# Inverse Spectral Problem of Discontinuous <br> Non-self-adjoint Operator Pencil with Almost Periodic Potentials 

R.F. Efendiev*, Sh. Annaghili


#### Abstract

Spectral characteristics for discontinuous non-self-adjoint operator pencil with almost periodic potentials are investigated. The spectrum of the operator is analysed, an effective algorithm for solving inverse problems using a part of spectral data is provided and uniqueness theorem is proved.


Key Words and Phrases: discontinuous equation, spectral singularities, inverse spectral problem, continuous spectrum.
2010 Mathematics Subject Classifications: 34A36, 34L05, 47A10, 47A70

## 1. Introduction

We deal with the spectral analysis of operator $L$ generated by the differential expression

$$
\begin{equation*}
l\left(\frac{d}{d x}, \lambda\right)=-\frac{d^{2}}{d x^{2}}+2 \lambda p(x)+q(x)-\lambda^{2} \rho(x) \tag{1}
\end{equation*}
$$

in the space $L_{2}(R)$ under the assumption that the potentials $p(x)$ and $q(x)$ have the form

$$
\begin{equation*}
p(x)=\sum_{n=1}^{\infty} p_{n} e^{i \alpha_{n} x}, q(x)=\sum_{n=1}^{\infty} q_{n} e^{i \alpha_{n} x}, \tag{2}
\end{equation*}
$$

where $\lambda$ is a complex parameter,

$$
\rho(x)=\left\{\begin{array}{cc}
1 & \text { for } x \geq 0,  \tag{3}\\
-1 & \text { for } x<0,
\end{array}\right.
$$

* Corresponding author.
http://www.azjm.org 156 (C) 2010 AZJM All rights reserved.
and the series

$$
\begin{equation*}
\sum_{n=1}^{\infty} \alpha_{n}\left|p_{n}\right|<\infty ; \quad \sum_{n=1}^{\infty}\left|q_{n}\right|<\infty \tag{4}
\end{equation*}
$$

converges.
Here the set of exponents $G=\left\{\alpha_{n}\right\}$ satisfies the following conditions:

1. $\alpha_{1}<\alpha_{2}<\ldots .<\alpha_{n}<\ldots, \quad \alpha_{n} \rightarrow \infty$
2. If $\alpha_{i}, \alpha_{j} \in G$, then $\alpha_{i}+\alpha_{j} \in G$

The functions (potentials) $p(x)$ and $q(x)$ are called Besicovitch almost periodic functions. They would be almost periodic if the following condition held:

If for every positive $\epsilon>0$ there exists a finite sum

$$
\sum_{j=1}^{n} r_{j} e^{i \alpha_{j} t} \equiv p(t)
$$

and for all $t \in R,|f(t)-p(t)|<\varepsilon$, then the function $f(t)$ is almost periodic.
In general, the function $f(x)$ is uniformly almost periodic if for every $\epsilon>$ 0 there is a finite linear combination of sine and cosine waves with respect to uniform norm.

The possibility of applying potentials (3) to investigation of differential equations with an almost periodic functions which are widely used and, undoubtedly, further development of the theory will lead to wider applications, can be justified by Riesz-Fischer theorem: Any trigonometric series $\sum_{n=1}^{\infty} A_{n} e^{i \Lambda_{n} x}$ subject to the single condition that the series $\sum_{n=1}^{\infty}\left|A_{n}\right|^{2}$ is convergent, is the Fourier series of an almost periodic function. [1,4,13,18]

Today, there are many publications dedicated to the spectral structure of selfadjoint operators with almost periodic coefficients in both one-dimensional and multidimensional cases. We note the works [3, 14], where the spectral theory of one-dimensional operators with limiting periodic potentials is considered. The operator $L$ is non- self-adjoint except for the trivial case $p(x)=q(x)=0$

The case where $\rho(x)$ can change sign, known as indefinite case, was firstly considered by Belishev [2], who solved inverse problem of reconstruction of $\rho(x)$ for $p(x)=q(x)=0$. As a rule, such problem is connected with discontinuities in the medium's physical characteristics.

Note that the case $\rho(x)=1$ was considered in $[10,14]$, where spectrum and resolvent in the space $L_{2}(R)$ have been investigated. It was shown that the spectrum of the operator is pure continuous. Moreover, simple spectral singularities can exist over the continuous spectrum.

In [16] the inverse problem for that case was solved using the normalizing numbers.
P. Sarnak in [17] investigated the case $p(x)=0, \rho(x)=1$ and $q(x)=$ $\sum_{i=1}^{n} q_{n} e^{i \alpha_{n} x}$. He proved that the spectrum of $L$ is $[0, \infty]$. The case $p(x)=0$ was investigated in [9], and the necessary and sufficient conditions for the normalising numbers to be the set of spectral data of the operator $L$ for that case have been found in [19].
R.F. Efendiev in [5, $6,7,8,9,10$ ] studied spectrum and inverse problem for $\alpha_{n}=$ $n, n \in N$. Note that the inverse problem for the operator $L$ differs from the case $\rho(x)=1$ because of the presence of a discontinuous coefficient $\rho(x)$, and we do not solve it, as in $[12,15,16]$, by normalizing numbers. We solve it by reflection coefficients, which is more natural for discontinuous problems.

In this work, we solve the inverse spectral problem of wave propagation with discontinuous wave speed in a one-dimensional layered-inhomogeneous medium in the frequency domain which is described by the Schroedinger equation

$$
\begin{equation*}
-y^{\prime \prime}(x, \lambda)+[2 \lambda p(x)+q(x)] y(x, \lambda)=\lambda^{2} \rho(x) y(x, \lambda) \tag{5}
\end{equation*}
$$

This work is organized as follows:
In Section 2 we investigate the representation of fundamental solutions. In Section 3 the properties of the spectrum are studied. It is proved that the continuous spectrum of the operator $L$ consists of axes $\{\operatorname{Re} \lambda=0\} \cup\{\operatorname{Im} \lambda=0\}$ and the continuous spectrum may have spectral singularities at the points $\pm \frac{\alpha_{n}}{2}, \pm \frac{i \alpha_{n}}{2} n \in$ $N$.

In Section 4 we give a formulation of the inverse problem and provide a constructive procedure for its solution.

## 2. Special solutions of equation $L y=0$

Let us denote the solutions to equation (5) by $f_{1}^{ \pm}(x, \lambda)$ and $f_{2}^{ \pm}(x, \lambda)$ which satisfy the following conditions:

$$
\begin{array}{ll}
\lim _{x \rightarrow \infty} f_{1}^{ \pm}(x, \lambda) e^{\mp i \lambda x}=1 & \text { for } \mp \operatorname{Im} \lambda>0, \\
\lim _{x \rightarrow \infty} f_{2}^{ \pm}(x, \lambda) e^{\mp \lambda x}=1 & \text { for } \pm \operatorname{Re} \lambda>0 .
\end{array}
$$

The existence of such solutions of equation (5) was considered in [5], where the following theorem was proved.

Theorem 1. Let $p(x)$ and $q(x)$ have the form (2),(4) and $\rho(x)$ satisfy the condition (3). Then equation (5) has particular solutions of the form

$$
\begin{align*}
& f_{1}^{ \pm}(x, \lambda)=e^{ \pm i \lambda x}\left(1+\sum_{n=1}^{\infty} V_{n}^{ \pm} e^{i \alpha_{n} x}+\sum_{n=1}^{\infty} \frac{1}{\alpha_{n} \pm 2 \lambda} \sum_{s=n}^{\infty} V_{n s}^{ \pm} e^{i \alpha_{s} x}\right) \text { for } x \geq 0,  \tag{6}\\
& f_{2}^{ \pm}(x, \lambda)=e^{ \pm \lambda x}\left(1+\sum_{n=1}^{\infty} V_{n}^{ \pm} e^{i \alpha_{n} x}+\sum_{n=1}^{\infty} \frac{1}{\alpha_{n} \mp 2 i \lambda} \sum_{s=n}^{\infty} V_{n s}^{ \pm} e^{i \alpha_{s} x}\right) \text { for } x<0 . \tag{7}
\end{align*}
$$

Here the numbers $V_{n \alpha}^{ \pm}$and $V_{n}^{ \pm}$are determined from the following relations:

$$
\begin{gather*}
\alpha_{s} V_{s}^{ \pm}+\alpha_{s} \sum_{n=1}^{s} V_{n s}^{ \pm}+\sum_{\substack{\alpha_{r}+\alpha_{k}=\alpha_{s} \\
k \geq n}}\left(q_{r} V_{k}^{ \pm} \pm \alpha_{n} p_{r}\right) V_{n k}^{ \pm}+q_{s}=0  \tag{8}\\
\alpha_{s}\left(\alpha_{s}-\alpha_{n}\right) V_{n s}^{ \pm}+\sum_{\substack{\alpha_{r}+\alpha_{k}=\alpha_{s} \\
k \geq n}}\left(q_{r} \mp \alpha_{n} p_{r}\right) V_{n k}^{ \pm}=0  \tag{9}\\
\alpha_{s} V_{s}^{ \pm} \pm \sum_{\alpha_{r}+\alpha_{k}=\alpha_{s}} p_{r} V_{k}^{ \pm} \pm p_{s}=0, \tag{10}
\end{gather*}
$$

where the series

$$
\begin{gather*}
\sum_{n=1}^{\infty} \frac{1}{\alpha_{n}} \sum_{s=n}^{\infty} \alpha_{s}\left|V_{n s}^{ \pm}\right|  \tag{11}\\
\sum_{n=1}^{\infty} \alpha_{n}^{2}\left|V_{n}^{ \pm}\right| \tag{12}
\end{gather*}
$$

converge.
We can easily see that at the points $\lambda=\mp \frac{\alpha_{n}}{2},\left(\lambda= \pm \frac{i \alpha_{n}}{2}\right), n \in N$, there can be simple poles of the function $f_{1}^{ \pm}(x, \lambda)\left(f_{2}^{ \pm}(x, \lambda)\right)$

Remark 1. If $\lambda \neq-\frac{\alpha_{n}}{2}$ and $\operatorname{Im} \lambda>0$, then $f_{1}^{+}(x, \lambda) \in L_{2}(0, \infty)$.
Remark 2. If $\lambda \neq-\frac{i \alpha_{n}}{2}$ and Re $\lambda>0$, then $f_{2}^{+}(x, \lambda) \in L_{2}(-\infty, 0)$.
Let us denote by

$$
W[f(x), g(x)]=f^{\prime}(x) g(x)-f(x) g^{\prime}(x)
$$

the Wronskian of the functions $f(x)$ and $g(x)$. Then

$$
\begin{aligned}
& W\left[f_{1}^{+}(x, \lambda), f_{1}^{-}(x, \lambda)\right]=2 i \lambda \neq 0, \text { for } \operatorname{Im} \lambda=0, \lambda \neq 0 \\
& W\left[f_{2}^{+}(x, \lambda), f_{2}^{-}(x, \lambda)\right]=-2 \lambda \neq 0, \text { for Re } \lambda=0, \lambda \neq 0
\end{aligned}
$$

for $\lambda \neq 0, \lambda_{n} \neq \pm \frac{\alpha_{n}}{2}, \lambda_{n} \neq \mp \frac{i \alpha_{n}}{2}$.
Therefore, the functions $f_{1}^{+}(x, \lambda), f_{1}^{-}(x, \lambda)$ and $f_{2}^{+}(x, \lambda), f_{2}^{-}(x, \lambda)$ are linearly independent for $\lambda \neq 0, \lambda_{n} \neq \pm \frac{\alpha_{n}}{2}, \lambda_{n} \neq \mp \frac{i \alpha_{n}}{2}, \operatorname{Im} \lambda=0$ and $\operatorname{Re} \lambda=0$, respectively.

Obviously, the functions

$$
\begin{align*}
& f_{n 1}^{ \pm}(x)=\lim _{\lambda \rightarrow \mp \frac{\alpha_{n}}{2}}\left(\alpha_{n} \pm 2 \lambda\right) f_{1}^{ \pm}(x, \lambda)=\sum_{s=n}^{\infty} V_{n s}^{ \pm} e^{i \alpha x} e^{-i \frac{\alpha_{n}}{2} x},  \tag{13}\\
& f_{n 2}^{ \pm}(x)=\lim _{\lambda \rightarrow \mp i \frac{\alpha_{n}}{2}}\left(\alpha_{n} \mp 2 i \lambda\right) f_{2}^{ \pm}(x, \lambda)=\sum_{s=n}^{\infty} V_{n s}^{ \pm} e^{i \alpha x} e^{-i \frac{\alpha_{n}}{2} x} \tag{14}
\end{align*}
$$

are the solutions of the equation (1) for $\lambda=\mp \frac{\alpha_{n}}{2}, \lambda=\mp i \frac{\alpha_{n}}{2}$.
From recurrent relations (8)-(10) and (11)-(12) it follows that $f_{n 1}^{ \pm}(x) \neq 0, f_{n 2}^{ \pm}(x) \neq$ 0 for $V_{n n}^{ \pm} \neq 0$. Then

$$
\begin{gathered}
W\left[f_{n 1}^{ \pm}(x), f_{1}^{ \pm}\left(x, \mp \frac{\alpha_{n}}{2}\right)\right]=0 \\
W\left[f_{n 2}^{ \pm}(x), f_{1}^{ \pm}\left(x, \mp i \frac{\alpha_{n}}{2}\right)\right]=0
\end{gathered}
$$

and consequently

$$
\begin{align*}
& f_{n 1}^{ \pm}(x)=S_{n 1}^{ \pm} f_{1}^{ \pm}\left(x, \mp \frac{\alpha_{n}}{2}\right)  \tag{15}\\
& f_{n 2}^{ \pm}(x)=S_{n 2}^{ \pm} f_{2}^{ \pm}\left(x, \mp i \frac{\alpha_{n}}{2}\right) .
\end{align*}
$$

Comparing these formulas we get

$$
S_{n 1}^{ \pm}=S_{n 2}^{ \pm}=V_{n n}^{ \pm} .
$$

Then (14) can be rewritten as follows:

$$
\begin{equation*}
V_{m \alpha+m}^{ \pm}=V_{m m}^{ \pm}\left(V_{\alpha}^{\mp}+\sum_{n=1}^{\alpha} \frac{V_{n \alpha}^{\mp}}{\alpha_{n}+\alpha_{m}}\right) . \tag{16}
\end{equation*}
$$

The formula (15) will play a crucial role in solving the inverse problem.

Note that by (12), the linearly independent solutions of (1), according to $\lambda= \pm \frac{\alpha_{n}}{2}, n \in N$, can be determined as

$$
\begin{aligned}
& \tilde{f}_{n 1}^{ \pm}(x)=\lim _{\lambda \rightarrow \mp \frac{\alpha_{n}}{2}}\left[f_{1}^{ \pm}(x, \lambda)+\frac{V_{n n}^{ \pm} f_{1}^{\mp}(x, \lambda)}{\alpha_{n} \pm 2 \lambda}\right]= \\
& =e^{-i \frac{\alpha_{n}}{2} x}\left(\varphi_{k n}^{ \pm}(x)+x \tilde{\varphi}_{k n}^{ \pm}(x)\right)
\end{aligned}
$$

where $\varphi_{k n}^{ \pm}(x)$ and $\tilde{\varphi}_{k n}^{ \pm}(x)$ are Bohr almost periodic functions. Obviously, $\tilde{f}_{n 1}^{ \pm}(x)$ and $f_{1}^{\mp}(x, \lambda)$ are linearly independent solutions of (1) for $\lambda= \pm \frac{\alpha_{n}}{2}, n \in N$. (Linearly independent solutions of (1) corresponding to $\lambda= \pm i \frac{\alpha_{n}}{2}, n \in N$ can be constructed analogously).

Linearly independent solutions of equation (1) corresponding to $\lambda=0$ are defined as $f_{k}^{ \pm}(x, \lambda)$ and $\frac{\partial f_{k}^{ \pm}(x, 0)}{\partial \lambda}$.

Hence, any solution of equation (1) for $\operatorname{Im} \lambda=0$ may be represented as a linear combination of the solutions $f_{1}^{+}(x, \lambda), f_{1}^{-}(x, \lambda)$ and for $\operatorname{Re} \lambda=0$ as a linear combination of the functions $f_{2}^{+}(x, \lambda), f_{2}^{-}(x, \lambda)$.

Then, in particular, we have

$$
\begin{align*}
& f_{2}^{+}(x, \lambda)=A^{+}(\lambda) f_{1}^{+}(x, \lambda)+B^{+}(\lambda) f_{1}^{-}(x, \lambda), \text { for } \lambda>0 \\
& f_{2}^{-}(x, \lambda)=A^{-}(\lambda) f_{1}^{+}(x, \lambda)+B^{-}(\lambda) f_{1}^{-}(x, \lambda), \text { for } \lambda<0  \tag{17}\\
& f_{1}^{+}(x, \lambda)=C^{+}(\lambda) f_{2}^{+}(x, \lambda)+D^{+}(\lambda) f_{2}^{-}(x, \lambda), \text { for } \operatorname{Im} \lambda>0, \operatorname{Re} \lambda=0 \\
& f_{1}^{-}(x, \lambda)=C^{-}(\lambda) f_{2}^{+}(x, \lambda)+D^{-}(\lambda) f_{2}^{-}(x, \lambda), \text { for } \operatorname{Im} \lambda<0, \operatorname{Re} \lambda=0,
\end{align*}
$$

where $A^{ \pm}(\lambda), B^{ \pm}(\lambda), C^{ \pm}(\lambda)$ and $D^{ \pm}(\lambda)$ are defined as follows:

$$
\begin{align*}
& A^{ \pm}(\lambda)=-\frac{1}{2 \lambda} W\left[f_{1}^{ \pm}(x, \lambda), f_{2}^{-}(x, \lambda)\right] \\
& B^{ \pm}(\lambda)=-\frac{1}{2 \lambda} W\left[f_{2}^{+}(x, \lambda) f_{1}^{ \pm}(x, \lambda)\right] \\
& C^{ \pm}(\lambda)=\frac{1}{2 i \lambda} W\left[f_{2}^{ \pm}(x, \lambda), f_{1}^{-}(x, \lambda)\right]  \tag{18}\\
& D^{ \pm}(\lambda)=\frac{1}{2 i \lambda} W\left[f_{1}^{+}(x, \lambda), f_{2}^{ \pm}(x, \lambda)\right] .
\end{align*}
$$

Taking into account formulas (17) and (18) we find

$$
\begin{align*}
& C(\lambda)=B^{+}(\lambda)=i D^{+}(\lambda)=\frac{W\left[f_{1}^{+}(x, \lambda), f_{2}^{+}(x, \lambda)\right]}{2 i \lambda} \text { for } \lambda \in S_{0} \\
& D(\lambda)=B^{-}(\lambda)=-i C^{+}(\lambda)=\frac{W\left[f_{1}^{+}(x, \lambda), f_{2}^{-}(x, \lambda)\right]}{2 i \lambda} \text { for } \lambda \in S_{1} \\
& B(\lambda)=A^{-}(\lambda)=i C^{-}(\lambda)=\frac{W\left[f_{2}^{-}(x, \lambda), f_{1}^{1}(x, \lambda)\right]}{2(2 i)} \text { for } \lambda \in S_{2}  \tag{19}\\
& A(\lambda)=A^{+}(\lambda)=i D^{-}(\lambda)=\frac{W\left[f_{2}^{+}(x, \lambda), f_{1}^{-}(x, \lambda)\right]}{2 i \lambda} \quad \text { for } \lambda \in S_{3},
\end{align*}
$$

where $S_{k}=\left\{\frac{k \pi}{2}<\arg \lambda<\frac{(k+1) \pi}{2}\right\}, k=\overline{0,3}$.
Thus, we find that eight coefficients $A^{ \pm}(\lambda), B^{ \pm}(\lambda), C^{ \pm}(\lambda)$ and $D^{ \pm}(\lambda)$ are actually expressed in terms of four complex-valued functions $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$. Then relation (12) takes the form

$$
\begin{align*}
& f_{2}^{+}(x, \lambda)=A(\lambda) f_{1}^{+}(x, \lambda)+C(\lambda) f_{1}^{-}(x, \lambda) \\
& f_{2}^{-}(x, \lambda)=B(\lambda) f_{1}^{+}(x, \lambda)+D(\lambda) f_{1}^{-}(x, \lambda) \\
& f_{1}^{+}(x, \lambda)=i D(\lambda) f_{2}^{+}(x, \lambda)-i C(\lambda) f_{2}^{-}(x, \lambda)  \tag{20}\\
& f_{1}^{-}(x, \lambda)=-i B(\lambda) f_{2}^{+}(x, \lambda)+i A(\lambda) f_{2}^{-}(x, \lambda)
\end{align*}
$$

## 3. The spectrum of the operator $L$

To study the spectrum of the operator $L$ generated by the differential expression (1), we first, we calculate the kernel of the resolvent of the operator $R_{\lambda}$. Let us prove the following theorem, from which we obtain the existence of the resolvent operator $R_{\lambda}$. We shall denote the resolvent set, spectrum, point spectrum, residual spectrum and continuous spectrum of $L$ by $\rho(L), \sigma(L), \sigma_{p}(L), \sigma_{r}(L)$ and $\sigma_{c}(L)$, respectively.

Theorem 2. The operator $L$ has no pure real and pure imaginary eigenvalues.
Proof. Equation (5) has the fundamental solutions $f_{1}^{+}(x, \lambda), f_{1}^{-}(x, \lambda)$ $\left(f_{2}^{+}(x, \lambda), f_{2}^{-}(x, \lambda)\right)$ on $|\operatorname{Im} \lambda|<\frac{\varepsilon}{2}\left(|\operatorname{Re} \lambda|<\frac{\varepsilon}{2}\right)$ and $\lambda \neq 0, \lambda \neq \pm \frac{\alpha_{n}}{2}, \lambda \neq$ $\pm \frac{i \alpha_{n}}{2}, n \in N$.

Then for $\operatorname{Im} \lambda=0$, the solution of equation (5) can be written as follows:
$y(x, \lambda)=C_{1} e^{i \operatorname{Re\lambda } x}\left(1+\sum_{n=1}^{\infty} V_{n}^{ \pm} e^{i \alpha_{n} x}+\sum_{n=1}^{\infty} \frac{1}{\alpha_{n}+2 \lambda} \sum_{s=n}^{\infty} V_{n s}^{ \pm} e^{i \alpha_{s} x}\right)+$
$+C_{2} e^{-i \operatorname{Re} \lambda x}\left(1+\sum_{n=1}^{\infty} V_{n}^{ \pm} e^{i \alpha_{n} x}+\sum_{n=1}^{\infty} \frac{1}{\alpha_{n}-2 \lambda} \sum_{s=n}^{\infty} V_{n s}^{ \pm} e^{i \alpha_{s} x}\right)$
Then $y(x, \lambda) \notin L_{2}(-\infty, \infty)$ except for $C_{1}=C_{2}=0$, because the principal parts of the solutions are periodic.

Analogously we can prove the case $\operatorname{Re} \lambda=0$. Hence $\sigma_{p}(L)=\emptyset$.
The theorem is proved.
Theorem 3. Residual spectrum of the operator $L$ is empty, $\sigma_{r}(L)=\emptyset$.
Proof. Let the function $g(x) \in L_{2}(R)$ be a solution of $L^{*}(\lambda)=0$ for $\lambda \in C$. Then $g(x)$ satisfies

$$
\begin{equation*}
-g^{\prime \prime}(x, \lambda)+[2 \lambda p(x)+q(x)] g(x, \lambda)=\lambda^{2} \rho(x) g(x, \lambda) \tag{21}
\end{equation*}
$$

Since (21) is of type (5), (21) cannot have a solution which belongs to $L_{2}(R)$. That means $\sigma_{p}\left(L^{*}\right)=\emptyset$ or $\sigma_{r}(L)=\emptyset$, so $\sigma(L)=\sigma_{c}(L)$ and $L^{-1}$ is defined in a dense set in $L_{2}(R)$ for $\forall \lambda \in C$.

The theorem proved.

In order to find $L^{-1}$ and resolvent set $\rho(L)$, let us investigate solution $y(x, \lambda) \in$ $L_{2}(R)$ of

$$
\begin{equation*}
-y^{\prime \prime}(x, \lambda)+[2 \lambda p(x)+q(x)] y(x, \lambda)-\lambda^{2} \rho(x) y(x, \lambda)=f(x) \tag{22}
\end{equation*}
$$

when $f(x) \in L_{2}(R)$.
If we apply the Lagrange method by using solutions of equation (5), then we find the solution of (22) as

$$
y(x, \lambda)=\int_{-\infty}^{\infty} R(x, t, \lambda) f(t) d t
$$

where the expression $R(x, t, \lambda)$ can be obviously written as

$$
R(x, t, \lambda)= \begin{cases}R_{11}(x, t, \lambda), & \lambda \in S_{0} \\ R_{12}(x, t, \lambda), & \lambda \in S_{1} \\ R_{21}(x, t, \lambda), & \lambda \in S_{2} \\ R_{22}(x, t, \lambda), & \lambda \in S_{3}\end{cases}
$$

where

$$
\begin{align*}
R_{11}(x, t, \lambda) & =-\frac{1}{2 i \lambda C(\lambda)}\left\{\begin{array}{ll}
f_{1}^{+}(x, \lambda) f_{2}^{+}(t, \lambda), & t \leq x \\
f_{1}^{+}(t, \lambda) f_{2}^{+}(x, \lambda), & t>x
\end{array} \quad \lambda \in S_{0}\right.  \tag{23}\\
R_{12}(x, t, \lambda) & =-\frac{1}{2 i \lambda D(\lambda)}\left\{\begin{array}{ll}
f_{1}^{+}(x, \lambda) f_{2}^{-}(t, \lambda), & t \leq x \\
f_{1}^{+}(t, \lambda) f_{2}^{-}(x, \lambda), & t>x
\end{array} \quad \lambda \in S_{1}\right.  \tag{24}\\
R_{13}(x, t, \lambda) & =\frac{1}{2 i \lambda B(\lambda)}\left\{\begin{array}{ll}
f_{1}^{-}(x, \lambda) f_{2}^{-}(t, \lambda), & t \leq x \\
f_{1}^{-}(t, \lambda) f_{2}^{-}(x, \lambda), & t>x
\end{array} \quad \lambda \in S_{2}\right.  \tag{25}\\
R_{14}(x, t, \lambda) & =\frac{1}{2 i \lambda A(\lambda)}\left\{\begin{array}{ll}
f_{1}^{-}(x, \lambda) f_{2}^{+}(t, \lambda), & t \leq x \\
f_{1}^{-}(t, \lambda) f_{2}^{+}(x, \lambda), & t>x
\end{array} \quad \lambda \in S_{3}\right. \tag{26}
\end{align*}
$$

with $S_{k}=\left\{\frac{k \pi}{2}<\arg \lambda<\frac{(k+1) \pi}{2}\right\}, k=\overline{0,3}$
By standard method (see [11, p. 302-304] it can be proved that the kernel $R(x, t, \lambda)$ of the operator $L^{-1}$ is bounded for $\lambda \notin\{\operatorname{Re} \lambda=0\} \cup\{\operatorname{Im} \lambda=0\}$ (i.e. for $\lambda \in \rho(L))$ and unbounded for $\lambda \in\{\operatorname{Re} \lambda=0\} \cup\{\operatorname{Im} \lambda=0\}$ (i.e. for $\left.\lambda \in \sigma_{c}(L)\right)$. This follows from the following lemma which was proved in [11, p. 302].

Lemma 1. For every $\tau>0$ the operators defined by the relations

$$
\begin{aligned}
& A f(x)=e^{-\tau x} \int_{-\infty}^{x} e^{\tau \xi} f(\xi) d \xi \\
& B f(x)=e^{\tau x} \int_{x}^{\infty} e^{-\tau \xi} f(\xi) d \xi
\end{aligned}
$$

are bounded in $L_{2}(-\infty, \infty)$ and

$$
\|A\| \leq \frac{1}{\tau},\|B\| \leq \frac{1}{\tau}
$$

Indeed, it suffices to take into account the following estimates:

$$
\left|f_{1}^{ \pm}(x, \lambda) f_{2}^{ \pm}(t, \lambda)\right| \leq C e^{-\tau|x-t|}
$$

where $C=C(\lambda), \tau=\min \{\operatorname{Im} \lambda, \operatorname{Re} \lambda\}, \forall x, t \in R$.
On the other hand, points $\lambda=0, \quad \lambda= \pm \frac{\alpha_{n}}{2}, \lambda= \pm \frac{i \alpha_{n}}{2}, n \in N$ can be simple pole points of $L^{-1}$. Since $L$ has no eigenvalue, there is no singularity at these points, too. So $\{\operatorname{Re} \lambda=0\} \cup\{\operatorname{Im} \lambda=0\}$ consists of continuous spectrum of $L$. Note that the kernel $R(x, t, \lambda)$ has simple poles at $\lambda=0, \quad \lambda= \pm \frac{\alpha_{n}}{2}, \quad \lambda= \pm \frac{i \alpha_{n}}{2}, n \in N$ on the spectrum which are called spectral singularities (in the sense of [11], p.306) of operator $L$.

The following theorem is true.
Theorem 4. The continuous spectrum of the operator $L$ consists of axes $\{R e \lambda=$ $0\} \cup\{\operatorname{Im} \lambda=0\}$ and it may have spectral singularities at the points $\pm \frac{\alpha_{n}}{2}, \pm \frac{i \alpha_{n}}{2} n \in$ $N$.

Theorem 5. The eigenvalues of the operator $L$ are finite and coincide with the squares of zeros of the functions $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$ from the sectors $S_{k}, k=0,1,2,3$ respectively.

Proof. For the solutions $f_{1}^{+}(0, \lambda), f_{2}^{+}(0, \lambda)$ we can obtain the asymptotic equalities

$$
\begin{aligned}
& f_{1}^{ \pm(j)}(0, \lambda)= \pm(i \lambda)^{j} C_{1}+o(1), \text { for }|\lambda| \rightarrow \infty, \quad j=0,1, \quad C_{1}>0 \\
& f_{2}^{ \pm(j)}(0, \lambda)= \pm(\lambda)^{j} C_{2}+o(1), \text { for }|\lambda| \rightarrow \infty, \quad j=0,1, \quad C_{2}>0
\end{aligned}
$$

For simplicity, we prove the first equality. Taking into account (6),(7) we have

$$
\begin{aligned}
& \left|f_{1}^{ \pm(j)}(0, \lambda)\right| \leq \pm(i \lambda)^{j}\left(1+\sum_{n=1}^{\infty}\left(i \alpha_{n}\right)^{j}\left|V_{n}^{ \pm}\right|+\right. \\
& +\sum_{n=1}^{\infty} \sum_{s=n}^{\infty} \frac{\left(i \alpha_{s}\right)^{j}\left|V_{n s}^{ \pm}\right|}{\left|\alpha_{s}+2 \lambda\right|} \leq \pm(i \lambda)^{j}\left(C_{1}+\frac{C_{2}}{|\operatorname{Im} \lambda|}\right)
\end{aligned}
$$

Therefore, as $|\lambda| \rightarrow \infty$, we obtain

$$
\left\{\begin{array}{cc}
f_{1}^{ \pm(j)}(0, \lambda)= \pm(i \lambda)^{j} C_{1}+o(1), & \text { for }|\lambda| \rightarrow \infty, \\
f_{2}^{ \pm(j)}(0, \lambda)= \pm(\lambda)^{j} C_{2}+o(1), & \text { for }|\lambda| \rightarrow \infty, \quad \\
& \quad C_{1}>0,1, \\
C_{2}>0
\end{array}\right.
$$

Then we get for the coefficients $A(\lambda), B(\lambda), C(\lambda), D(\lambda)$ the following asymptotic formulas:

$$
\begin{aligned}
& A(\lambda)=\frac{1-i}{2}+o(1) \lambda \in S_{4} \\
& B(\lambda)=\frac{1+i}{2}+o(1) \lambda \in S_{3} \\
& C(\lambda)=\frac{1+i}{2}+o(1) \lambda \in S_{1} \\
& D(\lambda)=\frac{1-i}{2}+o(1) \lambda \in S_{2}
\end{aligned}
$$

From this it follows that the zeros of the functions $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$ from the sectors $S_{k}, k=0,1,2,3$, respectively, are finite.

Note that, by generalizing the results of Theorems 2 and 5 , it is easy to see that the coefficients $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$ do not have zeros on the coordinate axes.

Then, dividing both sides of the first and third equality by $C(\lambda)$ and the second and fourth by $B(\lambda)$ we arrive at the solutions to the equation (5):

$$
\begin{array}{ll}
U_{1}^{+}(x, \lambda)=\frac{D(\lambda)}{C(\lambda)} f_{2}^{+}(x, \lambda)-f_{2}^{-}(x, \lambda), & \text { for } \operatorname{Re} \lambda=0 \\
U_{1}^{-}(x, \lambda)=\frac{A(\lambda)}{B(\lambda)} f_{2}^{-}(x, \lambda)-f_{2}^{+}(x, \lambda), & \text { for } \operatorname{Re} \lambda=0 \\
U_{2}^{+}(x, \lambda)=\frac{A(\lambda)}{C(\lambda)} f_{1}^{+}(x, \lambda)+f_{1}^{-}(x, \lambda), & \text { for } \operatorname{Im} \lambda=0 \\
U_{2}^{-}(x, \lambda)=\frac{D(\lambda)}{B(\lambda)} f_{1}^{-}(x, \lambda)+f_{1}^{+}(x, \lambda), & \text { for } \operatorname{Im} \lambda=0 .
\end{array}
$$

The functions $U_{1}^{-}(x, \lambda), U_{2}^{-}(x, \lambda)$ and $U_{1}^{+}(x, \lambda), U_{2}^{+}(x, \lambda)$ are called the eigenfunctions of the left and right spectral problems, respectively.

Definition 1. The functions

$$
S_{1}^{+}(\lambda)=\frac{D(\lambda)}{C(\lambda)}, S_{2}^{+}(\lambda)=\frac{A(\lambda)}{C(\lambda)}, S_{1}^{-}(\lambda)=\frac{A(\lambda)}{B(\lambda)}, S_{2}^{-}(\lambda)=\frac{D(\lambda)}{B(\lambda)}
$$

are be called the reflection coefficients for equation (5).

## 4. Formulation of the inverse problem

As follows from (23)-(26), the kernel $R(x, t, \lambda)$ for each $x$ and $t$ admits a meromorphic continuation from the sectors $S_{k}, k=\overline{0,3}$ and may have poles at the points $\pm \frac{\alpha_{n}}{2}, \pm i \frac{\alpha_{n}}{2}, \alpha_{n} \in G$, outside of $S_{k}, k=\overline{0,3}$.

These poles of the resolvent are called quasi-stationary states of the operator $L$. By analogy with the self-adjoint case, the functions $S_{k}^{ \pm}(\lambda), k=1,2$ and spectral singularities of the operator $L$ are called spectral data of the operator $L$.

Definition 2. The data $S_{k}^{ \pm}(\lambda), k=1,2$ and $\pm \frac{\alpha_{n}}{2}, \pm i \frac{\alpha_{n}}{2}, \alpha_{n} \in G$ are called the spectral data of $L$.

In our case, we will use a part of spectral data to construct the potentials $p(x), q(x)$.

### 4.1. Inverse problem.

Given the spectral data $S_{1}^{+}(\lambda)$ and $\pm i \frac{\alpha_{n}}{2}, \alpha_{n} \in G$, construct the potentials $p(x)$ and $q(x)$.

Let us first show that all numbers $V_{n n}^{ \pm}, n \in N$ can be defined by specifying a part of the spectral data $S_{1}^{+}(\lambda)$ and $\pm i \frac{\alpha_{n}}{2}, \alpha_{n} \in G$.

Indeed, taking into account (14), it is easy to see that

$$
\begin{aligned}
\lim _{\lambda \rightarrow i \frac{\alpha_{n}}{2}}\left(\alpha_{n}+2 i \lambda\right) \frac{D(\lambda)}{C(\lambda)} & =\lim _{\lambda \rightarrow i \frac{i n_{n}}{2}}\left(\alpha_{n}+2 i \lambda\right) \frac{W\left[f_{1}^{+}(x, \lambda), f_{2}^{-}(x, \lambda)\right]}{W\left[f_{1}^{+}(x, \lambda), f_{2}^{+}(x, \lambda)\right]}=V_{n n}^{-} \\
\lim _{\lambda \rightarrow-i \frac{\alpha n}{2}}\left(\alpha_{n}-2 i \lambda\right) \frac{C(\lambda)}{D(\lambda)} & =\lim _{\lambda \rightarrow-i \frac{\alpha n}{2}}\left(\alpha_{n}-2 i \lambda\right) \frac{W\left[f_{1}^{+}(x, \lambda), f_{2}^{+}(x, \lambda)\right]}{W\left[f_{1}^{+}(x, \lambda), f_{2}^{-}(x, \lambda)\right]}=V_{n n}^{+} .
\end{aligned}
$$

Lemma 2. For given numbers $V_{n n}^{ \pm}$, all numbers $V_{\alpha}^{ \pm}$and $V_{n \alpha}^{ \pm}, n<\alpha, n \in N$ are uniquely defined by the relation

$$
\begin{equation*}
V_{m \alpha+m}^{ \pm}=V_{m m}^{ \pm}\left(V_{\alpha}^{\mp}+\sum_{n=1}^{\alpha} \frac{V_{n \alpha}^{\mp}}{\alpha_{n}+\alpha_{m}}\right) . \tag{27}
\end{equation*}
$$

Proof.
Let

$$
V_{m, \alpha+m}^{ \pm}=S_{m}^{ \pm} V_{\alpha}^{\mp}+S_{m}^{ \pm} \sum_{n=1}^{\alpha} \frac{V_{n, \alpha}^{\mp}}{\alpha_{n}+\alpha_{m}} .
$$

We are going to show that all numbers $V_{n \alpha}^{ \pm}, n \in N, \alpha>n$ are uniquely determined by the given numbers $S_{n}^{ \pm}$. For this purpose, we consider the following
system of recurrent relations:

$$
\begin{gathered}
\bar{V}_{m, \alpha+m}^{ \pm}=S_{m}^{ \pm}+S_{m}^{ \pm} \sum_{n=1}^{\alpha} \frac{\bar{V}_{n, \alpha}^{\mp}}{\alpha_{n}+\alpha_{m}} \\
\overline{\bar{V}}_{m, \alpha+m}^{ \pm}= \pm i S_{m}^{ \pm}+S_{m}^{ \pm} \sum_{n=1}^{\alpha} \frac{\overline{\bar{V}}_{n, \alpha}^{\mp}}{\alpha_{n}+\alpha_{m}}
\end{gathered}
$$

where the numbers $S_{n}^{ \pm}$uniquely determine the numbers $\bar{V}_{n, \alpha}^{\mp}$ and $\overline{\bar{V}}_{n, \alpha}^{\mp}$. Let

$$
\begin{aligned}
& A_{m, \alpha+m}^{ \pm}=\frac{1}{2}\left[\bar{V}_{m, \alpha+m}^{ \pm} \mp i \overline{\bar{V}}_{m, \alpha+m}^{ \pm}\right]=S_{m}^{ \pm} \pm \frac{S_{m}^{ \pm}}{2}\left[\sum_{n=1}^{\alpha} \frac{\bar{V}_{n, \alpha}^{-}}{\alpha_{n}+\alpha_{m}}-i \sum_{n=1}^{\alpha} \frac{\overline{\bar{V}}_{n, \alpha}^{-}}{\alpha_{n}+\alpha_{m}}\right] \\
& B_{m, \alpha+m}^{ \pm}=\frac{1}{2}\left[\bar{V}_{m, \alpha+m}^{ \pm} \pm i \overline{\bar{V}}_{m, \alpha+m}^{ \pm}\right]=S_{m}^{ \pm} \pm \frac{S_{m}^{ \pm}}{2}\left[\sum_{n=1}^{\alpha} \frac{\bar{V}_{n, \alpha}^{-}}{\alpha_{n}+\alpha_{m}}+i \sum_{n=1}^{\alpha} \frac{\overline{\bar{V}}_{n, \alpha}^{-}}{\alpha_{n}+\alpha_{m}}\right]
\end{aligned}
$$

Then all numbers $V_{n, \alpha}^{ \pm}, \quad n \in N, \alpha>n$ are uniquely determined by the relation

$$
V_{m, \alpha+m}^{ \pm}=V_{\alpha}^{\mp} A_{m, \alpha+m}^{ \pm}+V_{\alpha}^{ \pm} B_{m, \alpha+m}^{\mp}
$$

Indeed,

$$
\begin{array}{r}
V_{n, \alpha}^{+}=V_{\alpha}^{-}\left[\frac{\bar{V}_{n, \alpha}^{+}-i \overline{\bar{V}}_{n, \alpha}^{+}}{2}\right]+V_{\alpha}^{+}\left[\frac{\bar{V}_{n, \alpha}^{+}+i \overline{\bar{V}}_{n, \alpha}^{+}}{2}\right]= \\
=V_{\alpha}^{-} S_{m}^{+}+\frac{S_{m}^{+}}{2}\left[\sum_{n=1}^{\alpha} \frac{V_{\alpha}^{ \pm} \bar{V}_{n, \alpha}^{-}}{\alpha_{n}+\alpha_{m}}-i \sum_{n=1}^{\alpha} \frac{V_{\alpha}^{ \pm} \overline{\bar{V}}_{n, \alpha}^{-}}{\alpha_{n}+\alpha_{m}}\right]+ \\
+\frac{S_{m}^{+}}{2}\left[\sum_{n=1}^{\alpha} \frac{V_{\alpha}^{ \pm} \bar{V}_{n, \alpha}^{-}}{\alpha_{n}+\alpha_{m}}+i \sum_{n=1}^{\alpha} \frac{V_{\alpha}^{ \pm} \overline{\bar{V}}_{n, \alpha}^{-}}{\alpha_{n}+\alpha_{m}}\right]= \\
=V_{\alpha}^{-} S_{m}^{+}+S_{m}^{+} \sum_{n=1}^{\alpha} \frac{V_{\alpha}^{-} B_{n, \alpha}^{+}+V_{\alpha}^{+} A_{n, \alpha}^{-}}{\alpha_{n}+\alpha_{m}}= \\
=V_{\alpha}^{-} S_{m}^{+}+S_{m}^{+} \sum_{n=1}^{\alpha} \frac{V_{n \alpha}^{-}}{\alpha_{n}+\alpha_{m}}
\end{array}
$$

Taking into account the relation

$$
\alpha_{s} V_{s}^{ \pm} \pm \sum_{\alpha_{r}+\alpha_{k}=\alpha_{s}} p_{r} V_{k}^{ \pm} \pm p_{s}=0
$$

we have

$$
\begin{aligned}
& \sum_{s=1}^{\infty} \alpha_{s} V_{s}^{+}+\sum_{s=1}^{\infty} \sum_{\alpha_{r}+\alpha_{k}=\alpha_{s}} V_{k}^{+} p_{k} \pm \sum_{s=1}^{\infty} p_{s}= \\
= & \sum_{s=1}^{\infty} \alpha_{s} V_{s}^{+}+\left(\sum_{s=1}^{\infty} V_{s}^{+}\right)\left(\sum_{s=1}^{\infty} p_{s}\right)+\sum_{s=1}^{\infty} p_{s}=0,
\end{aligned}
$$

whence

$$
\sum_{s=1}^{\infty} p_{s}=-\frac{\sum_{s=1}^{\infty} \alpha_{s} V_{s}^{+}}{1+\sum_{s=1}^{\infty} V_{s}^{+}}
$$

Then

$$
\sum_{s=1}^{\infty} \alpha_{s} V_{s}^{-}-\left(\sum_{s=1}^{\infty} V_{s}^{-}\right)\left(\frac{\sum_{s=1}^{\infty} \alpha_{s} V_{\alpha}^{+}}{1+\sum_{s=1}^{\infty} V_{s}^{+}}\right)-\frac{\sum_{s=1}^{\infty} \alpha_{s} V_{\alpha}^{+}}{1+\sum_{s=1}^{\infty} V_{s}^{+}}=0
$$

Hence,

$$
\left(1+\sum_{s=1}^{\infty} V_{s}^{+}\right) \sum_{s=1}^{\infty} \alpha_{s} V_{s}^{-}-\left(\sum_{s=1}^{\infty} V_{s}^{-}\right)\left(\sum_{s=1}^{\infty} \alpha_{s} V_{s}^{+}\right)-\sum_{s=1}^{\infty} \alpha_{s} V_{s}^{+}=0
$$

or

$$
\sum_{s=1}^{\infty} \alpha_{s} V_{s}^{-}+\sum_{s=1}^{\infty} \sum_{\alpha_{r}+\alpha_{k}=\alpha_{s}} V_{r}^{+} \alpha_{k} V_{k}^{-}-\sum_{s=1}^{\infty} \sum_{\alpha_{r}+\alpha_{k}=\alpha_{s}} V_{r}^{-} \alpha_{k} V_{k}^{+}-\sum_{s=1}^{\infty} \alpha_{s} V_{s}^{+}=0
$$

We obtain

$$
V_{\alpha}^{-}-V_{\alpha}^{+}+\sum_{s=1}^{\alpha-1} V_{s}^{+} V_{\alpha-s}^{-}=0
$$

and using the equality $f_{+}(0,0)=f_{-}(0,0)$ we get

$$
V_{\alpha}^{+}+\sum_{n=1}^{\alpha} \frac{V_{n, \alpha}^{+}}{\alpha_{n}}=V_{\alpha}^{-}+\sum_{n=1}^{\alpha} \frac{V_{n, \alpha}^{-}}{\alpha_{n}}
$$

Taking into account

$$
V_{m, \alpha+m}^{ \pm}=V_{\alpha}^{\mp} A_{m, \alpha+m}^{ \pm}+V_{\alpha}^{ \pm} B_{m, \alpha+m}^{\mp}
$$

it is easily seen that

$$
V_{\alpha}^{+}+\sum_{n=1}^{\alpha} \frac{V_{n, \alpha}^{+}}{\alpha_{n}}=V_{\alpha}^{-}+\sum_{n=1}^{\alpha} \frac{V_{n, \alpha}^{-}}{\alpha_{n}} .
$$

Then

$$
\begin{aligned}
\sum_{n=1}^{\alpha} \frac{V_{n, \alpha}^{+}-V_{n, \alpha}^{-}}{\alpha_{n}} & =\sum_{n=1}^{\alpha} \frac{V_{\alpha}^{-} A_{m, \alpha+m}^{+}+V_{\alpha}^{+} B_{m, \alpha+m}^{-}-V_{\alpha}^{+} A_{m, \alpha+m}^{-}-V_{\alpha}^{-} B_{m, \alpha+m}^{+}}{\alpha_{n}}= \\
& =V_{\alpha}^{-} \sum_{n=1}^{\alpha} \frac{A_{n, \alpha}^{+}-B_{n, \alpha}^{+}}{\alpha_{n}}+V_{\alpha}^{+} \sum_{n=1}^{\alpha} \frac{B_{n, \alpha}^{-}-A_{n, \alpha}^{-}}{\alpha_{n}}
\end{aligned}
$$

or

$$
V_{\alpha}^{+}=V_{\alpha}^{-} \frac{\left(1-\sum_{n=1}^{\alpha} \frac{A_{n, \alpha}^{+}-B_{n, \alpha}^{+}}{\alpha_{n}}\right)}{\left(1-\sum_{n=1}^{\alpha} \frac{A_{n, \alpha}^{-}-B_{n, \alpha}^{-}}{\alpha_{n}}\right)}=V_{\alpha}^{-} \Psi_{\alpha} .
$$

The relation

$$
V_{\alpha}^{-}-V_{\alpha}^{+}+\sum_{s=1}^{\alpha-1} V_{s}^{+} V_{\alpha-s}^{-}=0
$$

implies that

$$
V_{\alpha}^{-}-V_{\alpha}^{-} \Psi_{\alpha}+\sum_{s=1}^{\alpha-1} V_{s}^{-} \Psi_{s} V_{\alpha-s}^{-}=0
$$

and all numbers $V_{n}^{-}, n \in N$ are uniquely determined by the numbers $\Psi_{n}$. It means that all numbers $V_{n \alpha}^{ \pm}, V_{n}^{ \pm} n \in N, \alpha>n$, can be determined by the normalizing numbers $S_{n}^{ \pm}$.

From recurrent formulas (8)-(10), we find all numbers $p_{n}$ and $q_{n}$. So inverse problem has a unique solution and the numbers $p_{n}$ and $q_{n}$ are defined constructively by a part of the spectral data.

Theorem 6. The specification of the spectral data uniquely determines potentials $p(x), q(x)$.

## References

[1] J.A. Asadzadeh, A.N. Jabrailova, On Stability of Bases From Perturbed Exponential Systems in Orlicz Spaces, Azerbaijan Journal of Mathematics, 11(2), 2021, 196-213
[2] M.I. Belishev, Inverse spectral indefinite problem for the equation $y^{\prime \prime}+$ $\lambda p(x) y=0$ on an interval, Functional Analysis and Its Applications, 21(2), 1987, 146-148.
[3] V.A. Chulaevskii, On perturbations of a Schrödinger operator with periodic potential, Russian Mathematical Surveys, 36(5), 1981, 143.
[4] U. Demirbilek, Kh.R. Mamedov, On The Expansion Formula For A Singular Sturm-Liouville Operator, Journal of Science and Arts, 1(54), 2021, 67-76.
[5] R.F. Efendiev, Annaghili Floquet solutions of the operator pencil with almost periodic potentials, Proceeding of the 7 -th international conference on control and optimization with industrial applications, 1, 2020, 161-163.
[6] R.F. Efendiev, Spectral analysis for one class of second-order indefinite non-self-adjoint differential operator pencil, Applicable Analysis, 90(12), 2011, 1837-1849.
[7] R.F. Efendiev, Spectral analysis of a class of non-self-adjoint differential operator pencils with a generalized function, Teoret. Mat. Fiz. 145(1), 2005, 102-107 (in Russian) [Theor. Math. Phys. 1451(1), 1457-461 (in English)].
[8] R.F. Efendiev, Y.S. Gasimov, Inverse spectral problem for PT-symmetric Schrodinger operator on the graph with loop, Global and Stochastic Analysis, 9(2), 2022, 67-77.
[9] R.F. Efendiev, H.D. Orudzhev, S.J. Bahlulzade, Spectral Analysis of the discontinuous Sturm-Liouville operator with almost periodic potentials, Advanced Math, Models and Appl., 6(3), 2021, 266-277
[10] R.F. Efendiev, H.D. Orudzhev, Z.F. El-Raheem, Spectral analysis of wave propagation on branching strings, Boundary Value Problems, 2016(1), 2016, 1-18.
[11] M.A. Naimark, Linear differential operators, Part II: Linear differential operators in Hilbert space, Frederick Ungar Publishing Company, 1968.
[12] M.G. Gasymov, A.D. Orudzhev, Spectral properties of a class of differential operators with almost-periodic coefficients and their perturbations, Doklady Akademii Nauk, 287(4), 1986, 777-781.
[13] I. Krichever, S.P. Novikov, Periodic and almost-periodic potentials in inverse problems, Inverse Problems, 15(6), 1999, R117.
[14] S.A. Molchanov, V.A. Chulaevskii, Structure of the spectrum of a lacunary-limit-periodic Schrödinger operator, Funktsional'nyi analiz i ego prilozheniya, 18(4), 1984, 90-91.
[15] A.D. Orujov, On the spectrum of the bundle of second-order differential operators with almost periodic coefficients, Inter. J. of Pure and Appl. Math., 26(2), 2006, 195-204.
[16] A.D. Orujov An inverse problem for the quadratic pencil of differential operators with almost periodic coefficients, Proc. of the IMM, National Academy of Sciences of Azerbaijan, 48(1), 2022, 3-19.
[17] P. Sarnak, Spectral behaviour of quasi-periodic potentials, Communications in Mathematical Physics, 84(3), 1982, 377-401.
[18] B. Simon, Almost periodic Schrödinger operators: a review, Advances in Applied Mathematics, 3(4), 1982 463-490.
[19] M.N. Simbirskii, Inverse problem for Sturm-Liouville operator with almostperiodic potential having positive Fourier exponents only positive Fourier exponents, Advances in Soviet Mathematics, 11, 1992, 21-38.

Rakib F. Efendiev
Baku Engineering University, Department of Mathematics Teaching Khirdalan city, Baku, Absheron, Azerbaijan
E-mail: refendiyev@beu.edu.az

Shams Annaghili
Baku Engineering University, Department of Mathematics Teaching Khirdalan city, Baku, Absheron, Azerbaijan
E-mail: saannagili@beu.edu.az
Received 18 May 2022
Accepted 15 July 2022

