate kernel, Fourier series, classical solvability.

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1. Formulation of the problem

The theory of boundary value problems is currently one of the most important directions of the theory of higher order partial differential equations. A large number of research works are dedicated to the study of this theory (see, in particular, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]). Many problems of gas dynamics, theory of elasticity, theory of plates and shells are described by high-order partial differential equations. When the boundary of the flow domain of a physical process is unavailable for measurements, nonlocal conditions in integral form can serve as an information sufficient for the unique solvability of the problem [15]. Therefore, in recent years, the study of nonlocal boundary value problems for differential and integro-differential equations with integral conditions has been intensified (see, for example, [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]).

In this paper, we study the regular solvability of a boundary value problem for a fifth order integro-differential equation with an integral conditions, parameter and reflective time-argument. This paper is a further development of the works [31, 32, 33]. In studying one-valued solvability and constructing solutions, the presence of spectral parameter plays an important role.

In multi-dimensional domain $\Omega = \left\{ -T < t < T, 0 < x_1, ..., x_m < l \right\}$, a partial integro-differential equation of the following form is considered

$$U_{tt}(t,x) + \sum_{i=1}^{m} \left[\omega_1(t)\, U_{tt\, x_i\, x_i}(t,x) - \omega_2(t) \int\limits_{-T}^{T} U_{x_i\, x_i\, x_i\, x_i}(-\theta, x) d\theta \right]$$

$$= \nu \int\limits_{-T}^{T} K(t,s) \sum_{i=1}^{m} U_{t\, x_i\, x_i\, x_i\, x_i}(s,x) ds, \tag{1}$$

where T and l are the given positive real numbers, ν is a nonzero real parameter, $0 \le \omega_i(t) \in C\,[-T;T]$, $i = 1,2$, $x \in \mathbb{R}^m$, $0 \ne K(t,s) = \sum\limits_{i=1}^{k} a_i\,(t)\, b_i(s)$, $a_i(t)$, $b_i(s) \in C\,[-T;T]$. It is supposed that the system of functions $a_i(t)$, $i = \overline{1,\, k}$ and the system of functions $b_i(s)$, $i = \overline{1,\, k}$ are linearly independent.

Problem. Find in the domain Ω a function from the class

$$U(t,x) \in C\,(\bar{\Omega}) \cap C^{2,4}_{t,x}(\Omega) \cap C^{2+2+0+\,\ldots\,+0}_{t,x_1,x_2,\,\ldots\,,\,x_m}(\Omega)$$

$$\cap C^{2+0+2+0+\,\ldots\,+0}_{t,x_1,x_2,x_3,\,\ldots\,,\,x_m}(\Omega) \cap\, \ldots\, \cap C^{2+0+\,\ldots\,+0+2}_{t,\,x_1,\,\ldots\,,\,x_{m-1},\,x_m}(\Omega)$$

$$\cap C^{1+4+0+\,\ldots\,+0}_{t,\,x_1,\,x_2,\,\ldots\,,\,x_m}(\Omega) \cap C^{1+0+4+0+\,\ldots\,+0}_{t,\,x_1,\,x_2,\,x_3,\,\ldots\,,\,x_m}(\Omega) \cap\, \ldots\, \cap C^{1+0+\,\ldots\,+0+4}_{t,\,x_1,\,\ldots\,,\,x_{m-1},\,x_m}(\Omega), \tag{2}$$

satisfying the integro-differential equation (1) and the following boundary conditions

$$U(0,x) = \varphi(x), \quad 0 \le x \le l, \tag{3}$$

$$U_t(T,x) = \psi(x), \quad 0 \le x \le l, \tag{4}$$

$$U\big(t,0,x_2,x_3,\,\ldots\,,\,x_m\big) = U\big(t,l,x_2,x_3,\,\ldots\,,\,x_m\big)$$

$$= U\big(t,x_1,0,x_3,\,\ldots\,,\,x_m\big) = U\big(t,x_1,l,x_3,\,\ldots\,,\,x_m\big) = \ \ldots$$

$$= U\big(t,x_1,\,\ldots\,,\,x_{m-1},0\big) = U\big(t,x_1,\,\ldots\,,\,x_{m-1},l\big)$$

$$= U_{x_1x_1}\big(t,0,x_2,x_3\,,\,\ldots\,,\,x_m\big) = U_{x_1x_1}\big(t,l,x_2,x_3,\,\ldots\,,\,x_m\big)$$

$$= U_{x_1x_1}\big(t,x_1,0,x_3,\ldots\,,\,x_m\big) = U_{x_1x_1}\big(t,x_1,l,x_3,\,\ldots\,,\,x_m\big) = \ \ldots$$

$$= U_{x_1x_1}\big(t,x_1,\,\ldots\,,\,x_{m-1},0\big) = U_{x_1x_1}\big(t,x_1,\,\ldots\,,\,x_{m-1},l\big) = \ \ldots$$

$$= U_{x_mx_m}\big(t,0,x_2,x_3,\,\ldots\,,\,x_m\big) = U_{x_mx_m}\big(t,l,x_2,x_3,\,\ldots\,,\,x_m\big)$$

$$= U_{x_m x_m}\big(t, x_1, 0, x_3, \ldots, x_m\big) = U_{x_m x_m}\big(t, x_1, l, x_3, \ldots, x_m\big) = \ \ldots$$

$$= U_{x_m x_m}\big(t, x_1, \ldots, x_{m-1}, 0\big) = U_{x_m x_m}\big(t, x_1, \ldots, x_{m-1}, l\big) = 0, \qquad (5)$$

where $\psi(x), \varphi(x)$ are the given sufficiently smooth functions in the domain $\Omega_l^m = \{0 \le x_1, \ldots, x_m \le l\}$, $C^r(\Omega)$ is the class of functions $U\big(t, x_1, \ldots, x_m\big)$, possessing the derivatives $\frac{\partial^r U}{\partial t^r}, \frac{\partial^r U}{\partial x_1^r}, \ldots, \frac{\partial^r U}{\partial x_m^r}$ in Ω, $C_{t,x}^{r,s}(\Omega)$ is the class of functions $U\big(t, x_1, \ldots, x_m\big)$, possessing the continuous derivatives $\frac{\partial^r U}{\partial t^r}, \frac{\partial^s U}{\partial x_1^s}, \ldots, \frac{\partial^s U}{\partial x_m^s}$ in Ω, $C_{t,x_1,x_2,\ldots,x_m}^{r+r+0+\ldots+0}(\Omega)$ is the class of functions $U\big(t, x_1, \ldots, x_m\big)$, possessing the continuous derivative $\frac{\partial^{2r} U}{\partial t^r \partial x_1^r}$ in Ω, $\ldots$, $C_{t,x_1,\ldots,x_{m-1},x_m}^{r+0+\ldots+0+r}(\Omega)$ is the class of functions $U\big(t, x_1, \ldots, x_m\big)$, possessing the continuous derivative $\frac{\partial^{2r} U}{\partial t^r \partial x_m^r}$ in Ω, r, s are arbitrary natural numbers, $\bar{\Omega} = \{-T \le t \le T,\ 0 \le x_1, \ldots, x_m \le l\}$.

2. Formal solution of the problem

Taking into account the Dirichlet conditions (5), the nontrivial solutions of the problem (1)–(5) are sought as a following Fourier sine series

$$U(t,x) = \sum_{n_1,\ldots,n_m=1}^{\infty} u_{n_1,\ldots,n_m}(t)\, \vartheta_{n_1,\ldots,n_m}(x), \qquad (6)$$

where

$$u_{n_1,\ldots,n_m}(t) = \int_{\Omega_l^m} U(t,x)\, \vartheta_{n_1,\ldots,n_m}(x)\, dx, \qquad (7)$$

$$\int_{\Omega_l^m} U(t,x)\, \vartheta_{n_1,\ldots,n_m}(x)\, dx = \int_0^l \ldots \int_0^l U(t,x)\, \vartheta_{n_1,\ldots,n_m}(x)\, dx_1 \cdot \ldots \cdot dx_m,$$

$$\vartheta_{n_1,\ldots,n_m}(x) = \left(\sqrt{\frac{2}{l}}\right)^m \sin\frac{\pi n_1}{l} x_1 \cdot \ldots \cdot \sin\frac{\pi n_m}{l} x_m,$$

$$\Omega_l^m = [0; l]^m, \ \ n_1, \ldots, n_m = 1, 2, \ldots$$

We also suppose that the following functions can be expanded into Fourier series

$$\varphi(x) = \sum_{n_1,\ldots,n_m=1}^{\infty} \varphi_{n_1,\ldots,n_m} \vartheta_{n_1,\ldots,n_m}(x), \qquad (8)$$

$$\psi(x) = \sum_{n_1,\ldots,n_m=1}^{\infty} \psi_{n_1,\ldots,n_m}\, \vartheta_{n_1,\ldots,n_m}(x), \qquad (9)$$

where

$$\varphi_{n_1,\ldots,n_m} = \int\limits_{\Omega_l^m} \varphi(x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx, \tag{10}$$

$$\psi_{n_1,\ldots,n_m} = \int\limits_{\Omega_l^m} \psi(x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx. \tag{11}$$

Substituting Fourier series (6) into partial integro-differential equation (1), we obtain the countable system of ordinary integro-differential equations of second order

$$u''_{n_1,\ldots,n_m}(t) - \lambda^2_{n_1,\ldots,n_m}(t) \int\limits_{-T}^{T} u_{n_1,\ldots,n_m}(-\theta)\,d\theta$$

$$= \nu\,\lambda^2_{n_1,\ldots,n_m}(t) \int\limits_{-T}^{T} \sum_{i=1}^{k} a_i(t)\,b_i(s)\,u'_{n_1,\ldots,n_m}(s)\,ds, \tag{12}$$

where $\lambda^2_{n_1,\ldots,n_m}(t) = \frac{\omega_2(t)\,\mu^4_{n_1,\ldots,n_m}}{1+\omega_1(t)\,\mu^2_{n_1,\ldots,n_m}}$, $\mu_{n_1,\ldots,n_m} = \frac{\pi}{l}\sqrt{n_1^2 + \ldots + n_m^2}$. Using the notation

$$\tau_{i,n_1,\ldots,n_m} = \int\limits_{-T}^{T} b_i(s)\,u'_{n_1,\ldots,n_m}(s)\,ds, \tag{13}$$

the countable system (12) can be rewritten as

$$u''_{n_1,\ldots,n_m}(t) = \nu\,\lambda^2_{n_1,\ldots,n_m}(t) \sum_{i=1}^{k} a_i(t)\,\tau_{i,n_1,\ldots,n_m}$$

$$+\lambda^2_{n_1,\ldots,n_m}(t) \int\limits_{-T}^{T} u_{n_1,\ldots,n_m}(-\theta)\,d\theta. \tag{14}$$

The second order countable system of integro-differential equations (14) is solved by the method of variation of arbitrary constants

$$u_{n_1,\ldots,n_m}(t) = A_{1,n_1,\ldots,n_m}\,t + A_{2,n_1,\ldots,n_m} + \eta_{n_1,\ldots,n_m}(t), \tag{15}$$

where we have used the following notations:

$$\eta_{n_1,\ldots,n_m}(t) = \nu \sum_{i=1}^{k} \tau_{i,n_1,\ldots,n_m}\,h_{i,n_1,\ldots,n_m}(t) + \delta_{n_1,\ldots,n_m}(t) \int\limits_{-T}^{T} u_{n_1,\ldots,n_m}(-\theta)\,d\theta,$$

$$h_{i,n_1,\ldots,n_m}(t) = \int\limits_0^t (t-s)\,\lambda_{n_1,\ldots,n_m}(s)\,a_i(s)\,ds, \quad i=\overline{1,\,k},$$

$$\delta_{n_1,\ldots,n_m}(t) = \int\limits_0^t (t-s)\,\lambda_{n_1,\ldots,n_m}(s)\,ds.$$

By virtue of the Fourier coefficients (7), (10) and (11), the conditions (3) and (4) take the forms

$$u_{n_1,\ldots,n_m}(0) = \int\limits_{\Omega_l^m} U(0,x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx$$

$$= \int\limits_{\Omega_l^m} \varphi(x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx = \varphi_{n_1,\ldots,n_m}, \tag{16}$$

$$u'_{n_1,\ldots,n_m}(T) = \int\limits_{\Omega_l^m} U_t(T,x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx$$

$$= \int\limits_{\Omega_l^m} \psi(x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx = \psi_{n_1,\ldots,n_m}. \tag{17}$$

To find the unknown Fourier coefficients $A_{1,n_1,\ldots,n_m}$ and $A_{2,n_1,\ldots,n_m}$ in (15), we use the boundary value conditions (16) and (17). Applying (16) to representation (15), we find

$$A_{2,n_1,\ldots,n_m} = \varphi_{n_1,\ldots,n_m}. \tag{18}$$

After differentiating (15) once, by condition (17) we have

$$u'_{n_1,\ldots,n_m}(t) = A_{1,n_1,\ldots,n_m} +$$

$$+\nu \sum_{i=1}^{k} \tau_{i,n_1,\ldots,n_m}\, h'_{i,n_1,\ldots,n_m}(t) + \delta'_{n_1,\ldots,n_m}(t) \int\limits_{-T}^{T} u_{n_1,\ldots,n_m}(-\theta)\,d\theta, \tag{19}$$

where

$$h'_{i,n_1,\ldots,n_m}(t) = \int\limits_0^t \lambda_{n_1,\ldots,n_m}(s)\,a_i(s)\,ds, \;\; \delta'_{n_1,\ldots,n_m}(t) = \int\limits_0^t \lambda_{n_1,\ldots,n_m}(s)\,ds.$$

By virtue of condition (17), from (19) we obtain

$$A_{1,n_1,\dots,n_m} = \psi_{n_1,\dots,n_m}$$

$$-\nu \sum_{i=1}^{k} \tau_{i,n_1,\dots,n_m}\, h'_{i,n_1,\dots,n_m}(T) - \delta'_{n_1,\dots,n_m}(T) \int\limits_{-T}^{T} u_{n_1,\dots,n_m}(-\theta)\, d\theta. \qquad (20)$$

Substituting the determined Fourier coefficients (18) and (20) into presentation (15), we find

$$u_{n_1,\dots,n_m}(t) = \varphi_{n_1,\dots,n_m} + \psi_{n_1,\dots,n_m}\, t +$$

$$+\nu \sum_{i=1}^{k} \tau_{i,n_1,\dots,n_m}\, M_{i,n_1,\dots,n_m}(t) + N_{n_1,\dots,n_m}(t) \int\limits_{-T}^{T} u_{n_1,\dots,n_m}(-\theta)\, d\theta, \qquad (21)$$

where

$$M_{i,n_1,\dots,n_m}(t) = h_{i,n_1,\dots,n_m}(t) - t \cdot h'_{i,n_1,\dots,n_m}(T),$$

$$N_{n_1,\dots,n_m}(t) = \delta_{n_1,\dots,n_m}(t) - t \cdot \delta'_{i,n_1,\dots,n_m}(T),$$

$$h_{i,n_1,\dots,n_m}(t) = \int\limits_{0}^{t} (t-s)\, \lambda_{n_1,\dots,n_m}(s)\, a_i(s)\, ds, \quad i = \overline{1,\, k},$$

$$\delta_{n_1,\dots,n_m}(t) = \int\limits_{0}^{t} (t-s)\, \lambda_{n_1,\dots,n_m}(s)\, ds.$$

After differentiating (21) once, we have

$$u'_{n_1,\dots,n_m}(t) = \psi_{n_1,\dots,n_m} +$$

$$+\nu \sum_{i=1}^{k} \tau_{i,n_1,\dots,n_m}\, M'_{i,n_1,\dots,n_m}(t) + N'_{n_1,\dots,n_m}(t) \int\limits_{-T}^{T} u_{n_1,\dots,n_m}(-\theta)\, d\theta, \qquad (22)$$

where

$$M'_{i,n_1,\dots,n_m}(t) = h'_{i,n_1,\dots,n_m}(t) - h'_{i,n_1,\dots,n_m}(T),$$

$$N'_{n_1,\dots,n_m}(t) = \delta'_{n_1,\dots,n_m}(t) - \delta'_{i,n_1,\dots,n_m}(T),$$

$$h'_{i,n_1,\dots,n_m}(t) = \int\limits_{0}^{t} \lambda_{n_1,\dots,n_m}(s)\, a_i(s)\, ds, \quad \delta'_{n_1,\dots,n_m}(t) = \int\limits_{0}^{t} \lambda_{n_1,\dots,n_m}(s)\, ds.$$

Substituting the derivative (22) into (13), we obtain the system of algebraic equations (SAE)

$$\tau_{i,n_1,\ldots,n_m} + \nu \sum_{j=1}^{k} \tau_{i,n_1,\ldots,n_m} H_{i,j,n_1,\ldots,n_m}$$

$$= \psi_{n_1,\ldots,n_m} \Phi_{1,i,n_1,\ldots,n_m} + \Phi_{2,i,n_1,\ldots,n_m} \int_{-T}^{T} u_{n_1,\,\ldots\,,\,n_m}(-\theta)\, d\theta, \quad i = \overline{1,\, k}, \quad (23)$$

where

$$H_{i,j,n_1,\ldots,n_m} = \int_{-T}^{T} b_i(s)\, M'_{j,n_1,\ldots,n_m}(s)\, ds,$$

$$\Phi_{1,i,n_1,\ldots,n_m} = \int_{-T}^{T} b_i(s)\, ds, \;\; \Phi_{2,i,n_1,\ldots,n_m} = \int_{-T}^{T} b_i(s)\, N'_{n_1,\ldots,n_m}(s)\, ds.$$

We recall that the systems of functions $a_i(t)$, $i = \overline{1,\, k}$ and $b_i(s)$, $i = \overline{1,\, k}$ are linearly independent. Hence it follows that $H_{i,j,n_1,\ldots,n_m} \neq 0$. The SAE (23) is uniquely solvable for every bounded right-hand side of SAE, if the following Fredholm condition is fulfilled

$$\Delta(\nu) = \begin{vmatrix} 1+\nu H_{1\,1} & \nu H_{1\,2} & \ldots & \nu H_{1\,k} \\ \nu H_{2\,1} & 1+\nu H_{2\,2} & \ldots & \nu H_{2\,k} \\ \ldots & \ldots & \ldots & \ldots \\ \nu H_{k\,1} & \nu H_{k\,2} & \ldots & 1+\nu H_{k\,k} \end{vmatrix} \neq 0. \quad (24)$$

The values of the parameter ν, at which the condition (24) is satisfied, are called regular. On the set of regular values of the parameter ν, the solutions of the SAE (23) can be written as

$$\tau_{i,n_1,\ldots,n_m} = \psi_{n_1,\ldots,n_m} \frac{\Delta_{1,i}(\nu)}{\Delta(\nu)} + \frac{\Delta_{2,i}(\nu)}{\Delta(\nu)} \int_{-T}^{T} u_{n_1,\ldots,n_m}(-\theta)\, d\theta, \quad (25)$$

where

$$\Delta_{m,i}(\nu) = \begin{vmatrix} 1+\nu H_{1\,1} & \ldots & \nu H_{1\,(i-1)} & \Phi_{m\,1} & \nu H_{1\,(i+1)} & \ldots & \nu H_{1\,k} \\ \nu H_{2\,1} & \ldots & \nu H_{2\,(i-1)} & \Phi_{m\,2} & \nu H_{2\,(i+1)} & \ldots & \nu H_{2\,k} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \nu H_{k\,1} & \ldots & \nu H_{k\,(i-1)} & \Phi_{m\,k} & \nu H_{k\,(i+1)} & \ldots & 1+\nu H_{k\,k} \end{vmatrix},$$

$i = \overline{1,\, k}$, $m = 1,\, 2$.

Substituting the solutions (25) into representation (21), we obtain the countable system of integral equations (CSIE)

$$u_{n_1,\,\ldots,\,n_m}(t) = \varphi_{n_1,\,\ldots,\,n_m}$$

$$+\psi_{n_1,\,\ldots,\,n_m} V_{n_1,\,\ldots,\,n_m}(t) + W_{n_1,\,\ldots,\,n_m}(t) \int\limits_{-T}^{T} u_{n_1,\,\ldots,\,n_m}(-\theta)\, d\theta, \qquad (26)$$

where

$$V_{n_1,\,\ldots,\,n_m}(t) = t + \nu \sum_{i=1}^{k} \frac{\Delta_{1,\,i}(\nu)}{\Delta_1(\nu)}\, M_{i,n_1,\,\ldots,\,n_m}(t),$$

$$W_{n_1,\,\ldots,\,n_m}(t) = N(t) + \nu \sum_{i=1}^{k} \frac{\Delta_{2,i}(\nu)}{\Delta(\nu)}\, M_{i,n_1,\,\ldots,\,n_m}(t),$$

$\varphi_{n_1,\,\ldots,\,n_m}$ and $\psi_{n_1,\,\ldots,\,n_m}$ are Fourier coefficients in series (8), (9), and are determined from (10), (11), respectively.

Now, to obtain an expansion of the formal solution to the problem (1)–(5), we substitute the representation (26) into the Fourier series (6):

$$U(t,x) = \sum_{n_1,\,\ldots,\,n_m=1}^{\infty} \vartheta_{n_1,\,\ldots,\,n_m}(x)$$

$$\times \left[\varphi_{n_1,\ldots,n_m} + \psi_{n_1,\ldots,n_m} V_{n_1,\ldots,n_m}(t) + W_{n_1,\ldots,n_m}(t) \int\limits_{-T}^{T} u_{n_1,\ldots,n_m}(-\theta) d\theta \right]. \qquad (27)$$

One-valued solvability of CSIE (26)

We consider the concepts of the following well-known Banach spaces: the space $B_2(T)$ of sequences of continuous functions $\{u_{n_1,\,\ldots,\,n_m}(t)\}_{n_1,\,\ldots,\,n_m=1}^{\infty}$ on the segment $[-T;T]$ with norm

$$\|\, u(t)\,\|_{B_2(T)} = \sqrt{\sum_{n_1,\,\ldots,\,n_m=1}^{\infty} \left(\max_{t\in[-T;T]} |\, u_{n_1,\,\ldots,\,n_m}(t)\,| \right)^2} < \infty;$$

the Hilbert coordinate space ℓ_2 of number sequences $\{\varphi_{n_1,\,\ldots,\,n_m}\}_{n_1,\,\ldots,\,n_m=1}^{\infty}$ with norm

$$\|\,\varphi\,\|_{\ell_2} = \sqrt{\sum_{n_1,\,\ldots,\,n_m=1}^{\infty} |\,\varphi_{n_1,\,\ldots,\,n_m}|^2} < \infty;$$

the space $L_2\left(\Omega_l^m\right)$ of square-summable functions on the domain $\Omega_l^m = [0;l]^m$ with norm

$$\| \vartheta(x) \|_{L_2\left(\Omega_l^m\right)} = \sqrt{\int\limits_{\Omega_l^m} |\, \vartheta(y) \,|^2 \, dy} < \infty.$$

Assume that for the smooth functions $V_{n_1,\ldots,n_m}(t),\quad W_{n_1,\ldots,n_m}(t)$ from (26) it follows that the conditions

$$C_1 = \max_{n_1,\ldots,n_m} \left\{ \max_{t\in[-T;T]} |\, V_{n_1,\ldots,n_m}(t) \,| ; \max_{t\in[-T;T]} \left| V''_{n_1,\ldots,n_m}(t) \right| \right\} < \infty, \tag{28}$$

$$\sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty} \left(n_1^4 n_2^4 ... n_m^4 C_{2\, n_1,\ldots,n_m} \right)^2} < \infty, \tag{29}$$

hold, where $C_{2\, n_1,\ldots,n_m} = \max \left\{ \max\limits_{t\in[-T;\,T]} |\, W_{n_1,\ldots,n_m}(t) \,| ; \max\limits_{t\in[-T;\,T]} \left| W''_{n_1,\ldots,n_m}(t) \right| \right\}$.

Conditions of smoothness. Let the multidimensional functions $\varphi(x)$, $\psi(x) \in C^4\left(\Omega_l^m\right)$ in the given domain Ω_l^m have piecewise continuous derivatives up to the fifth order and

$$\begin{gathered}
\varphi\big(0, x_2, x_3, \ldots, x_m\big) = \varphi\big(l, x_2, x_3, \ldots, x_m\big) \\
= \varphi\big(x_1, 0, x_3, \ldots, x_m\big) = \varphi\big(x_1, l, x_3, \ldots, x_m\big) \\
= \ \ldots \ = \varphi\big(x_1, \ldots, x_{m-1}, 0\big) = \varphi\big(x_1, \ldots, x_{m-1}, l\big) \\
= \varphi_{x_1 x_1}\big(0, x_2, x_3, \ldots, x_m\big) = \varphi_{x_1 x_1}\big(l, x_2, x_3, \ldots, x_m\big) \\
= \varphi_{x_1 x_1}\big(x_1, 0, x_3, \ldots, x_m\big) = \varphi_{x_1 x_1}\big(x_1, l, x_3, \ldots, x_m\big) = \ \ldots \\
= \varphi_{x_1 x_1}\big(x_1, \ldots, x_{m-1}, 0\big) = \varphi_{x_1 x_1}\big(x_1, \ldots, x_{m-1}, l\big) = \ \ldots \\
= \varphi_{x_m x_m}\big(0, x_2, x_3, \ldots, x_m\big) = \varphi_{x_m x_m}\big(l, x_2, x_3, \ldots, x_m\big) \\
= \varphi_{x_m x_m}\big(x_1, 0, x_3, \ldots, x_m\big) = \varphi_{x_m x_m}\big(x_1, l, x_3, \ldots, x_m\big) = \ \ldots \\
= \varphi_{x_m x_m}\big(x_1, \ \ldots \ , x_{m-1}, 0\big) = \varphi_{x_m x_m}\big(x_1, \ldots \ , x_{m-1}, l\big) \\
= \varphi_{x_1 x_1 x_1 x_1}\big(0, x_2, x_3, \ldots, x_m\big) = \varphi_{x_1 x_1 x_1 x_1}\big(l, x_2, x_3, \ldots, x_m\big) \\
= \varphi_{x_1 x_1 x_1 x_1}\big(x_1, 0, x_3, \ldots, x_m\big) = \varphi_{x_1 x_1 x_1 x_1}\big(x_1, l, x_3, \ldots, x_m\big) = \ \ldots \\
= \varphi_{x_1 x_1 x_1 x_1}\big(x_1, \ldots, x_{m-1}, 0\big) = \varphi_{x_1 x_1 x_1 x_1}\big(x_1, \ldots, x_{m-1}, l\big) = \ \ldots \\
= \varphi_{x_m x_m x_m x_m}\big(0, x_2, x_3, \ldots, x_m\big) = \varphi_{x_m x_m x_m x_m}\big(l, x_2, x_3, \ldots, x_m\big) \\
= \varphi_{x_m x_m x_m x_m}\big(x_1, 0, x_3, \ldots, x_m\big) = \varphi_{x_m x_m x_m x_m}\big(x_1, l, x_3, \ldots, x_m\big) = \ \ldots
\end{gathered}$$

$$= \varphi_{x_m x_m x_m x_m}\big(x_1, \ldots, x_{m-1}, 0\big) = \varphi_{x_m x_m x_m x_m}\big(x_1, \ldots, x_{m-1}, l\big) = 0. \qquad (30)$$

Conditions similar to (30) also hold for the function $\psi(x)$. Then, after integrating the integrals (10), (11)

$$\varphi_{n_1, \ldots, n_m} = \int\limits_{\Omega_l^m} \varphi(x)\, \vartheta_{n_1, \ldots, n_m}(x)\, dx, \;\; \psi_{n_1, \ldots, n_m} = \int\limits_{\Omega_l^m} \psi(x)\, \vartheta_{n_1, \ldots, n_m}(x)\, dx$$

by parts five times over the variable x_1, we get

$$\varphi_{n_1, \ldots, n_m} = \left(\frac{l}{\pi}\right)^5 \frac{\varphi^{(5)}_{n_1, \ldots, n_m}}{n_1^5}, \;\; \psi_{n_1, \ldots, n_m} = \left(\frac{l}{\pi}\right)^5 \frac{\psi^{(5)}_{n_1, \ldots, n_m}}{n_1^5}, \qquad (31)$$

where

$$\varphi^{(5)}_{n_1, \ldots, n_m} = \int\limits_{\Omega_l^m} \frac{\partial^5 \varphi(x)}{\partial x_1^5}\, \vartheta_{n_1, \ldots, n_m}(x)\, dx, \qquad (32)$$

$$\psi^{(5)}_{n_1, \ldots, n_m} = \int\limits_{\Omega_l^m} \frac{\partial^5 \psi(x)}{\partial x_1^5}\, \vartheta_{n_1, \ldots, n_m}(x)\, dx. \qquad (33)$$

By integrating the integrals (32), (33) by parts five times over the variable x_2, we have

$$\varphi^{(5)}_{n_1, \ldots, n_m} = \left(\frac{l}{\pi}\right)^5 \frac{\varphi^{(10)}_{n_1, \ldots, n_m}}{n_2^5}, \;\; \psi^{(5)}_{n_1, \ldots, n_m} = \left(\frac{l}{\pi}\right)^5 \frac{\psi^{(10)}_{n_1, \ldots, n_m}}{n_2^5}, \qquad (34)$$

where

$$\varphi^{(10)}_{n_1, \ldots, n_m} = \int\limits_{\Omega_l^m} \frac{\partial^{10} \varphi(x)}{\partial x_1^5\, \partial x_2^5}\, \vartheta_{n_1, \ldots, n_m}(x)\, dx,$$

$$\psi^{(10)}_{n_1, \ldots, n_m} = \int\limits_{\Omega_l^m} \frac{\partial^{10} \psi(x)}{\partial x_1^5\, \partial x_2^5}\, \vartheta_{n_1, \ldots, n_m}(x)\, dx.$$

Continuing this process, we obtain

$$\varphi^{(5m-5)}_{n_1, \ldots, n_m} = \left(\frac{l}{\pi}\right)^5 \frac{\varphi^{(5m)}_{n_1, \ldots, n_m}}{n_m^5}, \;\; \psi^{(5m-5)}_{n_1, \ldots, n_m} = \left(\frac{l}{\pi}\right)^5 \frac{\psi^{(5m)}_{n_1, \ldots, n_m}}{n_m^5}, \qquad (35)$$

where

$$\varphi^{(5m)}_{n_1, \ldots, n_m} = \int\limits_{\Omega_l^m} \frac{\partial^{5m} \varphi(x)}{\partial x_1^5\, \partial x_2^5 \ldots \partial x_m^5}\, \vartheta_{n_1, \ldots, n_m}(x)\, dx,$$

$$\psi^{(5\,m)}_{n_1,\,\dots\,,\,n_m} = \int\limits_{\Omega_l^m} \frac{\partial^{5\,m}\,\psi(x)}{\partial\,x_1^5\,\partial\,x_2^5\dots\,\partial\,x_m^5}\,\vartheta_{n_1,\,\dots\,,\,n_m}(x)\,dx.$$

Then the Bessel inequalities are valid:

$$\sum_{n_1,\,\dots\,,\,n_m=1}^{\infty}\left[\varphi^{(5\,m)}_{n_1,\,\dots\,,\,n_m}\right]^2 \le \left(\frac{2}{l}\right)^m \int\limits_{\Omega_l^m}\left[\frac{\partial^{5\,m}\,\varphi(x)}{\partial\,x_1^5\partial\,x_2^5\dots\,\partial\,x_m^5}\right]^2 dx, \tag{36}$$

$$\sum_{n_1,\,\dots\,,\,n_m=1}^{\infty}\left[\psi^{(5\,m)}_{n_1,\,\dots\,,\,n_m}\right]^2 \le \left(\frac{2}{l}\right)^m \int\limits_{\Omega_l^m}\left[\frac{\partial^{5\,m}\,\psi(x)}{\partial\,x_1^5\,\partial\,x_2^5\dots\,\partial\,x_m^5}\right]^2 dx. \tag{37}$$

From the formulas (31), (34) and (35) we obtain the following relations

$$\varphi_{n_1,\,\dots\,,\,n_m} = \left(\frac{l}{\pi}\right)^{5\,m}\frac{\varphi^{(5\,m)}_{n_1,\,\dots\,,\,n_m}}{n_1^5\,.\,.\,.\,n_m^5},\ \ \psi_{n_1,\,\dots\,,\,n_m} = \left(\frac{l}{\pi}\right)^{5\,m}\frac{\psi^{(5\,m)}_{n_1,\,\dots\,,\,n_m}}{n_1^5\,.\,.\,.\,n_m^5}. \tag{38}$$

Now, let us prove the unique solvability of a countable system of linear Fredholm integral equations (26). We define the iterative Picard process for countable system (26) as follows:

$$\begin{cases} u^0_{n_1,\,\dots\,,\,n_m}(t) = \varphi_{n_1,\,\dots\,,\,n_m} + \psi_{n_1,\,\dots\,,\,n_m}\,V_{n_1,\,\dots\,,\,n_m}(t), \\ u^k_{n_1,\,\dots\,,\,n_m}(t) = u^0_{n_1,\,\dots\,,\,n_m}(t) + W_{n_1,\,\dots\,,\,n_m}(t)\int\limits_{-T}^{T} u^{k-1}_{n_1,\,\dots\,,\,n_m}(-\theta)\,d\theta. \end{cases} \tag{39}$$

Taking into account the formulas (28), (36)–(38) and applying the Cauchy–Schwarz and Bessel inequalities for the zero approximation, from (39) we obtain the estimate

$$\sum_{n_1,\,\dots\,,\,n_m=1}^{\infty}\max_{t\in[-T;\,T]}\left|\,u^0_{n_1,\,\dots\,,\,n_m}(t)\,\right|$$

$$\le \sum_{n_1,\,\dots\,,\,n_m=1}^{\infty}\left|\,\varphi_{n_1,\,\dots\,,\,n_m}\right| + \sum_{n_1,\,\dots\,,\,n_m=1}^{\infty}\max_{t\in[-T;\,T]}\left|\,V_{n_1,\,\dots\,,\,n_m}(t)\,\right|\cdot\left|\,\psi_{n_1,\,\dots\,,\,n_m}\right|$$

$$\le \sum_{n_1,\,\dots\,,\,n_m=1}^{\infty}\left|\,\varphi_{n_1,\,\dots\,,\,n_m}\right| + C_1\sum_{n_1,\,\dots\,,\,n_m=1}^{\infty}\left|\,\psi_{n_1,\,\dots\,,\,n_m}\right|$$

$$\le \sum_{n_1,\,\dots\,,\,n_m=1}^{\infty}\frac{1}{n_1^5\dots n_m^5}\left|\,\varphi^{(5\,m)}_{n_1,\,\dots\,,\,n_m}\right| + C_1\sum_{n_1,\,\dots\,,\,n_m=1}^{\infty}\frac{1}{n_1^5\,.\,.\,.\,n_m^5}\left|\,\psi^{(5\,m)}_{n_1,\,\dots\,,\,n_m}\right|$$

$$\leq \sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{n_1^{10}\ldots n_m^{10}}} \sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty} \left[\varphi_{n_1,\ldots,n_m}^{(5\,m)}\right]^2}$$

$$\left. + C_1 \sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{n_1^{10}\ldots n_m^{10}}} \sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty} \left[\psi_{n_1,\ldots,n_m}^{(5\,m)}\right]^2}\,\right]$$

$$\leq \left(\frac{2}{l}\right)^m \sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{n_1^{10}\ldots n_m^{10}}} \left[\sqrt{\int_{\Omega_l^m} \left[\frac{\partial^{5\,m}\varphi(x)}{\partial x_1^5\,\partial x_2^5\ldots\partial x_m^5}\right]^2 dx}\right.$$

$$\left. + C_1 \sqrt{\int_{\Omega_l^m} \left[\frac{\partial^{5\,m}\psi(x)}{\partial x_1^5\,\partial x_2^5\ldots\partial x_m^5}\right]^2 dx}\,\right] < \infty. \tag{40}$$

Taking into account (29) and applying the Cauchy–Schwarz inequality for an arbitrary difference of the approximation (39), we obtain the estimate

$$\sum_{n_1,\ldots,n_m=1}^{\infty} \max_{t\in[-T;T]} \left| u_{n_1,\ldots,n_m}^{k}(t) - u_{n_1,\ldots,n_m}^{k-1}(t) \right|$$

$$\leq \max_{t\in[-T;T]} \sum_{n_1,\ldots,n_m=1}^{\infty} |\,W_{n_1,\ldots,n_m}(t)\,| \cdot \int_{-T}^{T} \left| u_{n_1,\ldots,n_m}^{k-1}(-t) - u_{n_1,\ldots,n_m}^{k-2}(-t) \right| dt$$

$$\leq \int_{-T}^{T} \sum_{n_1,\ldots,n_m=1}^{\infty} C_{2\,n_1,\ldots,n_m} \left| u_{n_1,\ldots,n_m}^{k-1}(-t) - u_{n_1,\ldots,n_m}^{k-2}(-t) \right| dt$$

$$\leq 2\,T\, \|C_2\|_{\ell_2} \left\| u^{k-1}(-t) - u^{k-2}(-t) \right\|_{B_2(T)}, \quad k = 2,\,3,\,\ldots \tag{41}$$

Similarly to (41), we obtain

$$\sum_{n_1,\ldots,n_m=1}^{\infty} \left| u_{n_1,\ldots,n_m}^{k}(-t) - u_{n_1,\ldots,n_m}^{k-1}(-t) \right|$$

$$\leq 2\,T\, \|C_2\|_{\ell_2} \left\| u^{k-1}(t) - u^{k-2}(t) \right\|_{B_2(T)}, \quad k = 2,\,3,\,\ldots \tag{42}$$

From the estimates (41) and (42) we have

$$\left\| U^{k}(t) - U^{k-1}(t) \right\|_{B_2(T)} \leq \rho \left\| U^{k-1}(t) - U^{k-2}(t) \right\|_{B_2(T)}, \tag{43}$$

where $\rho = 2\,T\,\|C_2\|_{\ell_2} < 1$, $\left\| U^k(t) - U^{k-1}(t) \right\|_{B_2(T)} =$

$$= \max\left\{ \left\| u^k(t) - u^{k-1}(t) \right\|_{B_2(T)};\ \left\| u^k(-t) - u^{k-1}(-t) \right\|_{B_2(T)} \right\}.$$

Since $\left\| u^k(t) - u^{k-1}(t) \right\|_{B_2(T)} \le \left\| U^k(t) - U^{k-1}(t) \right\|_{B_2(T)}$, it follows from (43) that the operator on the right-hand side of (26) is contracting. And, from (40) and (43) it follows that there is a unique fixed point $u(t) \in B_2(T)$ on the segment $[-T;\, T]$. This implies the unique solvability of the countable system of integral equations (26) on the interval $[-T;\, T]$, if there have been fulfilled the conditions (24), (28), (29), smoothness and $2\,T\,\|C_2\|_{\ell_2} < 1$. Consequently, the iterative Picard process (39) converges absolutely and uniformly to the function $u(t) \in B_2(T)$, if the conditions (24), (28), (29), smoothness and $2\,T\,\|C_2\|_{\ell_2} < 1$ are fulfilled.

3. Convergence of Fourier series

If the conditions (24), (28), (29), smoothness and $2\,T\,\|C_2\|_{\ell_2} < 1$ are fulfilled, then we can show absolute and uniform convergence of series (27). Indeed, we have

$$|\,U(t,\,x)\,| \le \sum_{n_1,\,\ldots,\,n_m=1}^{\infty} |\,\vartheta_{\,n_1,\ldots,n_m}(x)\,|$$

$$\times \left| \varphi_{\,n_1,\ldots,n_m} + \psi_{n_1,\ldots,n_m} V_{n_1,\ldots,n_m}(t) + W_{n_1,\ldots,n_m}(t) \int_{-T}^{T} u_{n_1,\,\ldots\,,\,n_m}(-\theta)\,d\theta \right|$$

$$\le \left(\sqrt{\frac{2}{l}}\right)^m \sum_{n_1,\,\ldots,\,n_m=1}^{\infty} |\,\varphi_{n_1,\ldots,n_m}\,| + C_1 \left(\sqrt{\frac{2}{l}}\right)^m \sum_{n_1,\,\ldots,\,n_m=1}^{\infty} |\,\psi_{n_1,\ldots,n_m}\,|$$

$$+ \left(\sqrt{\frac{2}{l}}\right)^m \int_{-T}^{T} \sum_{n_1,\,\ldots,\,n_m=1}^{\infty} C_{2n_1,\ldots,n_m}\,|\,u_{n_1,\ldots,n_m}(-t)\,|\;dt$$

$$\le \left(\frac{2}{l}\right)^{\frac{3\,m}{2}} \gamma_1 \left[\left\| \frac{\partial^{5\,m}\varphi(x)}{\partial x_1^5\,\partial x_2^5 \ldots \partial x_m^5} \right\|_{L_2(\Omega_l^m)} + \left\| \frac{\partial^{5\,m}\psi(x)}{\partial x_1^5\,\partial x_2^5 \ldots \partial x_m^5} \right\|_{L_2(\Omega_l^m)} \right]$$

$$+2\,T \left(\sqrt{\frac{2}{l}}\right)^m \|C_2\|_{\ell_2}\,\|\,u(-t)\,\|_{B_2(T)} < \infty,$$

where $\gamma_1 = \sqrt{\sum\limits_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{n_1^{10}\ldots n_m^{10}}}$.

The function (27) is formally differentiated the required number of times with respect to the required arguments

$$U_{tt}(t,x) = \sum_{n_1,\ldots,n_m=1}^{\infty} \vartheta_{n_1,\ldots,n_m}(x)\Big[\varphi_{n_1,\ldots,n_m}$$

$$+\psi_{n_1,\ldots,n_m} V''_{n_1,\ldots,n_m}(t) + W''_{n_1,\ldots,n_m}(t) \int_{-T}^{T} u_{n_1,\ldots,n_m}(-\theta)\,d\theta\Big], \tag{44}$$

$$U_{x_1x_1x_1x_1}(t,x) = \sum_{n_1,\ldots,n_m=1}^{\infty} \left(\frac{\pi n_1}{l}\right)^4 \vartheta_{n_1,\ldots,n_m}(x)\Big[\varphi_{n_1,\ldots,n_m}$$

$$+\psi_{n_1,\ldots,n_m} V_{n_1,\ldots,n_m}(t) + W_{n_1,\ldots,n_m}(t) \int_{-T}^{T} u_{n_1,\ldots,n_m}(-\theta)\,d\theta\Big], \tag{45}$$

$$U_{x_2x_2x_2x_2}(t,x) = \sum_{n_1,\ldots,n_m=1}^{\infty} \left(\frac{\pi n_2}{l}\right)^4 \vartheta_{n_1,\ldots,n_m}(x)\Big[\varphi_{n_1,\ldots,n_m}$$

$$+\psi_{n_1,\ldots,n_m} V_{n_1,\ldots,n_m}(t) + W_{n_1,\ldots,n_m}(t) \int_{-T}^{T} u_{n_1,\ldots,n_m}(-\theta)\,d\theta\Big]. \tag{46}$$

Similar to (44)–(46), we define the expansions of the functions

$$U_{x_3x_3x_3x_3}(t,x),\ \ldots,\ U_{x_mx_mx_mx_m}(t,x),\ U_{ttx_1x_1}(t,x),\ U_{ttx_2x_2}(t,x),\ \ldots,\ U_{ttx_mx_m}(t,x)$$

into Fourier series.

For the function (27), it is easy to verify that all derivatives, appearing in the equation (1) are continuous. The proof of the convergence of functions (44) exactly coincides with the proof of convergence of the function (27). For the functions (45) and (46), we present a proof of the convergence

$$|\,U_{x_1x_1x_1x_1}(t,x)\,| \le \frac{\pi^4}{l^4} \sum_{n_1,\ldots,n_m=1}^{\infty} n_1^4\,|\,u_{n_1,\ldots,n_m}(t)\,| \cdot |\,\vartheta_{n_1,\ldots,n_m}(x)\,|$$

$$\le \left(\sqrt{\frac{2}{l}}\right)^m \frac{\pi^4}{l^4} \sum_{n_1,\ldots,n_m=1}^{\infty} n_1^4\,|\,\varphi_{n_1,\ldots,n_m}\,|$$

$$+C_1 \left(\sqrt{\frac{2}{l}}\right)^m \frac{\pi^4}{l^4} \sum_{n_1,\ldots,n_m=1}^{\infty} n_1^4 \,|\, \psi_{n_1,\ldots,n_m} \,|$$

$$+\left(\sqrt{\frac{2}{l}}\right)^m \frac{\pi^4}{l^4} \int_{-T}^{T} \sum_{n_1,\ldots,n_m=1}^{\infty} n_1^4 C_{2\,n_1,\ldots,n_m} \,|\, u_{n_1,\ldots,n_m}(-t) \,|\, dt$$

$$\leq \gamma_2 \sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{n_1^2 \ldots n_m^{10}}}$$

$$\times \left[\left\| \frac{\partial^{5\,m} \varphi(x)}{\partial\, x_1^5 \,\partial\, x_2^5 \ldots \,\partial\, x_m^5} \right\|_{L_2\left(\Omega_l^m\right)} + C_1 \left\| \frac{\partial^{5\,m} \psi(x)}{\partial\, x_1^5 \,\partial\, x_2^5 \ldots \,\partial\, x_m^5} \right\|_{L_2\left(\Omega_l^m\right)} \right]$$

$$+\left(\sqrt{\frac{2}{l}}\right)^m \frac{\pi^4}{l^4} \sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty} \left(n_1^4 C_{2\,n_1,\ldots,n_m}\right)^2} \,\|\, u(-t) \,\|_{B_2(T)} < \infty,$$

$$|\, U_{x_2\,x_2\,x_2\,x_2}(t,x) \,| \leq \frac{\pi^4}{l^4} \sum_{n_1,\ldots,n_m=1}^{\infty} n_2^4 \,|\, u_{n_1,\ldots,n_m}(t,\nu) \,| \cdot |\, \vartheta_{n_1,\ldots,n_m}(x) \,|$$

$$\leq \left(\sqrt{\frac{2}{l}}\right)^m \frac{\pi^4}{l^4} \sum_{n_1,\ldots,n_m=1}^{\infty} n_2^4 \,|\, \varphi_{n_1,\ldots,n_m} \,|$$

$$+C_1 \left(\sqrt{\frac{2}{l}}\right)^m \frac{\pi^4}{l^4} \sum_{n_1,\ldots,n_m=1}^{\infty} n_2^4 \,|\, \psi_{n_1,\ldots,n_m} \,|$$

$$+\left(\sqrt{\frac{2}{l}}\right)^m \frac{\pi^4}{l^4} \int_{-T}^{T} \sum_{n_1,\ldots,n_m=1}^{\infty} n_2^4 C_{2\,n_1,\ldots,n_m} \,|\, u_{n_1,\ldots,n_m}(-t) \,|\, dt$$

$$\leq \gamma_2 \sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{n_1^{10} n_2^2 n_3^{10} \ldots n_m^{10}}}$$

$$\times \left[\left\| \frac{\partial^{5\,m} \varphi(x)}{\partial\, x_1^5 \,\partial\, x_2^5 \ldots \,\partial\, x_m^5} \right\|_{L_2\left(\Omega_l^m\right)} + C_1 \left\| \frac{\partial^{5\,m} \psi(x)}{\partial\, x_1^5 \,\partial\, x_2^5 \ldots \partial\, x_m^5} \right\|_{L_2\left(\Omega_l^m\right)} \right]$$

$$+\left(\sqrt{\frac{2}{l}}\right)^m \frac{\pi^4}{l^4} \sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty} \left(n_1^4\, C_{2\,n_1,\ldots,n_m}\right)^2} \,\|\, u(-t) \,\|_{B_2(T)} < \infty,$$

where $\gamma_2 = \left(\frac{2}{l}\right)^{\frac{3m}{2}} \frac{\pi^4}{l^4}$.

The convergence of the following functions is proved in exactly the same way as above

$$|\,U_{x_3\,x_3\,x_3\,x_3}(t,x)\,| < \infty, \ \ldots, \ |\,U_{x_m\,x_m\,x_m\,x_m}(t,x)\,| < \infty, \ |\,U_{t\,t\,x_1\,x_1}(t,x)\,| < \infty,$$

$$|\,U_{t\,t\,x_2\,x_2}(t,x)\,| < \infty, \ \ldots, \ |\,U_{t\,t\,x_m\,x_m}(t,x)\,| < \infty.$$

Consequently, in the domain Ω, the function $U(t,x)$, defined by the series (27), satisfies the conditions (2) of the problem (1)–(5) for all possible $n_1, \ldots, n_m$.

4. Formulation of the theorem

So, we have proved the following

Theorem 1. *Let the conditions* (24), (28), (29), *smoothness and* $2\,T\,\|C_2\|_{\ell_2} < 1$ *be fulfilled. Then the boundary value problem* (1)-(5) *is uniquely solvable in the domain* Ω. *The solution is determined by the series* (27). *In this case, the function* (27) *is differentiable with respect to all variables, i.e. the derivatives of the solution* (27) *included in equation* (1) *exist and are continuous.*

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