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Strong Solvability of a Nonlocal Problem for the Laplace Equation in Weighted Grand Sobolev Spaces

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Abstract. We consider a nonlocal boundary value problem for the Laplace equation in an unbounded domain in Sobolev spaces generated by the norm of the weighted grand Lebesgue space. The notion of strong solvability of this problem is defined and its correct solvability is proved. At the same time, the basis property of one trigonometric system in separable weighted grand Lebesgue spaces is proved, and this fact is used to establish the correct solvability. Note that earlier this problem was considered by E.I.Moiseev [10] in the classical formulation.

Key Words and Phrases: Laplace equation, nonlocal problem, weighted grand Lebesgue space, strong solvability

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1. Introduction

The theory of elliptic equations for classical spaces (Hölder classes, Lebesgue spaces) is well developed, and detailed information about it can be found, for example, in [1, 4, 5]. At the same time, there are boundary value problems (for example, nonlocal problems, etc.) that do not fit this theory, but still have a scientific interest in the context of applications. One of such problems is the following (which has been considered formally):

$$y^m u_{xx} + u_{yy} = 0,\ 0 < x < 2\pi,\ y > 0, \tag{1}$$

$$\left. \begin{array}{l} u(x;0) = f(x),\ u(0;y) = u(2\pi;y), \\ u_x(0;y) = 0,\ 0 < x < 2\pi,\ y > 0, \end{array} \right\} \tag{2}$$

where $m > -2$ is some number. It is easy to see that this problem is nonlocal, the boundary condition are supported by semi-infinite lines, and a normal derivative is given on one of them. Therefore, such problems have specific features compared to with problems with local conditions. Problems with nonlocal conditions were previously considered by F.I. Franklin, [6], [7, pp.453-456] for mixed type equations and by Bitsadze-Samarskii [8] for elliptic equations. N.I. Ionkin and E.I. Moiseev [9] considered a boundary value problem with a nonlocal condition of the form (2) for a multidimensional parabolic equation. In the classical formulation, problem (1), (2) was also considered by E.I. Moiseev [10] and M.E. Lerner & O.A.Repin [11].

Lately, interest in nonstandard function spaces has greatly increased in connection with their applications in mechanics, mathematical physics and pure mathematical problems. Such spaces include Lebesgue spaces with variable summability index, Morrey spaces, grand Lebesgue spaces, Orlicz, Lorents, Martsinkevich, etc.. Numerous works have been dedicated to this field, and this trend is increasing over time. More detailed information can be found, for example, in [12, 13, 14, 15, 16]. Problems of the theory of partial differential equations in Sobolev spaces generated by the norms of the above spaces also began to be studied (see, e.g., [2, 3, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]). This article is also dedicated to this field.

In this work, we consider a nonlocal boundary value problem for the Laplace equation in an unbounded domain in Sobolev spaces generated by the norm of the weighted grand Lebesgue space. The notion of strong solvability of this problem is defined and its correct solvability is proved. At the same time, the basis property of one trigonometric system in separable weighted grand Lebesgue spaces is proved, and this fact is used to establish the correct solvability. Note that earlier this problem was considered by E.I.Moiseev [10] in the classical formulation.

2. Needful information

We will use the following notations. Let N be natural numbers and $Z_+ = \{0\} \bigcup N$. Let $\alpha = (\alpha_1;\alpha_2) \in Z_+ \times Z_+$ be a multiindex and $\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$, where $|\alpha| = \alpha_1 + \alpha_2$. Let $|M|$ denote the Lebesgue measure of the set $M \subset R$ (R is a real axis). p' be the number conjugate to p: $\frac{1}{p'} + \frac{1}{p} = 1$. We also denote $p_\varepsilon = p - \varepsilon$. Also, let X^* denote the dual space of X.

Let $\nu : R \to R_+ = (0, +\infty)$ be some weight function, that is $\left|\nu^{-1}\{0; +\infty\}\right| = 0$. We say that $\nu(\cdot)$ belongs to the Muckenhoupt class $A_p(J)$ $(J = (0, 2\pi))$ if it is periodic on R with period 2π and satisfies the condition

$$\sup_{I \subset J} \left(\frac{1}{|I|} \int_I \nu(t)\, dt \right) \left(\frac{1}{|I|} \int_I |\nu(t)|^{-\frac{1}{p-1}}\, dt \right)^{p-1} < +\infty,$$

where sup is taken over all intervals $I \subset J$. Assume that $\Pi = J \times R_+$ and $J_0 = \{(0;y) : y > 0\}$, $J_{2\pi} = \{(2\pi;y) : y > 0\}$.

Let us define the spaces we need. The (weighted) grand Lebesgue space $L_{p),\nu}(J)$ is the Banach space of measurable (in the Lebesgue sense) functions on J with the norm

$$\|f\|_{L_{p),\nu}(J)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_J |f|^{p-\varepsilon} \nu dx \right)^{\frac{1}{p-\varepsilon}}, \ 1 < p < +\infty.$$

We also define a weighted space $L_{p),\nu}(\Pi)$ with the norm

$$\|u\|_{L_{p),\nu}(\Pi)} = \sup_{0<\varepsilon<p-1} \int_0^{+\infty} \left(\varepsilon \int_J |u(x;y)|^{p-\varepsilon} \nu(x) \, dx \right)^{\frac{1}{p-\varepsilon}} dy.$$

The corresponding Sobolev space $W^m_{p),\nu}(\Pi)$ is defined by the norm

$$\|u\|_{W^m_{p),\nu}(\Pi)} = \sum_{|\alpha| \le m} \|\partial^\alpha u\|_{L_{p),\nu}(\Pi)}.$$

These spaces are nonseparable, and therefore, the method of biorthogonal expansion (essentially the spectral method) is not applicable for studying the solvability of differential equations for these spaces. Then we select the subspace $N_{p),\nu}(\Pi) \subset L_{p),\nu}(\Pi)$ (separable) based on the shift operator T_δ:

$$(T_\delta u)(x;y) = \begin{cases} u(x+\delta;y), & (x+\delta;y) \in \Pi, \\ 0 , & (x+\delta;y) \notin \Pi. \end{cases}$$

So, let us assume

$$N^m_{p),\nu}(\Pi) = \left\{ u \in W^m_{p),\nu}(\Pi) : \sum_{|\alpha| \le m} \|T_\delta(\partial^\alpha u) - \partial^\alpha u\|_{L_{p),\nu}(\Pi)} \to 0, \ \delta \to 0 \right\}.$$

Let $N^0_{p),\nu}(\Pi) = N_{p),\nu}(\Pi)$. In a similar way, we define the Sobolev space $N^m_{p),\nu}(J)$ ($N^0_{p),\nu}(J) = N_{p),\nu}(J)$ on the interval J. Let $V_{p)}(J)$ denote the following class of weights:

$$V_{p)}(J) = \bigcup_{\varepsilon \in (0,p-1)} L_{{p'}_\varepsilon - 1}(J).$$

The following lemma is true.

Lemma 1. *[29] Let* $\nu \in L_1(J) \, \& \nu^{-1} \in V_{p)}(J)$, $1 < p < +\infty$. *Then the following statements are true: i) there is a continuous inclusion* $L_{p),\nu}(J) \subset L_1(J)$*; ii)* $\overline{C_0^\infty(J)} = N_{p),\nu}(J)$*, where the closure is taken in the norm of* $\|\cdot\|_{L_{p),\nu}(J)}$.

We will largely use the following result from [32].

Theorem 1. *Let* $\nu \in A_p(J)$, $1 < p < +\infty$. *Then the trigonometric system* $\{1; \cos nx; \sin nx\}_{n\in N}$ *forms a basis for* $N_{p),\nu}(J)$.

Let us consider the following systems of functions:

$$\{1, \cos nx; x \sin nx\}_{n\in N}, \tag{3}$$

$$\{u_0(x); u_n(x); \vartheta_n(x)\}_{n\in N}, \tag{4}$$

where

$$u_0(x) = \tfrac{1}{2\pi}(2\pi - x); \ u_n(x) = \tfrac{1}{\pi^2}(2\pi - x)\cos nx;$$

$$\vartheta_n(x) = \tfrac{1}{\pi}\sin nx, \ n \in N.$$

The following theorem is true.

Theorem 2. *Let* $\nu \in A_p(J)$, $1 < p < +\infty$. *Then the system* (3) *forms a basis for* $N_{p),\nu}(J)$.

Proof. Let us assume

$$(g; f) = \int_J f(x)\, g(x)\, dx,$$

and denote the functional generated by the function g by b_g, that is, $b_g(f) = (g, f)$. Let us consider the functionals $\{b_{u_0}; b_{u_n}; b_{\vartheta_n}\}_{n\in N}$. Let $\varepsilon_0 \in (0, p-1)$ be some number. We have

$$|b_{u_n}(f)| \le \int_J |f|\, \nu^{\frac{1}{p_{\varepsilon_0}}} \nu^{-\frac{1}{p_{\varepsilon_0}}} dx \le /\text{Hölder's} \quad \text{inequality}/ \le$$

$$\le \left(\int_J |f|^{p_{\varepsilon_0}} \nu dx\right)^{\frac{1}{p_{\varepsilon_0}}} \left(\int_J \nu^{-\frac{p'_{\varepsilon_0}}{p_\varepsilon}} dx\right)^{-\frac{1}{p'_{\varepsilon_0}}} \le$$

$$\le \varepsilon_0^{-\frac{1}{p_{\varepsilon_0}}} \left(\int_J \nu^{-\frac{1}{p_{\varepsilon_0}-1}} dx\right)^{-\frac{1}{p'_{\varepsilon_0}}} \|f\|_{L_{p),\nu}(J)}. \tag{5}$$

It is well known that (see, e.g., [33, p.395]) if $\nu \in A_p(J)$, then $\exists \varepsilon_0 > 0$ (sufficiently small): $\nu \in A_{p_{\varepsilon_0}}(J)$. Choosing ε_0 in (5) based on the condition $\nu \in A_{p\varepsilon_0}(J)$, we obtain $b_{u_n} \in \left(L_{p),\nu}(J)\right)^*$, $\forall n \ge 0$. Similar considerations also imply $b_{\vartheta_n} \in \left(L_{p),\nu}(J)\right)^*$, $\forall n \ge 1$. Therefore, according to the results of [10], system (4) is biorthogonal to system (3) in $L_{p),\nu}(J)$, and therefore, system (3) is minimal in

$L_{p),\nu}(J)$. Let us prove that it is also complete in $N_{p),\nu}(J)$. From $\nu \in A_p(J) \Rightarrow \nu \in L_1(J) \& \nu^{-1} \in V_{p)}(J)$. Then Lemma 1 implies that $C_0^{\infty}(J)$ is dense in $N_{p),\nu}(J)$. Therefore, it suffices to prove that an arbitrary function from $C_0^{\infty}(J)$ can be approximated by linear combinations of the system (4) in $L_{p),\nu}(J)$.

So, let $f \in C_0^{\infty}(J)$ be an arbitrary function and let us assume that $g(x) = \frac{2\pi - x}{\pi^2} f(x)$. It is clear that $g \in C_0^{\infty}(J)$. We have

$$f_n^+ = \frac{1}{\pi^2}\int_0^{2\pi} f(x)(2\pi - x)\cos nx dx = \int_J g(x)\cos nx dx = \frac{1}{n}\int_0^{2\pi} g'(x)\sin nx dx =$$

$$= -\frac{1}{n^2}\int_0^{2\pi} g''(x)\cos nx dx \Rightarrow |f_n^+| \le \frac{c}{n^2},\ \forall n \in N.$$

Similarly,

$$\left|f_n^-\right| \le \frac{c}{n^2},\ \forall n \in N,$$

where

$$f_n^- = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx.$$

As a result, the series

$$F(x) = \frac{1}{2\pi}\int_0^{2\pi}(2\pi - x) f(x) dx + \sum_{n=1}^{\infty}\left(f_n^+\cos nx + f_n^- x\sin nx\right), \qquad (6)$$

converges uniformly on J. According to the results of [10], system (3) forms a basis for $L_2(J)$, and therefore, it is clear that $F = f$. It follows from the uniform convergence that the series (6) converges to f in $L_{p),\nu}(J)$. This implies the completeness of system (3) in $N_{p),\nu}(J)$.

Let us prove that the system (3) is a basis in $N_{p),\nu}(J)$. Consider the following projectors:

$$S_{n;m}(f) = \sum_{k=0}^{n}(u_k; f)\cos kx + \sum_{k=1}^{m}(\vartheta_k; f)\, x\sin kx,\ \forall n \in Z_+;\ \forall m \in N.$$

We have

$$(u_0; f) = \left(\frac{1}{2\pi}; F\right),\ (u_k; f) = \left(\frac{1}{\pi^2}\cos kx; F\right),$$

where $F(x) = (2\pi - x) f(x)$. Taking into account these relations for $S_{n;m}$, we obtain the following estimate:

$$\|S_{n;m}(f)\|_{L_{p),\nu}(J)} \le \left\|\left(\frac{1}{2\pi}; F\right) + \sum_{k=1}^{n}\left(\frac{1}{\pi^2}\cos kx; F\right)\cos kx\right\|_{L_{p),\nu}(J)} +$$

$$+\left\| x\sum_{k=1}^{m}\left(\frac{1}{\pi}\sin kx;4\pi f\right)\sin kx\right\|_{L_{p),\nu}(J)}\leq$$

$$\leq /from\ the\ basicity\ of\ system\ \{1;\cos nx;\sin nx\}_{n\in N}\ in\ N_{p),\nu}\left(J\right)/\leq$$

$$\leq c\left(\|F\|_{L_{p),\nu}(J)}+2\pi\left\|4\pi F\right\|_{L_{p),\nu}(J)}\right)\leq$$

$$\leq c\left\|f\right\|_{L_{p),\nu}(J)},\ \forall n\in Z_{+};\ \forall m\in N,$$

where $c>0$ is a constant which is independent of f and may be different in different places. This implies that the projectors $\{S_{n;m}\}$ are uniformly bounded in $L_{p),\nu}\left(J\right)$ and, as a result, system (3) forms a basis for $N_{p),\nu}\left(J\right)$.

Theorem is proved. ◄

In what follows, we will use an analog of Minkowski's integral inequality with respect to the grand Lebesgue norm. For this purpose, we need some concepts and facts from the theory of Banach function spaces (see, e.g., [34]).

So, let $S_{p),\nu}$ be the unit ball in $L_{p),\nu}\left(J\right)$, that is

$$S_{p),\nu}=\left\{f\in L_{p),\nu}\left(J\right):\|f\|_{L_{p),\nu}(J)}\leq 1\right\}.$$

We denote the associative space of $L_{p),\nu}\left(J\right)$ by $L'_{p),\nu}\left(J\right)$, i.e.

$$L'_{p),\nu}\left(J\right)=\left\{g\in F\left(J\right):\sup_{f\in S_{p),\nu}}\left|\int_J f\left(x\right)\overline{g}\left(x\right)dx\right|<+\infty\right\},$$

with the norm

$$\|g\|_{L'_{p),\nu}(J)}=\sup_{f\in S_{p),\nu}}\left|\int_J f\overline{g}dx\right|,$$

where $F\left(J\right)$ is the set of all measurable functions (in the sense of Lebesgue) on J. Let $S'_{p),\nu}$ be the unit ball in $L'_{p),\nu}\left(J\right)$, i.e.

$$S'_{p),\nu}=\left\{g\in L'_{p),\nu}\left(J\right):\|g\|_{L'_{p),\nu}(J)}\leq 1\right\}.$$

As established in the monograph [34] (Theorem 2.9; p.13),

$$\|f\|_{L_{p),\nu}(J)}=\sup_{g\in S'_{p),\nu}}\left|\int_J f\overline{g}dx\right|.$$

Let us prove the following

Proposition 1. *(Minkowski's inequality) Let* $f \in F(J \times R_+)$ *(*$F(J \times R_+)$ *be the set of Lebesgue measurable functions on* $J \times R_+$*). Then the following inequality holds:*

$$\left\| \int_{R_+} f(\cdot; y)\, dy \right\|_{L_{p),\nu}(J)} \le \int_{R_+} \|f(\cdot; y)\|_{L_{p),\nu}(J)}\, dy. \tag{7}$$

Proof. Let $g \in S'_{p),\nu}$ be an arbitrary function. We have

$$\int_J \left| \int_{R_+} f(x; y)\, dy \overline{g}(x) \right| dx \le \int_J \int_{R_+} |f(x; y)\, \overline{g}(x)|\, dydx =$$

$$= /Fubini's\ theorem/ = \int_{R_+} \int_J |f(x; y)\, \overline{g}(x)|\, dxdy \le$$

$$\le \int_{R_+} \sup_{g \in S'_{p),\nu}} \int_J |f(x; y)\, \overline{g}(x)|\, dxdy = \int_{R_+} \|f(\cdot; y)\|_{L_{p),\nu}(J)}\, dy.$$

Consequently,

$$\sup_{g \in S'_{p),\nu}} \int_J \left| \int_{R_+} f(x; y)\, dy \overline{g}(x) \right| dx \le \int_{R_+} \|f(\cdot; y)\|_{L_{p),\nu}(J)}\, dy.$$

This immediately implies inequality (7).

Proposition is proved. ◀

3. The main results

Let us consider the following nonlocal problem for the Laplace equation:

$$\Delta u = 0,\ (x; y) \in \Pi; \tag{8}$$

$$u/_J = f;\ u/_{J_0} = u/_{J_{2\pi}};\ u_x/_J = 0. \tag{9}$$

By a solution of the problem (8), (9), we mean a function $u \in N^2_{p),\nu}(\Pi)$ satisfying the equation (8) a.e. in Π, for which the relations (9) hold on the boundary $\partial\Pi = J \bigcup J_0 \bigcup J_{2\pi}$ (it is assumed that these relations make sense). First, we prove the uniqueness of the solution to this problem. So, the following is true.

Theorem 3. *Let* $\nu \in A_p(J)$, $1 < p < +\infty$, $f \in W^2_{p;\nu}(J)\, \& f(0) = f(2\pi) = f'(0) = 0$. *If problem* (8), (9) *is solvable in* $N^2_{p),\nu}(\Pi)$, *then its solution is unique.*

Proof. Let all conditions of the theorem be satisfied and let $u \in N^2_{p),\nu}(\Pi)$ be some solution of problem (8), (9). Following the estimate (5), we establish

$$\int_0^\pi |u(x;y)|\, dx \le c \|u(\cdot;y)\|_{L_{p),\nu}(J)}$$

in exactly the same way, where $c > 0$ is a constant is independent of $u(\cdot;\cdot)$. From this estimate we directly obtain

$$\|u\|_{W^2_1(\Pi)} \le c \|u\|_{W^2_{p),\nu}(\Pi)} \tag{10}$$

and $u \in W^2_1(\Pi)$. Let $\xi > 0$ be an arbitrary number and let us suppose that

$$J_\xi = \{(x;\xi) : x \in J\}\,;\ \Pi_\xi = \{(x;y) : x \in J \& y \in (0,\xi)\}\,.$$

Denote the trace of the function (as an element of the space $W^2_1(\Pi_\xi)$) $u(\cdot;\cdot)$ on J_ξ by $u_\xi(\cdot)$, i.e., $u_\xi = u/_{J_\xi}$. Let us show that $u_\xi \in L_{p),\nu}(J)$. It is clear that $u \in W^2_{p),\nu}(\Pi_\xi)$. Denote the closure of $C^\infty\left(\overline{\Pi}_\xi\right)$ in $W^2_{p),\nu}(\Pi_\xi)$ by $N^2_{p),\nu}(\Pi_\xi)$. First, let us consider the case $u \in C^\infty\left(\overline{\Pi}_\xi\right)$. Without loss of generality, we assume that $u/_J = 0$. We have

$$u_\xi(x) = u(x;\xi) = \int_0^\xi \frac{\partial u(x;y)}{\partial y} dy.$$

Applying the Minkowski inequality (Proposition 1), we obtain

$$\|u_\xi\|_{L_{p),\nu}(J)} \le \int_0^\xi \left\|\frac{\partial u(\cdot;y)}{\partial y}\right\|_{L_{p),\nu}(J)} dy \le \|u\|_{W^2_{p),\nu}(J)}\,,\ \forall u \in C^\infty\left(\overline{\Pi}_\xi\right).$$

Proceeding from this estimate and using the fact that $C^\infty\left(\overline{\Pi}_\xi\right)$ is dense in $N^2_{p),\nu}(\Pi_\xi)$, we establish that the trace of an arbitrary function $u \in N^2_{p),\nu}(\Pi_\xi)$ satisfies the estimate

$$\|u_\xi\|_{L_{p),\nu}(J)} \le \|u\|_{W^2_{p),\nu}(\Pi)}\,,\ \forall u \in N^2_{p),\nu}(\Pi)\,.$$

If $u(\cdot;\cdot)$ satisfies equation (8), then it is clear that $u \in C^\infty(\Pi) \Rightarrow u_\xi(x) = u(x;\xi)\,,\ \forall x \in J$.

So, let $u \in N^2_{p),\nu}(\Pi)$ be a solution to problem (8), (9). Consider the relations

$$u_0(y) = \tfrac{1}{2\pi}\textstyle\int_0^{2\pi} u(x;y)(2\pi - x)\, dx,$$

$$u_n(y) = \tfrac{1}{\pi^2}\textstyle\int_0^{2\pi} u(x;y)(2\pi - x)\cos nx dx,$$
$$\vartheta_n(y) = \tfrac{1}{\pi}\textstyle\int_0^{2\pi} u(x;y)\sin nx dx,\ \forall n \in N,$$

for $\forall y \in R_+$. It is clear that the Newton–Leibniz formula

$$u(x;y+h) - u(x;y) = \int_y^{y+h} \frac{\partial u(x;t)}{\partial t} dt, \ \forall y > 0,$$

holds for a.e. $x \in J$. As already established, from $\nu \in A_p(J) \Rightarrow L_{p),\nu}(\Pi) \subset L_1(\Pi)$. Therefore, $\frac{\partial u}{\partial y} \in L_1(\Pi)$, and as a result, it follows from Theorem 1.1.1 of the monograph [35, p.13] that the functions $\{u_n; \vartheta_n\}$ are twice differentiable and can be differentiated under the integral sign. Let us consider ϑ_n, $n \in N$. Multiplying the equation by $\sin nx$ and integrating it over J, we obtain

$$\vartheta_n''(y) - n^2 \vartheta_n(y) = 0, \ y > 0, \tag{11}$$

for $\vartheta_n(\cdot)$.

Let $\alpha \in C^\infty(R)$ be such that $\alpha(y) \equiv 1$ in a sufficiently small neighborhood of the point $y = 0$ and $\alpha(y) = 0, \ \forall y : |y| \geq 1$. Considering the function $F(x;y) = \alpha(y) u(x;y)$, we obtain $F(x;y) = 0, \ \forall y \geq 1$. Therefore, without loss of generality, we will assume $u(x;y) = 0, \ \forall y \geq 1$ in the calculations below. So, we have

$$u(x;y) = -\int_y^1 \frac{\partial u(x;t)}{\partial t} dt, \ a.e.\, x \in J$$

$$\Rightarrow f(x) = u(x;0) = -\int_0^1 \frac{\partial u(x;t)}{\partial t}, \quad a.e. \ \ x \in J.$$

Consequently,

$$|u(x;y) - f(x)| \leq \int_0^y \left| \frac{\partial u(x;t)}{\partial t} \right|, \quad a.e. \quad x \in J,$$

and, as a result, we obtain

$$\int_J |u(x;y) - f(x)|\, dx \leq \int_J \int_0^y \left| \frac{\partial u(x;t)}{\partial t} \right| dtdx.$$

Since $|\{(x;t) : (x;t) \in J \times (0,y)\}| \to 0, \ y \to +0$, it is clear that $u_y(\cdot) \to f(\cdot), \ y \to +0$, in $L_1(J)$. It is easy to see that $\vartheta_n(\cdot) \in W_1^2(R_+)$, and therefore, $\exists \lim\limits_{y \to +0} \vartheta_n(y) = \vartheta_n(0), \ \forall n \in N$. From these relations we directly obtain

$$\vartheta_n(0) = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx, \ \forall n \in N. \tag{12}$$

On the other hand, we have

$$\vartheta_n(y) - \vartheta_n(0) = \frac{1}{\pi}\int_0^{2\pi}(u(x;y) - u(x;0))\sin nx dx =$$

$$= \frac{1}{\pi}\int_0^{2\pi}\int_0^y \frac{\partial u(x;t)}{\partial t}\sin nx dt dx \Rightarrow |\vartheta_n(y) - \vartheta_n(0)| \le$$

$$\le \frac{1}{\pi}\iint_\Pi \left|\frac{\partial u}{\partial y}\right| dx dy < +\infty.$$

From here we directly obtain

$$\sup_{y>0}|\vartheta_n(y)| < +\infty. \tag{13}$$

The only solution to problem (11)–(13) is

$$\vartheta_n(y) = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx\, e^{-ny},\ \forall n \in N. \tag{14}$$

From similar considerations for u_n, we obtain

$$u_0(y) = \frac{1}{2\pi}\int_0^{2\pi}(2\pi - x) f(x) dx,$$

$$u_n(y) = \frac{1}{\pi^2}\int_0^{2\pi}(2\pi - x) f(x)\cos nx dx\, e^{-ny} + \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx\, y e^{-ny},\ \forall n \in N. \tag{15}$$

Thus, if the function $u \in N^2_{p),\nu}(\Pi)$ is a solution to problem (8), (9), then the biorthogonal coefficients of the function $u(\cdot;y)$ in system (3) satisfy expressions (14), (15). This immediately implies the uniqueness of the solution of problem (8), (9). In fact, if $f = 0$, then formulas (14), (15) imply that $u_0(y) = u_n(y) = \vartheta_n(y) = 0$, $\forall n \in N$, $\forall y \in R_+$. Since $u_y \in N_{p),\nu}(J)$, $\forall y \in R_+$, and system (3) forms a basis for $N_{p),\nu}(J)$, it follows that $u_y(x) = u(x;y) = 0$, $a.e.\, x \in J$, $\forall y \in R_+ \Rightarrow u(x;y) = 0$, $a.e.\ (x;y) \in \Pi$. Therefore, the homogeneous problem has only a trivial solution.

Theorem is proved. ◄

Now let us move on to the existence of a solution. The following is true.

Theorem 4. *Let $\nu \in A_p(J)$, $1 < p < +\infty$, and the boundary function f satisfy the conditions*

$$f \in N^2_{p),\nu}(J)\,\&\, f(0) = f(2\pi) = f'(0) = 0.$$

Then problem (8), (9) has a (unique) solution in the space $N^2_{p),\nu}(\Pi)$.

Proof. Let f satisfy all conditions of the theorem and consider the function

$$u(x;y) = u_0(y) + \sum_{n=1}^{\infty} (u_n(y)\cos nx + \vartheta_n(y)\sin nx), \ (x;y) \in \Pi,$$

where the coefficients $u_0(\cdot), u_n(\cdot), \vartheta_n(\cdot), n \in N$, are defined by the expressions (14), (15). Let us show that $u \in N^2_{p),\nu}(\Pi)$. Firstly, let us consider the series

$$u_1(x;y) = \sum_{n=1}^{\infty} \vartheta_n(y)\, x \sin nx.$$

Formally differentiating term by term, we have

$$\frac{\partial^2 u_1}{\partial y^2} = \sum_{n=1}^{\infty} \vartheta_n''(y)\, x \sin nx,$$

$$\frac{\partial u_1}{\partial x} = \sum_{n=1}^{\infty} \vartheta_n(y)\sin nx + \sum_{n=1}^{\infty} n\vartheta_n(y)\, x\cos nx,$$

$$\frac{\partial^2 u_1}{\partial x^2} = 2\sum_{n=1}^{\infty} n\vartheta_n(y)\cos nx - \sum_{n=1}^{\infty} n^2\vartheta_n(y)\, x \sin nx.$$

Let

$$w(x;y) = \sum_{n=1}^{\infty} n^2 \vartheta_n(y)\, x \sin nx.$$

Let us show that the function $w(\cdot;\cdot)$ belongs to the space $N_{p),\nu}(\Pi)$. Suppose

$$f_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx.$$

Consequently,

$$\vartheta_n(y) = f_n e^{-ny}, \ n \in N.$$

Taking into account the conditions on the function f, we have

$$f_n = -\frac{1}{\pi n}\int_0^{2\pi} f(x)\, d\cos nx = \frac{1}{\pi n}\int_0^{2\pi} f'(x)\cos nx dx =$$

$$= \frac{1}{\pi n^2}\int_0^{2\pi} f''(x)\sin nx dx = \frac{1}{n^2} f_n'',$$

where

$$f_n'' = (f'')_n = \frac{1}{\pi}\int_0^{2\pi} f''(x)\sin nx dx.$$

Thus,

$$w(x;y)=\sum_{n=1}^{\infty} f_n^{''} x \sin nx\, e^{-ny}.$$

It is known that if $\nu \in A_p(J)$, $1<p<+\infty$, then $\exists\delta>0 : \nu \in L_{1+\delta}(J)$ (see, e.g., [33, p.395]). Let $\alpha = 1+\delta$ and $\frac{1}{\alpha}+\frac{1}{\alpha'}=1$. Let $\varepsilon \in (0,p-1)$ be an arbitrary number. We have the following continuous embeddings: $L_p(J) \subset L_{p-\varepsilon}(J) \subset L_1(J)$. Let us suppose that $\beta = \frac{p}{p-\varepsilon} \Rightarrow \frac{1}{\beta'} = 1-\frac{p-\varepsilon}{p} = \frac{\varepsilon}{p} \Rightarrow \beta' = \frac{p}{\varepsilon}$. Applying Hölder inequality, we have

$$\int_0^{2\pi} |f|^{p-\varepsilon} \nu dx = \int_0^{2\pi} |f|^{p-\varepsilon} \nu^{\frac{1}{\beta}} \nu^{\frac{1}{\beta'}} dx \le$$

$$\left(\int_0^{2\pi} |f|^{p} \nu dx\right)^{\frac{1}{\beta}} \left(\int_0^{2\pi} \nu dx\right)^{\frac{1}{\beta'}} \Rightarrow \left(\varepsilon \int_0^{2\pi} |f|^{p-\varepsilon} \nu dx\right)^{\frac{1}{p-\varepsilon}} \le$$

$$\le \left(\int_0^{2\pi} |f|^{p} \nu dx\right)^{\frac{1}{p}} \left(\int_0^{2\pi} \nu dx\right)^{\frac{\varepsilon}{p-\varepsilon}\cdot\frac{1}{p}} \varepsilon^{\frac{1}{p-\varepsilon}} \le c \left(\int_0^{2\pi} |f|^{p} \nu dx\right)^{\frac{1}{p}},$$

where $c>0$ is a constant independent of f and ε. This immediately gives

$$\|f\|_{L_{p),\nu}(J)} \le c\, \|f\|_{L_{p,\nu}(J)}, \ \forall f \in L_{p),\nu}(J). \tag{16}$$

Applying the Hölder inequality again, we obtain

$$\int_0^{2\pi} |f|^p \nu(x)\, dx \le c \left(\int_0^{2\pi} |f|^{p\alpha'} dx\right)^{\frac{1}{\alpha'}}, \tag{17}$$

where c is a constant independent of f. Let us consider the following special cases.

I. $p \ge 2$. We have $p_1 = p\alpha' > 2$. Applying the classical Hausdorff–Young theorem (see, e.g., [36, p.154]) to the function $w(\cdot;y)$, from inequalities (16), (17) we have

$$\|w(\cdot;y)\|_{L_{p),\nu}(J)} \le c\, \|w(\cdot;y)\|_{L_{p,\nu}(J)} \le$$

$$\le c \left(\int_0^{2\pi} |w(x;y)|^{p_1} dx\right)^{\frac{1}{p_1}} \le c \left(\sum_{n=1}^{\infty} \left|f_n^{''} e^{-ny}\right|^{p_1'}\right)^{\frac{1}{p_1'}} \le$$
$$\le c \sum_{n=1}^{\infty} \left|f_n^{''} e^{-ny}\right|, \ \forall y \in R_+.$$

Integrating this inequality with respect to y over R_+, we obtain

$$\|w\|_{L_{p),\nu}(\Pi)} = \int_0^{+\infty} \left(\int_J |w(x,y)|^P \nu(x)\, dx\right)^{\frac{1}{p}} dy \le$$

$$\leq c \sum_{n=1}^{\infty} \left| f_n^{''} \right| \int_0^{+\infty} e^{-ny} dy = c \sum_{n=1}^{\infty} \frac{\left| f_n^{''} \right|}{n}.$$

This implies

$$\|w\|_{L_{p),\nu}(\Pi)} \leq c \left(\sum_{n=1}^{\infty} \frac{1}{n^{\beta'}} \right)^{\frac{1}{\beta'}} \left(\sum_{n=1}^{\infty} \left| f_n^{''} \right|^{\beta} \right)^{\frac{1}{\beta}} \leq$$

$$\leq /\text{Hausdorff-Young inequality}/ \leq c \left\| f^{''} \right\|_{L_{\beta'}(J)}, \tag{18}$$

where $\beta \in [2, +\infty]$ is some number with $\frac{1}{\beta} + \frac{1}{\beta'} = 1$.

Further, it is known that if $\nu \in A_p(J)$, $1 < p < +\infty$, then $\exists q : 1 < q < p - \varepsilon < p \Rightarrow \nu \in A_q(J)$. Let us suppose that $r = \frac{p_\varepsilon}{q} \Rightarrow 1 < r < p_\varepsilon$. Then

$$\int_J |g|^r \, dx = \int_J |g|^{\frac{p_\varepsilon}{q}} \nu^{\frac{1}{q}} \nu^{-\frac{1}{q}} dx \leq \left(\int_J \nu^{-\frac{q'}{q}} dx \right)^{\frac{1}{q'}} \left(\int_J |g|^{p_\varepsilon} \nu dx \right)^{\frac{1}{q}}.$$

Taking into account that $-\frac{q'}{q} = -\frac{1}{q-1}$, the relation $\nu^{-\frac{1}{q-1}} \in L_1(J)$ follows from $\nu \in A_q(J)$. Then from the previous inequality we directly obtain

$$\|g\|_{L_r(J)} \leq c \|g\|_{L_{p_\varepsilon,\nu}(J)}, \tag{19}$$

where $c > 0$ is a constant independent of g. Let us take β so large that $1 < \beta' < r \Rightarrow \|g\|_{L_{\beta'}(J)} \leq c \|g\|_{L_r(J)}$. Then from inequalities (18), (19), we have

$$\|w\|_{L_{p),\nu}(\Pi)} \leq c \|f''\|_{L_{\beta'}(J)} \leq c \|f''\|_{L_r(J)} \leq c \|f''\|_{L_{p_\varepsilon,\nu}(J)} \leq c \|f''\|_{L_{p),\nu}(J)},$$

where $c > 0$ is a constant independent of f.

II. $p \in (1, 2)$. Following the definition of the number α, we choose $\delta > 0$ so small that $p_1 = p\alpha' > 2$ (since $\alpha \to 1+0 \Rightarrow \alpha' \to +\infty$, this is possible). Based on this inequality, the further reasoning is carried out in a completely similar way to case I.

Other series in the expression $u(\cdot;\cdot)$ can be estimated in a similar way, and, as a result, we obtain the following estimate:

$$\|u\|_{W^2_{p),\nu}(\Pi)} \leq c \|f''\|_{L_{p),\nu}(J)} \leq c \|f\|_{W^2_{p),\nu}(J)},$$

where $c > 0$ is a constant independent of f. $u(\cdot;\cdot)$ satisfying the equation (8) is verified directly. Let us show that it also satisfies the boundary conditions. Denote the trace operators on the boundaries $J_0; J_{2\pi}$ and J by $\theta_0; \theta_{2\pi}$ and θ_J, respectively. Let us show that $\theta_J u = f$. In fact, let $\nu \in A_p(J)$, $1 < p < +\infty \Rightarrow \exists \varepsilon > 0$ (sufficiently small): $\nu \in A_{p_\varepsilon}$, $p_\varepsilon = p - \varepsilon > 1$. From $f \in L_{p),\nu}(J) \Rightarrow f \in L_{p_\varepsilon,\nu}(J)$. It is clear that $\theta_J u; f \in L_1(J)$. Therefore, it suffices to prove that $\theta_J u = f$, a.e. on J.

Let us introduce the following function:

$$u_m(x;y) = u_0(y) + \sum_{n=1}^{m} \left(u_n(y)\cos nx + \vartheta_n(y)\, x \sin nx\right),\ \forall (x;y) \in \Pi,\ m \in N.$$

We have

$$(\theta_J u_m)(x) = u_m(x;0) = u_0(0) +$$

$$+ \sum_{n=1}^{m} \left(u_n(0)\cos nx + \vartheta_n(0)\, x \sin nx\right) =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(x)(2\pi - x)\, dx +$$

$$+ \sum_{n=1}^{m} \left(\frac{1}{\pi^2}\int_0^{2\pi} f(x)(2\pi - x)\cos nx dx \cos nx + \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx x \sin nx\right). \tag{20}$$

Also, $\theta_J \in \left[W^2_{p_\varepsilon,\nu}(\Pi); L_{p_\varepsilon,\nu}(J)\right]$. On the other hand, it follows from $\nu \in A_{p_\varepsilon)}(J)$ that the system (3) forms a basis for $L_{p_\varepsilon,\nu}(J)$. Then it directly follows from (20) that $\theta_J u_m \to f$, $m \to \infty$, in $L_{p_\varepsilon,\nu}(J)$ and, as a result, it is clear that $\theta_J u = f$, a.e. on J.

Let us consider the boundary conditions (9). Assume

$$\Pi_\delta = \{(x;y) : x \in J \& y \in (0,\delta)\},\ \forall \delta > 0.$$

It is easy to see that $u_m \in C^\infty\left(\overline{\Pi}_\delta\right)$ and moreover $u_m(0;y) = u_m(2\pi;y)$, $\forall y > 0$; $\forall m \in N$, and also $\frac{\partial x_m}{\partial x}(0;y) = 0$, $\forall y > 0$. From these relations and from $\theta_J \in \left[W^2_{p_\varepsilon,\nu}(\Pi_\delta); L_{p_\varepsilon,\nu}(J)\right]$, where $\varepsilon \in (0, p-1)$ is a sufficiently small fixed number, it follows $u(0;y) = u(2\pi;y) = 0$, $u'_x(0,y) = 0$, $\forall y > 0$ (since $\delta > 0$ is an arbitrary number). Thus, the boundary conditions (9) are satisfied.

Theorem is proved. ◄

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