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The Solution of a Mixed Problem for a Parabolic Type Equation with General Form Coefficients Under Unconventional Boundary Conditions

H.I. Ahmadov

Abstract. We study one-dimensional mixed problem for a parabolic type equation with time-advance constant coefficients in the boundary conditions. Under minimum conditions on the initial data we prove the existence and unigueness of the considered mixed problem and obtain explicit analytical representation for the solution.

Key Words and Phrases: mixed problem, time advance, residue method.

2010 Mathematics Subject Classifications: 35R30, 35L70, 65M06

1. Introduction

In this paper we consider a mixed problem for a parabolic type equation with constant coefficients that have time-advance in the boundary conditions.

In [1], mixed problems have been considered for one-dimensional heat-conductivity equation with partially determined boundary regime under certain conditions on the initial data. The existence and uniqueness of the solution of the given problem represented in the form of a contour integral have been proved.

The works [2,3] consider mixed problems for a heat-conductivity equation that has a more general form time-advance in the boundary conditions. A unique solvability of the considered problems is proved and the solutions are represented in the form of a contour integral. In [4], a problem for a parabolic-hyperbolic equation with heat conductivity operators and strings in a rectangular domain, with Samarskiy-Ionkin nonlocal boundary condition, has been studied. The criterion of uniqueness of solutions has been proved by the spectral expansions method. The paper [5] studies a boundary value problem for a mixed type equation with Lavrentyev-Bitsadze operator in the principal part, with lead-lag arguments and closed-change line. A uniqueness theorem has been proved under the restriction on deviation value of arguments, and explicit integral representations for the solutions have been found.

Unlike all above-mentioned papers, in the present work we consider a more general mixed problem for an equation with constant coefficients that have a time deviation in the boundary conditions.

2. Problem statement

Let

$$L\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) u\left(x, t\right) = au_{xx}\left(x,\ t\right) + bu_x\left(x,\ t\right) + cu\left(x,\ t\right) - u_t(x,\ t)$$

$$l_j u\left(x, t\right) = u\left(x,\ t + \left(1 - j\right) \omega\right) + \alpha_j u\left(1 - x,\ t + j\omega\right), \quad j = 0,\ 1$$

$$l_j u\left(x, t\right) = a_{j-2} u_x^{(j-2)}(x,\ t) + b_{j-2} u_x^{(j-2)}\left(1 - x,\ t\right), \quad j = 2,\ 3,$$

where a, b, c, ω, α_j, a_j, b_j $(j = 0,\ 1)$ are real constants, $a > 0$, $\omega > 0$, $\alpha_0 \alpha_1 \neq 0$.

On the semistrip $\Pi = \{(x,\ t):\ \ 0 < x < 1,\ \ t > 0\}$, we consider the following mixed problem:

$$L\ u\left(x,\ t\right) = 0, \qquad (x,\ t) \in \Pi, \tag{1}$$

$$u\left(x, 0\right) = \varphi\left(x\right), \qquad 0 < x < 1, \tag{2}$$

$$l_j u|_{x=0} = 0, \qquad t > 0, \quad j = 0,\ 1, \tag{3}$$

$$l_j u|_{x=0} = 0, \qquad 0 < t \leq \omega, \quad j = 2,\ 3, \tag{4}$$

where $\varphi(x)$ is a given, and $u(x, t)$ is a sought function.

The solution of the problem (1)-(4) is a function $u(x,\ t)$, satisfying the following conditions:

1) $u(x, t) \in C^{2,1}(\Pi) \cap C(0 < x < 1,\ \ t \geq 0)$; $\int\limits_0^t u(x,\ \tau) d\tau \in C\left(0 \leq x \leq 1,\ \ t \geq 0\right)$;
2) $l_j u\left(x,\ t\right) \in C\left(0 \leq x < 1,\ \ \ t > 0\right), \quad j = 0,\ 1$;
3) $l_j u\left(x,\ t\right) \in C\left(0 \leq x < 1,\ \ 0 < t \leq \omega\right), \quad j = 2,\ 3$;
4) $u(x,\ t)$ satisfies the equalities (1)-(4) in the usual sense.

As noted in [2], the stated problem can be solved step by step. First, by solving in $\{(x,\ t):\ 0 < x < 1,\ 0 < t < \omega\}$ the ordinary mixed problem (1), (2), (4), we can find the initial, and then, using conditions (3), the boundary state for the rectangle $\{(x,\, t) : 0 < x < 1,\ \omega < t < 2\omega\}$, etc. But such an approach is very cumbersome and each time requires to clarify conditions on initial and boundary data $u\,(x,\, k\omega)\,,\, u\,(0,\, t)\,,\, u(1,\, t)$, providing unique solbability in $(0,\, 1) \times (k\omega,\, (k+1)\,\omega)$ $(k = 1,\, 2, \dots)$, and also preservation of smoothness when going from one layer to another.

Not dividing the problem (1)-(4) into numerous problems, by combining the contour integral method and Rasulov residue method [6, 7], the existence and uniqueness of the solution were proved, and analytic representation was obtained for it.

3. Uniqueness of the solution

We call the problem

$$L\left(\frac{d}{dx},\ \mu^2\right) y\,(x,\, \mu) = 0, \quad l_j y|_{x=0} = 0, \ \ j = 2,\, 3 \tag{5}$$

the first spectral problem with a complex parameter μ, corresponding to the problem (1), (2), (4).

It is known that [8] if $a_0 b_1 + a_1 b_0 \neq 0$, then for all complex values of μ, not belonging to the set $S = \{\mu_\nu :\ \ \nu = 1,\, 2, ...\}$, there exists the Green function $G_1(x, \xi, \mu)$ of the problem (5), analytic everywhere with respect to μ except for the points of the set S, that are its poles and have the asymptotic representation

$$\mu_\nu = \sqrt{a}\ \pi\nu i + \frac{(-1)^\nu (a_0 a_1 + b_0 b_1)}{2(a_0 b_1 + a_1 b_0)} + O\left(\frac{1}{\nu}\right), \ \ \nu \to \infty.$$

Renumbering the points in S in an ascending order of their modules taking into account their multiplicity, we denote $S = \{\mu_\nu,\ \ \nu = 1,\, 2, \dots\}$, with $|\mu_1| \leq |\mu_2| \leq \dots$, μ_ν has the multiplicity χ_ν, with $\chi_\nu = 1$ or $\chi_\nu = 2$. It is clear that $|\mu_\nu| \to \infty$ $(\nu \to \infty)$. There exist $h,\, \delta > 0$ such that

$$-h < \operatorname{Re}\, \mu_\nu < h, \quad |\mu_{\nu+1} - \mu_\nu| > 2\delta \ \ (\nu = 1,\, 2, \dots)\,. \tag{6}$$

Outside the δ neighnbourhood of the points μ_ν, the following estimations are valid:

$$\left|\frac{\partial^k G_1(x,\, \xi,\, \mu)}{\partial x^k}\right| \leq c|\mu|^{k-1}, \ \ c > 0, \ \ k = 0, 1, 2. \tag{7}$$

For any function $f(x)$ from the domain of the operator of the first spectral problem, we have

$$f(x) = -\int_0^1 G_1(x,\, \xi,\, \mu)\,\varphi(\xi)\, d\xi =$$

$$= -\int_0^1 G_1(x,\, \xi,\, \mu)(af''(\xi) + bf'(\xi) + cf(\xi) - \mu^2 f(\xi))d\xi.$$

Hence we obtain

$$\int_0^1 G_1(x,\, \xi,\, \mu)\, f(\xi)\, d\xi = \frac{f(x)}{\mu^2} + \frac{c}{\mu^2}\int_0^1 G_1(x, \xi, \mu)\, f(\xi)\, d\xi +$$

$$+\frac{1}{\mu^2}\int_0^1 G_1(x,\, \xi,\, \mu)\,(af''(\xi) + bf'(\xi))d\xi. \tag{8}$$

Let us accept some notations that will be used later: let $c > 0$, $r > 0$ be some numbers, z be a complex variable, $\Im_c = \{z : \mathrm{Re} z^2 = c\}$ be a hyperbola with the branches $\Im_c^{\pm} = \{z : \mathrm{Re} z^2 = c,\ \pm \mathrm{Re} z > 0\}$, $\Omega_z = \{z :\ |z| = r\}$, $\Omega_r(\theta_1, \theta_2)$ be an arch of the circle Ω_r, enclosed between the rays $z = \sigma\, e^{i\theta j}$ $(0 \le \sigma < \infty,\ i = \sqrt{-1},\ j = 1, 2)$. Note that the arches connecting the branches and the sides of the hyperbola $\Im_c$ $\{z : |z| = r,\ Rez^2 \ge c,\ Rez > 0\}$, $\{z :\ |z| = r,$ $Rez^2 \le c,\ Imz > 0\}$, $\{z : |z| = r,\ Rez^2 \ge c,\ Rez < 0\}$ and $\{z : |z| = r,\ Rez^2 \le c,\ Imz < 0\}$, in our notations will be

$$\Omega_r(-\theta_{c,r},\ \theta_{c,r}),\ \Omega_r(\theta_{c,r},\ -\theta_{c,r} + \pi),\ \Omega_r(-\theta_{c,r} + \pi,\ \theta_{c,r} + \pi),$$

$$\Omega_r(\theta_{c,r} + \pi,\ -\theta_{c,r} + 2\pi),$$

where $\theta_{c,r} = \mathrm{arctg}\sqrt{\frac{r^2-c}{r^2+c}}$.

We introduce the contours (broken)

$$\hat{\Im}_c = \hat{\Im}_c^{+} \cup \hat{\Im}_c^{-},\ \ \hat{\Im}_c^{\pm} = \left\{z :\ \pm z = \sigma\ e^{-\frac{3\pi}{8}i}, \sigma \in \left(2c\sqrt{1+\sqrt{2}},\ \infty\right] \right\} \cup$$

$$\left\{z :\ \pm z = c(1 + i\eta), \eta \in \left[-1 - \sqrt{2},\ 1 + \sqrt{2}\right] \right\} \cup$$

$$\cup \left\{z :\ \pm z = \sigma\, e^{-\frac{3\pi}{8}i}, \sigma \in \left[2c\sqrt{1+\sqrt{2}},\ \infty\right]\right\}.$$

We denote a part of contours $\Im_c$, $\hat{\Im}_c^{\pm}$, $\hat{\Im}_c$, $\hat{\Im}_c^{\pm}$, enclosed inside the circle Ω_r, by $\Im_{c,r}$, $\Im_{c,r}^{\pm}$, $\hat{\Im}_{c,r}$, $\hat{\Im}_{c,r}^{\pm}$. Finally, by $\Gamma_{c,r}$, $\Gamma_{c,r}^{+}$, $\widehat{\Gamma}_{c,r}$, $\widehat{\Gamma}_{c,r}^{+}$ for $r \geq 2c\sqrt{1+\frac{\sqrt{2}}{2}}$ we denote the closed contours

$$\Gamma_{c,r} = \Omega_r\left(\theta_{c,r}+\pi,\ -\theta_{c,r}+2\pi\right) \cup \Im_{c,r}^{+} \cup \Omega_r\left(\theta_{c,r},\ -\theta_{c,r}+\pi\ \right) \cup \Im_{c,r}^{-},$$

$$\Gamma_{c,r}^{+} = \Im_{c,r}^{+} \cup \Omega_r\left(-\theta_{c,r},\ \theta_{c,r}\right), \quad \widehat{\Gamma}_{c,r}^{+} = \hat{\Im}_{c,r}^{+} \cup \Omega_r\left(-\frac{3\pi}{8}, \frac{3\pi}{8}\right),$$

$$\widehat{\Gamma}_{c,r} = \Omega_r\left(-\frac{5\pi}{8}, -\frac{3\pi}{8}\right) \cup \hat{\Im}_{c,r}^{+} \cup \Omega\left(\frac{3\pi}{8}, \frac{5\pi}{8}\right) \cup \hat{\Im}_{c,r}^{-}.$$

In the sequel, we will consider the counter-clockwise direction as a positive direction.

Let $\{r_n\}$ be a sequence of numbers such that

$$0 < r_1 < r_2 < \cdots < r_n < \ldots, \quad \lim_{n\to\infty} r_n \ = \infty,$$

The circles Ω_{r_n} do not intersect the δ-vicinity (δ is a rather small, fixed number) of the points $\mu_\nu \in S$. In view of the structure of S, the existence of such a number δ and such a sequence $\{r_n\}$ in undeniable. We denote the number of the points μ_ν lying inside to $\widehat{\Gamma}_{h,r_n}$ by m_n. It is seen from (8) that for any function $f(x)$ from the domain of the operator of the first spectral problem, i.e. $f(x) \in c^2\,[0,\ 1]$, $l_j f|_{x=0} = 0$ $(j = 2,\ 3)$, we have the following relation:

$$\frac{1}{2\pi i}\int\limits_{\widehat{\Gamma}_{h,r_n}} \mu d\mu \int\limits_0^1 G_1\left(x,\xi,\mu\right) f\left(\xi\right) d\xi = \frac{1}{2\pi i}\int\limits_{\widehat{\Gamma}_{h,r_n}} \mu^{-1} f(x) d\mu +$$

$$\frac{c}{2\pi i}\int\limits_{\widehat{\Gamma}_{h,r_n}} \mu^{-1} d\mu \int\limits_0^1 G_1\left(x,\xi,\mu\right) f\left(\xi\right) d\xi +$$

$$+\frac{1}{2\pi i}\int\limits_{\widehat{\Gamma}_{h,r_n}} \mu^{-1} d\mu \int\limits_0^1 G_1\left(x,\xi,\mu\right) \left(af''\left(\xi\right) + bf'(\xi)\right) d\xi,$$

$$\lim_{n\to\infty}\frac{1}{2\pi i}\int\limits_{\widehat{\Gamma}_{h,r_n}} \mu^{-1} f(x) d\mu \ =$$

$$= \frac{f(x)}{2\pi i} \lim_{n\to\infty} \int\limits_{\Omega_{r_n}} \frac{d\mu}{\mu} = \frac{f(x)}{2\pi i} \lim_{n\to\infty} \int\limits_0^{2\pi} \frac{ir_n e^{i\varphi} d\varphi}{r_n e^{i\varphi}} = \frac{f(x)}{2\pi} \cdot 2\pi = f\,(x)\,.$$

$$\lim_{n\to\infty} \int\limits_{\widehat{\Gamma}_{h,r_n}} \mu^{-1} d\mu \int\limits_0^1 G_1(x,\,\xi,\,\mu) f(\xi) d\xi = \lim_{n\to\infty} \int\limits_{\Omega_{r_n}} \mu^{-1} d\mu \int\limits_0^1 G_1(x,\,\xi,\,\mu) f(\xi) d\xi,$$

$$\left| \int\limits_{\Omega_n} \mu^{-1} d\mu \int\limits_0^1 G_1(x,\,\xi,\,\mu) f(\xi) d\xi \right| \le \int\limits_0^{2\pi} \left| \frac{d\mu}{\mu} \right| \cdot \frac{c}{|\mu|} \cdot c_0 \le$$

$$c_1 \int\limits_0^{2\pi} \left| \frac{d\mu}{|\mu|^2} \right| = c_1 \int\limits_0^{2\pi} \left| \frac{ir_n e^{i\varphi} d\varphi}{r_n^2 e^{2i\varphi}} \right| = c_1 \int\limits_0^{2\pi} \frac{1}{r_n} d\varphi \to 0 \quad (n \to \infty)$$

consequently.

In the similar way we prove

$$\lim_{n\to\infty} \int\limits_{\widehat{\Gamma}_{h,r_n}} \mu^{-1} d\mu \int\limits_0^1 G_1\,(x,\,\xi,\,\lambda)\,(af''\,(\xi) + bf'(\xi)) d\xi = 0,$$

consequently,

$$f\,(x) = \lim_{n\to\infty} \frac{1}{2\pi i} \int\limits_{\widehat{\Gamma}_{h,r_n}} \mu d\mu \int\limits_0^1 G_1\,(x,\xi,\lambda)\, d\xi = \sum_{\nu=1}^{\infty} \operatorname*{res}_{\mu\nu} \mu \int\limits_0^1 G_1\,(x,\xi,\,\mu)\, f(\xi) d\xi \tag{9}$$

We have the following theorem.

Theorem 1. *Let* $a_0 b_1 + a_1 b_0 \ne 0$, $\varphi(x) \in C^2\,[0,1]$ *and* $l_j \varphi|_{x=0} = 0$ $(j = 2,\ 3)$. *The problem (1)-(4) may have at most one solution.*

Proof. We introduce the operators

$$A_{\nu s} f\,(x) = \operatorname*{res}_{\mu\nu} \mu^{2S+1} \int\limits_0^1 G_1\,(x,\xi,\,\mu)\, f(\xi) d\xi = f_{\nu s}\,(x)\,, \tag{10}$$

mapping every function $f(x) \in C[0,\ 1]$ to $f_{\nu S}(x) \in C^2[0,\ 1]$, $l_j f_{\nu s}|_{x=0} = 0$ $(j = 2,\ 3)$. It is seen from (9) that if $f(x) \in C^2[0,\ 1]$ and $l_j f|_{x=0} = 0$ $(j = 2,\ 3)$, then

$$\sum_{\nu=1}^{\infty} f_{\nu 0}(x) = f(x). \tag{11}$$

Obviously, if the problem (1)-(4) has some solution $u(x,t)$, then this function is also the solution of the problem (1), (2), (4) in the domain $\{(x,t) : 0 < x < 1;\ 0 < t \le \omega\}$. Applying the operators $A_{\nu s}$ to (1), (2) we obtain

$$\begin{aligned} \frac{\partial u_{\nu s}(x,t)}{\partial t} &= \operatorname*{res}_{\mu\nu} \mu^{2s+1} \int_0^1 G_1(x,\xi,\ \mu)\,\mu^2 u(\xi,\ t)d\xi = \\ &= \operatorname*{res}_{\mu\nu} \mu^{2(S+1)+1} \int_0^1 G_1(x,\ \xi,\ \mu)\,u(\xi,\ t)d\xi = u_{\nu s+1}(x,\ t), \end{aligned} \tag{12}$$

$$u_{\nu s}(x,0) = \varphi_{\nu s}(x). \tag{13}$$

If μ_ν is a simple pole $(\chi_\nu = 1)$ of the function $G_1(x,\xi,\mu)$, then

$$\operatorname*{res}_{\mu\nu} \mu^{2s+1} \left(\mu^2 - \mu_\nu^2\right) \int_0^1 G_1(x,\xi,\mu)\,\mu^2 u(\xi,t)d\xi = 0,$$

i.e.

$$u_{\nu s+1}(x,t) = \mu_\nu^2 u_{\nu s}(x,t). \tag{14}$$

But if $\chi_2 = 2$, then

$$\operatorname*{res}_{\mu\nu} \mu^{2s+1} \left(\mu^2 - \mu_\nu^2\right)^2 \int_0^1 G_1(x,\xi,\ \mu)\,u(\xi,\ t)d\xi = 0.$$

Consequently,

$$u_{\nu s+2}(x,t) - 2\mu_\nu^2 u_{\nu s+1}(x,t) + \mu_\nu u_{\nu s}(x,t) = 0. \tag{15}$$

For $\chi_\nu = 1$ from (13), (14) we obtain

$$\begin{aligned} &\frac{\partial u_{\nu 0}(x,t)}{\partial t} = \mu_\nu^2 u_{\nu 0}(x,t) \\ &u_{\nu 0}(x,0) = \varphi_{\nu 0}(x). \end{aligned} \tag{16}$$

But if $\chi_\nu = 2$, assuming in (12) $s = 0$ and $s = 1$, while in (15) $s = 0$, we have

$$\begin{aligned} &\tfrac{\partial u_{\nu 0}(x,t)}{\partial t} = u_{\nu 1}(x,t)\,, \quad \tfrac{\partial u_{\nu 1}(x,t)}{\partial t} = -\mu^4 u_{\nu 0}(x,t) + 2\mu_\nu^2 u_{\nu 1}(x,t)\,, \\ &u_{\nu 0}(x,0) = \varphi_{\nu 0}(x), \quad u_{\nu 1}(x,0) = \varphi_{\nu 1}(x). \end{aligned} \tag{17}$$

Obviously, the problems (16) and (17) have a unique solution

$$u_{\nu s}(x,\, t) = \operatorname*{res}_{\mu\nu} \mu^{2s+1}\ e^{\mu^2 t} \int\limits_0^1 G_1(x,\xi,\mu)\,\varphi(\xi)d\xi,$$

where $s = 0$ for the problem (16) and $s = 0,\, 1$ for the problem (17).

The allowing for (11), we find

$$u(x,t) = \sum_{\nu=1}^{\infty} \operatorname{res}\mu\ e^{\mu^2 t} \int\limits_0^1 G_1(x,\xi,\mu)\,\varphi(\xi)d\xi \tag{18}$$

for $0 \le x \le 1$, $0 \le t \le \omega$. This implies the validity of the theorem statement. Indeed, if the problem (1)-(4) had two solutions $u_1(x,t)$, $u_2(x,t)$, then their difference would be the solution of the problem (1)-(4) $\upsilon(x,t) = u_1(x,t) - u_2(x,t)$ with $\varphi(x) \equiv 0$, and $\upsilon(0,t) \equiv \upsilon(1,t) = 0$, for $t \ge 0$. In connection with this and condition 1), it is easy to see that the function

$$w(x,t) = \int\limits_0^t \upsilon(x,\tau)\,d\tau$$

is the solution of the homogeneous problem $w_t = a^2 w_{xx}$ $(0 < x < 1,\ t \ge \omega)$, $w(x,\omega) = 0$ $(0 \le x \le 1)$, $w(0,t) = w(1,t) = 0$ $(t > \omega)$, continuous in $\{0 \le x \le 1,\ t \ge \omega\}$, whence, by the maximum principle [9, 10] we conclude that $w(x,t) \equiv 0$, $(0 \le x \le 1,\ \ t \ge \omega)$, consequently, $\upsilon(x,t) = 0$ $(0 \le x \le 1,\ t \ge 0)$.

Under the conditions of Theorem 1 and allowing for the equality (8), we can reduce the formula (18) to the following form:

$$\begin{aligned} &u(x,t) = \lim_{n\to\infty} \tfrac{1}{2\pi i} \int\limits_{\widehat{\Gamma}_{h,r_n}} \mu e^{\mu^2 t} d\mu \int\limits_0^1 G_1(x,\xi,\mu)\,\varphi(\xi)\,d\xi = \\ &\varphi(x) + \lim_{n\to\infty} \tfrac{1}{2\pi i} \int\limits_{\widehat{\Gamma}_{h,r_n}} \mu^{-1} e^{\mu^2 t} d\mu \int\limits_0^1 G_1(x,\xi,\mu)\,[a\varphi''(\xi) + b\varphi'(\xi) + c\varphi(\xi)]\,d\xi. \end{aligned} \tag{19}$$

It is known that $\widehat{\Gamma}_{h,r_n} = \Omega_{r_n}\left(-\frac{5\pi}{8}, -\frac{3\pi}{8}\right) \cup \hat{\Im}^{+}_{h,r_n} \cup \Omega_{r_n}\left(\frac{3\pi}{8}, \frac{5\pi}{8}\right) \cup \hat{\Im}^{-}_{h,r_n}$.

On the archs $\Omega_{r_n}\left(-\frac{5\pi}{8} + j\pi, -\frac{3\pi}{8} + j\pi\right)$ $(j = 0,\ 1)$

$$\mathrm{Re}\mu^2 t = |\mu|^2 t\cos\left(-\frac{3\pi}{4}\right) \le -\frac{|\mu|^2 t\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}|\mu|^2 t.$$

So,

$$\lim_{n\to\infty} \int\limits_{\Omega_{r_n}\left(-\frac{5\pi}{8}+j\pi, -\frac{3\pi}{8}+j\pi\right)} \mu^{-1} e^{\mu^2 t} d\mu \int\limits_0^1 G_1(x,\xi,\mu)\left[a\varphi''(\xi) + \right.$$

$$\left. + b\varphi'(\xi) + c\varphi(\xi)\right] d\xi = 0,\ \ (j = 0,\ 1).$$

Consequently,

$$u(x,t) = \varphi(x) + \frac{1}{2\pi i}\int\limits_{\hat{\Im}_h} \mu^{-1} e^{\mu^2 t} d\mu \int\limits_0^1 G_1(x,\xi,\mu)\left[a\varphi''(\xi) + b\varphi'(\xi) + c\varphi(\xi)\right] d\xi,$$

and using the property $G_1(x,\xi,-\mu) \equiv G_1(x,\xi,\mu)$, the solution of the problem (1), (2), (4) can be represented by the formula

$$u(x,t) = \varphi(x) + \frac{1}{\pi i}\int\limits_{\hat{\Im}^{+}_h} \mu^{-1} e^{\mu^2 t} d\mu \int\limits_0^1 G_1(x,\xi,\mu)\left[a\varphi''(\xi) + b\varphi'\xi) + c\varphi(\xi)\right] d\xi \tag{20}$$

For $|\mu| > 2h\sqrt{1 + \frac{\sqrt{2}}{2}}$, i.e. on the distant parts of the contour $\hat{Z}^{+}_h$, the following inequality is fulfilled:

$$\begin{gathered} \left| \frac{\partial^{k+m}}{\partial t^k \partial x^m} \mu^{-1} e^{\mu^2 t} \int\limits_0^1 G_1(x,\xi,\mu)\left[a\varphi''(\xi) + b\varphi'(\xi) + c\varphi(\xi)\right] d\xi \right| \le \\ C|\mu|^{2k+m-2} e^{-\frac{\sqrt{2}}{2}t|\mu|^2} \quad (2k + m \le 2)\,. \end{gathered} \tag{21}$$

Then for $0 \le x \le 1$, $0 \le t \le \omega$ the operators $L\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial t}\right)$, $l_j u|_{x=0}$ $(j = 2, 3)$ can be taken under the integral sign in (18), and allowing for (5) we have

$$L\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) u(x,\, t) = L\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) \times$$

$$\left\{\varphi(x)+\frac{1}{\pi i}\int\limits_{\hat{\Im}_h^+}\mu^{-1}e^{\mu^{2t}}d\mu\int\limits_0^1 G_1(x,\,\xi,\,\mu)[a\varphi''(\xi)+b\varphi'(\xi)+c\varphi(\xi)]d\xi\right\}=$$

$$=-(a\varphi''+b\varphi'+c\varphi)+\frac{a\varphi''+b\varphi'+c\varphi}{\pi i}\int\limits_{\hat{\Im}_h^+}\mu^{-1}e^{\mu^{2t}}d\mu=$$

$$=-(a\varphi''+b\varphi'+c\varphi)+\frac{a\varphi''+b\varphi'+c\varphi}{2\pi i}\lim_{r\to\infty}\int\limits_{\widehat{\Gamma}_{h,r}}\mu^{-1}e^{\mu^2 t}d\mu\ =$$

$$=-(a\varphi''+b\varphi'+c\varphi)+a\varphi''+b\varphi'+c\varphi=0,$$

$$l_j u\,(x,t)|_{x\to 0}=l_j\varphi\,(x)|_{x\to 0}+\frac{1}{\pi i}\int\limits_{\hat{\Im}_h^+}\mu^{-1}e^{\mu^{2t}}d\mu\times$$

$$\times\int\limits_0^1 l_j G_1\,(x,\xi,\mu)|_{x=0}\left[a\varphi''\,(\xi)+b\varphi'(\xi)+c\varphi(\xi)\right]d\xi=0,\quad (j=2,\,3)\,.$$

It is seen from the estimate (21) that limit as $t\to 0$ can also be taken under the integral sign for all $x\in[0,1]$:

$$u\,(x,0)=\lim_{t\to 0}u(x,t)\ =\varphi\,(x)+\frac{1}{\pi i}\int\limits_{\hat{\Im}_h^+}\mu^{-1}d\mu\times$$

$$\times\int\limits_0^1 G_1\,(x,\xi,\mu)\left[a\varphi''\,(\xi)+b\varphi'(\xi)+c\varphi(\xi)\right]d\xi=\varphi\,(x)+$$

$$+\frac{1}{\pi i}\lim_{r\to\infty}\int\limits_{\Omega_r\left(-\frac{3\pi}{8},\frac{3\pi}{8}\right)}\mu^{-1}d\mu\int\limits_0^1 G_1\,(x,\xi,\mu)\left[a\varphi''\,(\xi)+b\varphi'(\xi)+c\varphi(\xi)\right]d\xi=\varphi\,(x)$$

We denote the boundary value of the solution (18) by $\gamma_s(t)$:

$$\gamma_s\left(t\right)=u\left(s,t\right)=\varphi(s)+\frac{1}{\pi i}\int\limits_{\hat{\Im}_h^+}\mu^{-1}e^{\mu^2 t}d\mu\times$$

$$\times\int\limits_0^1 G_1\left(s,\xi,\mu\right)\left[a\varphi''\left(\xi\right)+b\varphi'(\xi)+c\varphi(\xi)\right]d\xi. \tag{22}$$

Under the conditions of Theorem 1, if the problem (1)-(4) has a solution, then on the parts $\{(s,\,t):\;0\le t\le\omega,\;s=0,1\}$ of the lateral side of the domain $\{(x,t):\quad 0<x<1,\quad t>0\}$, it takes boundary values $\gamma_s(t)\in c\,[0,\omega]\bigcap C^{\infty}(0,\,\omega)$ $(s=0,1)$, determined by formula (19). ◄

4. Studying the existence of the solution of the main mixed problem

Applying the integral operator $A\left[f\right]=\int\limits_0^{\infty}e^{-\lambda^2 t}f\left(t\right)dt$ (see [11]) to the equation (1) and boundary condition (3), we obtain the following second spectral problem with a complex parameter λ:

$$L\left(\frac{d}{dx},\lambda^2\right)z\left(x,\lambda\right)=-\varphi\left(x\right), \tag{23}$$

$$\begin{cases} e^{\lambda^2\omega}z\left(0,\lambda\right)+\alpha_0 z\left(1,\lambda\right)=A\left(\lambda\right), \\ z\left(0,\lambda\right)+\alpha_1 e^{\lambda^2\omega}z\left(1,\lambda\right)=B\left(\lambda\right), \end{cases} \tag{24}$$

where

$$L\left(\frac{d}{dx},\lambda^2\right)z\left(x,\lambda\right)=az''+bz'+(c-\lambda^2)z,$$

$$A\left(\lambda\right)=e^{\lambda^2\omega}\int\limits_0^{\omega}e^{-\lambda^2 t}u\left(0,t\right)dt,$$

$$B\left(\lambda\right)=\alpha_1 e^{\lambda^2\omega}\int\limits_0^{\omega}e^{-\lambda^2 t}u\left(1,t\right)dt.$$

The boundary conditions (24) can be reduced to the form

$$z\,(0,\lambda) = m\,(\lambda)\,,\ \ z\,(1,\,\lambda) = n\,(\lambda)\,, \tag{25}$$

where

$$\begin{cases} m\,(\lambda) = z_0\,(\lambda) = \left[\alpha_1 e^{2\lambda^2\omega} - \alpha_0\right]^{-1}\left(A\,(\lambda)\,\alpha_1 e^{2\lambda^2\omega} - \alpha_0 B\,(\lambda)\right) \\ \\ n\,(\lambda) = z_1\,(\lambda) = \left[\alpha_1 e^{2\lambda^2\omega} - \alpha_0\right]^{-1}\left(B\,(\lambda)\,e^{\lambda^2\omega} - A\,(\lambda)\right). \end{cases} \tag{26}$$

The solution of the problem (23),(24) can be represented as a sum of solutions of two problems:
A. $L\left(\frac{d}{dx},\lambda^2\right)z\,(x,\lambda) = 0, \qquad z\,(0,\lambda) = m(\lambda), \quad z\,(1,\lambda) = n(\lambda);$
B. $L\left(\frac{d}{dx},\lambda^2\right)z\,(x,\lambda) = -\varphi(x), \qquad z\,(0,\lambda) = 0, \quad z\,(1,\lambda) = 0.$

The solution of the problem A is represented by the formula

$$Q\,(x,\lambda,m,n) = \left[e^{-\left(\frac{b}{2a}+\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)-\left(\frac{b}{2a}-\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)}\right]^{-1}\times$$
$$\left[\left(m\,(\lambda)\,e^{-\left(\frac{b}{2a}+\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)} - n(\lambda)\right)e^{-\left(\frac{b}{2a}-\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)x}+\right.$$
$$\left.\left(n\,(\lambda) - m(\lambda)e^{-\left(\frac{b}{2a}-\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)}\right)e^{-\left(\frac{b}{2a}+\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)x}\right], \tag{27}$$

where $m(\lambda)$ and $n(\lambda)$ are determined by formula (23).

If $m = z_0(\lambda)$, $n = z_1(\lambda)$, then the function $Q\,(x,\,\lambda,\,m,\,n)$ is everywhere analytic with respect to λ except for the points $\lambda_\nu = \sqrt{a}\nu\pi i + O\left(\frac{1}{\nu}\right)$ $(\nu = 0,\,\pm1,\,\pm2,\dots)$, and the points $\lambda_m^{\pm} = \pm\left[\frac{1}{2\omega}\left(ln\left|\frac{\alpha_0}{\alpha_1}\right| + 2\pi mi\right)\right]^{1/2}$ $(m = 0,\,\pm1,\dots)$ are its poles.

If $m = \varphi(0)$, $n = \varphi(1)$, i.e. m and n are constants, then the function $Q\,(x,\,\lambda,\,m,\,n)$ is everywhere analytic except for the points $\lambda_\nu = \sqrt{a}\nu\pi i + O\left(\frac{1}{\nu}\right)$.

Obviously, at all the points λ, where $Q\,(x,\lambda,m,n)$ exists, the following identities are valid:

$$L\left(\frac{d}{dx},\lambda^2\right)Q\,(x,\lambda,m,n) = 0, \tag{23_1}$$

$$Q\,(0,\lambda,m,n) = m, \qquad Q\,(1,\lambda,m,n) = n.$$

We build the solution of the problem B by means of the Green function denoted by $G_2(x,\,\xi,\,\lambda)$. This function is everywhere analytic with respect to λ except for the points $\lambda_\nu = \sqrt{a}\nu\pi i + O\left(\frac{1}{\nu}\right)$, that are its simple poles.

Let us note some known facts about the Green function $G_2(x,\xi,\lambda)$: there exists $\delta>0$ such that on the λ plane outside the set $\bigcup\limits_{\nu=1}^{\infty}\{\lambda:\ \ |\lambda-\lambda_\nu|<\delta\}$ the following estimate is valid:

$$\left|\frac{\partial^k G_2(x,\xi,\lambda)}{\partial x^k}\right|\leq c_0|\lambda|^{k-1},\quad c_0>0,\quad k=0,\,1,\,2,\ \text{for all}\ x,\,\xi\in[0,\ 1];$$

for $\lambda\neq\lambda_\nu$ $(\nu=0,\ \pm1,\dots)$

$$L\left(\frac{d}{dx},\ \lambda^2\right)\int\limits_0^1 G_2\left(x,\xi,\lambda\right)\varphi\left(\xi\right)d\xi=-\varphi\left(x\right),$$

$$G_2\left(0,\xi,\lambda\right)=G_2\left(1,\xi,\ \lambda\right)=0.$$

Obviously, the solution of the second spectral problem is represented by the sum of two solutions (problem A and problem B):

$$z\left(x,\lambda\right)=-\int\limits_0^1 G_2\left(x,\xi,\lambda\right)\varphi\left(\xi\right)d\xi+Q\left(x,\lambda,\ m,\ n\right). \tag{28}$$

For any function $\varphi(x)$ from the domain of the operator of the second spectral problem, we have the equality :

$$\int\limits_0^1 G_2\left(x,\xi,\lambda\right)\varphi\left(\xi\right)d\xi=-\frac{\varphi\left(x\right)}{\lambda^2}+\frac{c}{\lambda^2}\int\limits_0^1 G_2\left(x,\xi,\lambda\right)\varphi\left(\xi\right)d\xi+$$

$$+\frac{1}{\lambda^2}\int\limits_0^1 G_2\left(x,\xi,\lambda\right)\left[a\varphi''\left(\xi\right)+b\varphi'(\xi)\right]d\xi+\frac{Q(x,\lambda,\varphi\left(0\right),\varphi\left(1\right))}{\lambda^2}.$$

Then, formula (28) becomes

$$z\left(x,\lambda\right)=\tfrac{\varphi(x)}{\lambda^2}-\tfrac{c}{\lambda^2}\int\limits_0^1 G_2\left(x,\xi,\lambda\right)\varphi\left(\xi\right)d\xi-\tfrac{1}{\lambda^2}\int\limits_0^1 G_2\left(x,\xi,\lambda\right)\left[a\varphi''\left(\xi\right)+b\varphi'(\xi)\right]d\xi-$$

$$-\tfrac{Q(x,\lambda,\varphi(0),\varphi(1))}{\lambda^2}+Q\left(x,\lambda,\ m,\ n\right). \tag{29}$$

We fix the number $c_1>\max\left(0,\ \ln\left|\frac{\alpha_0}{\alpha_1}\right|\right)$.

Theorem 2. *Let* $a_0b_1 + a_1b_0 \neq 0$, $\varphi(x) \in C^2[0,1]$ *and* $l_j\varphi|_{x=0} = 0$ *(*$j = 2,\ 3$*). Then the problem (1)-(4) has the solution*

$$u(x,t) = \varphi(x) + \tfrac{1}{\pi i} \int\limits_{\hat{\Im}^+_{c_1}} \lambda^{-1} e^{\lambda^2 t} \left[\int\limits_0^1 G_2(x,\,\xi,\,\lambda)(a\varphi''(\xi) + b\varphi'(\xi) + c\varphi(\xi))d\xi \right.$$

$$\left. -Q(x, \lambda,\, \varphi(0), \varphi(1)) \right] d\lambda + \tfrac{1}{\pi i} \int\limits_{\Im^+_{c_1}} \lambda e^{\lambda^2 t} Q(x,\,\lambda,\, m(\lambda),\, n(\lambda)) d\lambda. \tag{30}$$

The three integrals contained in (30) are treated in the same way. For example, let us consider the second integral

$$u_2\left(x,t\right) = -\frac{1}{\pi i} \int\limits_{\hat{\Im}^+_{c_1}} \lambda^{-1} e^{\lambda^2 t} Q\left(x, \lambda, \varphi\left(0\right), \varphi\left(1\right)\right) d\lambda. \tag{30_1}$$

On the distant parts of the contour $\hat{\Im}^+_{c_1}$ ($\mathrm{Re}\ \lambda > c_1$)

$$\left| e^{\lambda^2 t} \right| = e^{t \mathrm{Re} \lambda^2} = e^{t\ |\lambda|^2 \cos 2\ \arg\lambda} = e^{t|\lambda|^2 \cos\left(\pm\frac{3\pi}{4}\right)} = e^{-\frac{\sqrt{2}}{2} t\ |\lambda|^2}. \tag{31}$$

Further, from formula (27), the function $Q\left(x,\ \lambda,\ \varphi\left(0\right),\ \varphi\left(1\right)\right)$ is analytic in the domain $\mathrm{Re}\ \lambda > C_1$, and the following estimates are valid for it:

$$\left| \frac{\partial^k Q(x,\ \lambda,\ \varphi(0),\ \varphi(1)}{\partial x^k} \right| \leq c|\lambda|^k + \frac{c_0}{|\lambda|^k}, \quad (k = 0,\ 1,\ 2) \text{ for all } x \in [0,\ 1]. \tag{32}$$

On the distant parts of the contour $\hat{\Im}^+_{c_1}$ ($\mathrm{Re}\ \lambda > c_1$) and on the arches $\Omega_r\left(-\frac{3\pi}{8},\ \frac{3\pi}{8}\right)$ ($r > 2c_1\sqrt{1+\sqrt{2}}$) we have the estimate

$$\left|Q(x, \lambda, \varphi\left(0\right), \varphi\left(1\right))\right| \leq C_1 e^{-\left|\frac{\lambda}{\sqrt{a}}\right|(1-x)\cos\frac{3\pi}{8}} + C_2 e^{-\left|\frac{\lambda}{\sqrt{a}}\right| x \cos\frac{3\pi}{8}} + \frac{C_3}{|\lambda|}. \tag{33}$$

(31) and (33) yield

$$u_2\left(x,t\right) \in C^{2,1}\ \left(0 \leq x \leq 1,\ \ t > 0\right), \tag{34}$$

In (30_1) for $t > 0$ the operators $L\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial t}\right)$, as $x \to 0$, $x \to 1$, can be taken under integral sign. Then, allowing for (23_1), we obtain

$$L\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) u_2\left(x,t\right) = 0,$$

$$u_2(0,t) = -\frac{\varphi(0)}{\pi i}\int\limits_{\hat{\Im}_{c_1}^{+}} \lambda^{-1}e^{\lambda^2 t}d\lambda = -\frac{\varphi(0)}{2\pi i}\lim_{r\to\infty}\int\limits_{\hat{\Im}_{c_1,r}^{+}} \lambda^{-1}e^{\lambda^2 t}d\lambda = -\varphi(0),$$

$$u_2(1,t) = -\frac{\varphi(1)}{\pi i}\int\limits_{\hat{\Im}_{c_1}^{+}} \lambda^{-1}e^{\lambda^2 t}d\lambda = -\frac{\varphi(1)}{2\pi i}\lim_{r\to\infty}\int\limits_{\hat{\Im}_{c_1,r}^{+}} \lambda^{-1}e^{\lambda^2 t}d\lambda = -\varphi(1),$$

From the equality (33) it is seen that for x, belonging to any segment $[x_1, x_2] \subset (0,\ 1)$, the integral (31_1) converges uniformly with respect to $t \geq 0$.

Then

$$u_2(x,t) \in C(0 < x < 1,\ \ t \geq 0), \tag{35}$$

while for $x \in [x_1,\ x_2]$

$$u_2(x,0) = \frac{1}{\pi i}\int\limits_{\hat{\Im}_{c_1}^{+}} \lambda^{-1}Q(x,\lambda,\varphi(0),\varphi(1))\,d\lambda =$$

$$= \frac{1}{\pi i}\lim_{r\to\infty}\left[\int\limits_{\hat{\Im}_{c_1,r}^{+}} \lambda^{-1}Q(x,\lambda,\varphi(0),\varphi(1))\,d\lambda + \right.$$

$$\left. + \int\limits_{\Omega_r\left(-\frac{3\pi}{8},\frac{3\pi}{8}\right)} \lambda^{-1}Q(x,\lambda,\varphi(0),\varphi(1))\,d\lambda\right] = 0 \tag{36}$$

by the analyticity of $Q(x,\ \lambda,\varphi(0),\ \varphi(1))$ inside the closed contour $\widehat{\Gamma}_{c_1,r}^{+}$.

Combining Theorems 1 and 2, we arrive at the final statement:

Theorem 3. *Let $a_0b_1 + a_1b_0 \neq 0$, $\varphi(x) \in C^2[0,1]$ and $l_j\varphi|_{x=0} = 0$ ($j = 2,\ 3$). Then problem (1)-(4) has a unique solution represented by formula (30).*

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Hikmat I. Ahmadov
Department of Equation of Mathematical Physics, Baku State University, Baku, Azerbaijan
E-mail: hikmatahmadov@yahoo.com

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