# On the Solvability of an Inverse Problem for a Hyperbolic Heat Equation 

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#### Abstract

The paper considers the inverse problem of determining the unknown coefficient on the right-hand side of the hyperbolic heat equation. An additional condition for finding the unknown coefficient, which depends on the time variable, is given in integral form. Theorems on the uniqueness, stability and existence of the solution are proved.


Key Words and Phrases: inverse problem, hyperbolic heat equation, uniqueness, "conditional" stability, existence.
2010 Mathematics Subject Classifications: 35R30, 35L70, 65M06

## 1. Introduction

We consider the following inverse problem of determining a pair of functions $\{f(t), u(x, t)\}$ :

$$
\begin{gather*}
u_{t}+\nu u_{t t}-u_{x x}=f(t) g(x),(x, t) \in D=(0,1) \times(0, T]  \tag{1}\\
u(x, 0)=\varphi(x), u_{t}(x, 0)=\psi(x), x \in[0,1]  \tag{2}\\
u(0, t)=u(1, t)=0, t \in[0, T]  \tag{3}\\
\int_{0}^{1} u(x, t) d x=h(t), t \in[0, T] \tag{4}
\end{gather*}
$$

where $g(x), \varphi(x), \psi(x), h(t)$ are the given functions, $\nu>0$ is a relaxation coefficient, and $u_{t}=\frac{\partial u}{\partial t}, u_{t t}=\frac{\partial^{2} u}{\partial t^{2}}, u_{x}=\frac{\partial u}{\partial x}, u_{x x}=\frac{\partial^{2} u}{\partial x^{2}}$.

Direct problems for hyperbolic equations have been studied in $[1,2,3]$ etc. .

[^0]The coefficient inverse problem for a hyperbolic equation has been studied in $[4,5,6]$. Inverse problems for the hyperbolic heat equation have been considered in $[7,8]$

Problem (1)-(4) belongs to the class of Hadamard ill-posed problems. Therefore, this problem should be treated proceeding from the general concepts of the theory of ill-posed problems. We make the following assumptions on the data of problem (1)-(4):

$$
\begin{aligned}
& 1^{0} \cdot g(x) \in C[0,1], \int_{0}^{1} g(x) d x=g_{0} \neq 0 \\
& 2^{0} \cdot \varphi(x) \in C^{2}[0,1], \psi(x) \in C^{1}[0,1], \int_{0}^{1} \varphi(x) d x=h(0), \int_{0}^{1} \psi(x) d x=h^{\prime}(0) ; \\
& 3^{0} . h(t) \in C^{2}[0, T]
\end{aligned}
$$

Definition 1. The pair of functions $\{f(t), u(x, t)\}$ is called the solution of problem (1)-(4) if :

1) $f(t) \in C[0, T]$;
2) $u(x, t) \in C^{2,2}(\bar{D})$;
3) the conditions (1)-(4) hold for these functions.

First we reduce the problem (1)-(4) to an equivalent one.
Lemma 1. Let the conditions $1^{0}-3^{0}$ be satisfied. Then the problems (1)-(4) and (1), (2), (3)

$$
\begin{equation*}
f(t)=\left[h^{\prime}(t)+\nu h^{\prime \prime}(t)-u_{x}(1, t)+u_{x}(0, t)\right] / g_{0}, t \in[0, T], \tag{5}
\end{equation*}
$$

which require finding the pair $\{f(t), u(x, t)\}$, are equivalent, where $h^{\prime}(t)=\frac{d h(t)}{d t}, h^{\prime \prime}(t)=$ $\frac{d^{2} h(t)}{d t^{2}}$.

Proof. Let the pair of functions $\{f(t), u(x, t)\}$ be the solution of problem (1)(4) in the sense of Definition 1. If we integrate equation (1) in the interval $(0,1)$ with respect to the variable $x$, we get:

$$
\begin{equation*}
\int_{0}^{1} u_{t} d x+\nu \int_{0}^{1} u_{t t} d x-\int_{0}^{1} u_{x x} d x=f(t) \int_{0}^{1} g(x) d x . \tag{6}
\end{equation*}
$$

Taking into account the conditions of Lemma 1, we obtain

$$
h^{\prime}(t)+\nu h^{\prime \prime}(t)-u_{x}(1, t)+u_{x}(0, t)=f(t) g_{0} .
$$

Hence the validity of the formula (5) is obvious.
Now suppose that the pair of functions $\{f(t), u(x, t)\}$ is the classical solution of problem (1),(2),(3), (5). If we take into account formula (5) in (6), then for $y(t)=\int_{0}^{1} u(x, t) d x-h(t)$ we can write

$$
\begin{gathered}
\nu y^{\prime \prime}+y^{\prime}=0, \\
y(0)=0, y^{\prime}(0)=0 .
\end{gathered}
$$

It is clear that the only solution to this problem is $y(t) \equiv 0$. From here we get $\int_{0}^{1} u(x, t) d x=h(t), t \in[0, T]$.

The Lemma 1 is proved.
The uniqueness theorem and estimation of stability for the solutions of inverse problems occupy a central place in investigation of their well-posedness. Define the following set:

$$
\begin{aligned}
& K=\left\{( f , u ) \left|f(t) \in C[0, T], u(x, t) \in C^{2,2}(\bar{D}),|f(t)| \leq c_{1},\left|u_{x}(x, t)\right| \leq c_{2},\right.\right. \\
& (x, t) \in \bar{D}, u_{1 x}(0, t)=u_{2 x}(0, t), u_{1 x}(1, t)=u_{2 x}(1, t), t \in[0, T], \\
& \left.\forall\left(f_{1}, u_{1}\right),\left(f_{2}, u_{2}\right) \in K, c_{1}, c_{2}=\text { const }>0\right\}
\end{aligned}
$$

Let us assume that the two input sets, $\left\{g_{1}(x), \varphi_{1}(x), \psi_{1}(x), h_{1}(t)\right\}$ and $\left\{g_{2}(x)\right.$, $\left.\varphi_{2}(x), \psi_{2}(x), h_{2}(t)\right\}$ are given for problem (1),(2),(3),(5). For brevity, we will call the problem with the first input set problem $I_{1}$, while the one with the second input set will be called problem $I_{2}$. Let $\left\{f_{1}(t), u_{1}(x, t)\right\}$ and $\left\{f_{2}(t), u_{2}(x, t)\right\}$ be solutions of problems $I_{1}$ and $I_{2}$, respectively.

## Theorem 1. Let the following conditions hold:

1) the functions $g_{i}(x), \varphi_{i}(x), \psi_{i}(x), h_{i}(t), i=1,2$, satisfy conditions $1^{0}-3^{0}$, respectively;
2) Solutions of problems $I_{1}$ and $I_{2}$ exist in the sense of Definition 1 and they belong to the set $K$.

Then there exists a $T^{*}\left(0<T^{*} \leq T\right)$ such that for $(x, t) \in \bar{D}_{*}=[0,1] \times\left[0, T^{*}\right]$ the solution of problem (1),(2),(3),(5) is unique, and the stability estimate

$$
\begin{align*}
& \int_{0}^{1}\left[\nu\left(u_{1 t}(x, t)-u_{2 t}(x, t)\right)^{2}+\left(u_{1 x}(x, t)-u_{2 x}(x, t)\right)^{2}\right] d x+ \\
& +c_{3}\left\|f_{1}(t)-f_{2}(t)\right\|_{0}^{2} \leq c_{4}\left\{\int _ { 0 } ^ { 1 } \left[\left(g_{1}(x)-g_{2}(x)\right)^{2}+\left(\varphi_{1 x}(x)-\varphi_{2 x}(x)\right)^{2}+\right.\right.  \tag{7}\\
& \left.\left.+\left(\psi_{1}(x)-\psi_{2}(x)\right)^{2}\right] d x+\left\|h_{1 t}^{\prime}(t)-h_{2 t}^{\prime}(t)\right\|_{0}^{2}+\left\|h_{1 t t}^{\prime \prime}(t)-h_{2 t t}^{\prime \prime}(t)\right\|_{0}^{2}\right\}
\end{align*}
$$

is valid, where $c_{3}, c_{4}>0$ depend on the data of problems $I_{1}$ and $I_{2}$ in the set $K$, $\|q(t)\|_{0}=\max _{[0, T]}|q(t)|$.

Proof. First, we prove inequality (7) under the condition $g_{1}=g_{2}, \varphi_{1}=$ $\varphi_{2}, \psi_{1}=\psi_{2}, h_{1}=h_{2}$.

Denote

$$
\begin{gathered}
z(x, t)=u_{1}(x, t)-u_{2}(x, t), \lambda(t)=f_{1}(t)-f_{2}(t), \delta_{1}(x)=g_{1}(x)-g_{2}(x) \\
\delta_{2}(x)=\varphi_{1}(x)-\varphi_{2}(x), \delta_{3}(x)=\psi_{1}(x)-\psi_{2}(x), \delta_{4}(t)=h_{1}(t)-h_{2}(t)
\end{gathered}
$$

Subtracting from the relations of problem $I_{1}$ the corresponding relations of problem $I_{2}$, we obtain the problem of determining a pair of functions $\{\lambda(t), z(x, t)\}$ :

$$
\begin{gather*}
z_{t}+\nu z_{t t}-z_{x x}=\lambda(t) g_{1}(x)+f_{2}(t) \delta_{1}(x),(x, t) \in D  \tag{8}\\
z(x, 0)=\delta_{2}(x), z_{t}(x, 0)=\delta_{3}(x), x \in[0,1]  \tag{9}\\
z(0, t)=z(1, t), t \in[0, T]  \tag{10}\\
\lambda(t)=\left[\delta_{4}^{\prime}(t)+\nu \delta_{4}^{\prime \prime}(t)-z_{x}(1, t)+z_{x}(0, t)\right] \backslash g_{01}+H(t), t \in[0, T], \tag{11}
\end{gather*}
$$

where $g_{0 i}=\int_{0}^{1} g_{i}(x) d x, i=1,2, H(t)=\left[h_{2}^{\prime}(t)+\nu h_{2}^{\prime \prime}(t)-u_{2 x}(1, t)+u_{2 x}(0, t)\right] \times$ $\times\left(g_{02}-g_{01}\right) \backslash\left(g_{01} \cdot g_{02}\right)$

Multiply equations (8) by $2 z_{t}(x, t)$ and integrate over the domain $D$ :

$$
\begin{equation*}
2 \int_{0}^{t} \int_{0}^{1}\left[z_{t}+\nu z_{t t}-z_{x x}\right] z_{t} d x d t=2 \int_{0}^{t} \int_{0}^{1}\left[\lambda(t) g_{1}(x)+f_{2}(t) \delta_{1}(x)\right] z_{t} d x d t \tag{12}
\end{equation*}
$$

If we consider

$$
\begin{gathered}
2 \int_{0}^{t} \int_{0}^{1} z_{t} z_{t} d x d t=2 \int_{0}^{t} \int_{0}^{1} z_{t}^{2} d x d t \\
2 \nu \int_{0}^{t} \int_{0}^{1} z_{t t} z_{t} d x d t=\nu \int_{0}^{1}\left[z_{t}^{2}(x, t)-\delta_{3}^{2}(x)\right] d x \\
2 \int_{0}^{t} \int_{0}^{1} z_{x x} z_{t} d x d t=-\int_{0}^{1}\left[z_{x}^{2}(x, t)-\delta_{2 x}^{2}(x)\right] d x \\
2 \int_{0}^{t} \int_{0}^{1}\left[\lambda(t) g_{1}(x)+f_{2}(t) \delta_{2}(x)\right] z_{t} d x d t \leq \int_{0}^{t} \int_{0}^{1} \lambda^{2}(t) g_{1}^{2} d x d t+ \\
+\int_{0}^{t} \int_{0}^{1} f_{1}^{2}(t) \delta_{1}^{2}(x) d x d t+2 \int_{0}^{t} \int_{0}^{1} z_{1}^{2}(t) d x d t
\end{gathered}
$$

then from (12) we get:

$$
\begin{gather*}
\int_{0}^{1}\left[\nu z_{t}^{2}(x, t)+z_{x}^{2}(x, t)\right] d x d t \leq \\
\leq \int_{0}^{1}\left[\nu \delta_{3}^{2}(x)+\delta_{2 x}^{2}(x)\right] d x+c_{5} \int_{0}^{1} \delta_{1}^{2}(x) d x+c_{6} t\|\lambda\|_{0}^{2}, \tag{13}
\end{gather*}
$$

where $c_{5}, c_{6}>0$ depend on the data of problems $I_{1}$ and $I_{2}$ in the set $K$. Let us estimate the function $\lambda(t)$. From (11) we have

$$
\begin{gathered}
|\lambda(t)| \leq\left[\left|\delta_{4}^{\prime}(t)\right|+\nu\left|\delta_{4}^{\prime \prime}(t)\right|\right] /\left|g_{01}\right|+|H(t)|\left|g_{02}-g_{01}\right| /\left|g_{01} \cdot g_{02}\right| \\
\lambda(t)^{2} \leq c_{7}\left[\left\|\delta_{4}^{\prime}\right\|_{0}^{2}+\left\|\delta_{4}^{\prime \prime}\right\|_{0}^{2}\right]+c_{8} \int_{0}^{1} \delta_{1}^{2}(x) d x
\end{gathered}
$$

The last inequality is satisfied for each $t \in[0, T]$, so it must be satisfied for the maximum value of the left-hand side:

$$
\begin{equation*}
\|\lambda\|_{0}^{2} \leq c_{7}\left[\left\|\delta_{4}^{\prime}\right\|_{0}^{2}+\left\|\delta_{4}^{\prime \prime}\right\|_{0}^{2}\right]+c_{8} \int_{0}^{1} \delta_{1}^{2}(x) d x \tag{14}
\end{equation*}
$$

From (13) and (14) we get:

$$
\begin{align*}
& \int_{0}^{1}\left[\nu z_{t}^{2}(x, t)+z_{x}^{2}(x, t)\right] d x+\|\lambda\|_{0}^{2} \leq \int_{0}^{1}\left[\nu \delta_{3}^{2}(x)+\delta_{2 x}^{2}(x)\right] d x+ \\
& +c_{9} \int_{0}^{1} \delta_{1}^{2}(x) d x+c_{6} t\|\lambda\|_{0}^{2}+c_{7}\left[\left\|\delta_{4}^{\prime}\right\|_{0}^{2}+\left\|\delta_{4}^{\prime \prime}\right\|_{0}^{2}\right] \tag{15}
\end{align*}
$$

Let $T * \in(0, T]$ be a number such that $c_{6} T^{*}<1$. Then the stability estimate (7) is true in the domain $\bar{D}_{*}=[0,1] \times\left[0, T^{*}\right]$.

The uniqueness of the solution of problem (1), (2), (3), (5) is obtained from inequality (7) for $g_{1}(x)=g_{2}(x), \varphi_{1}(x)=\varphi_{2}(x), \psi_{1}(x)=\psi_{2}(x), h_{1}(t)=h_{2}(t)$.

Theorem 1 is proved.
For A.N. Tikhonov correct problems, the existence of a solution is a priori assumed and justified by the physical meaning of the problem under consideration.

Despite the fact that the proof of the existence of a solution to ill-posed problems requires some additional conditions on the input data, from the point of view of constructing algorithims for exact or appoximate solution of the problem, it is certainly of practical interest.

Theorem 2. Let

1) $g(x) \in C^{1}[0,1], g(0)=g(1)=0, \int_{0}^{1} g(x) d x=g_{0} \neq 0$;
2) $\varphi(x) \in C^{2}[0,1], \varphi(0)=\varphi(1)=0, \varphi^{\prime \prime}(0)=\varphi^{\prime \prime}(1)=0, \int_{0}^{1} \varphi(x) d x=h(0)$;
3) $\psi(x) \in C^{1}[0,1], \psi(0)=\psi(1)=0, \int_{0}^{1} \psi(x) d x=h_{t}^{\prime}(0)$;
4) $h(t) \in C^{2}[0, T]$

The problem (1), (2), (3), (5) in $\bar{D}=[0,1] \times[0, T]$ has a solution in the sense of Definition 1.

Proof. For a given $f(t) \in C[0, T]$, the solution of problem (1),(2),(3) will be sought in the form

$$
u(x, t)=\vartheta(x, t)+w(x, t)
$$

Here $\vartheta(x, t)$ is the solution of the following problem:

$$
\begin{gather*}
\nu \vartheta_{t t}+\vartheta_{t}-\vartheta_{x x}=0,(x, t) \in D,  \tag{16}\\
\vartheta(x, 0)=\varphi(x), \vartheta_{t}(x, 0)=\psi(x), x \in[0,1],  \tag{17}\\
\vartheta(0, t)=\vartheta(1, t)=0, t \in[0, T], \tag{18}
\end{gather*}
$$

and $w(x, t)$ is the solution of the following problem:

$$
\begin{gather*}
\nu w_{t t}+w_{t}-w_{x x}=f(t) g(x),(x, t) \in D,  \tag{19}\\
w(x, 0)=w_{t}(x, 0)=0, x \in[0,1]  \tag{20}\\
w(0, t)=w(1, t)=0, t \in[0, T] \tag{21}
\end{gather*}
$$

Let us first consider a homogeneous equation (16). We will look for all solutions of this equation that can be represented in the form $\vartheta(x, t)=y(t) q(x)$ and satisfy the boundary condition (18).

We can say that each of the functions

$$
\vartheta_{n}(x, t)=e^{-\frac{t}{2 \nu}}\left(A_{n} \cos \sigma_{n} t+B_{n} \sin \sigma_{n} t\right) \sin \lambda_{n} x, n=1,2, \ldots
$$

(for any constants $A_{n}$ and $B_{n}$ ) is a solution to the equation (16) that satisfies the boundary condition (18).

Here $\lambda_{n}=n \pi, n=1,2, \ldots, \sigma_{n}=\frac{\sqrt{4 \nu \lambda_{n}^{2}-1}}{2 \nu}$. Note that if $4 \nu \lambda_{n}^{2}-1>0$, then this inequality will hold for a finite number $\lambda_{n}$.

In this case, the corresponding eigen functions of the form $y_{n}(t)=$ $=e^{-\frac{t}{2 \nu}}\left(A_{n} e^{\sigma_{n} t}+B_{n} e^{-\sigma_{n} t}\right), n=1, \ldots, n_{0}, y_{n_{0}+1}(t)=A_{n_{0}+1}+B_{n_{0}+1} t\left(\sigma_{n}=0\right)$ do not affect both the scheme of the proof and the assertions of Theorem 2.

Taking into account the initial conditions leads us to the following expression for the coefficients $A_{n}$ and $B_{n}$ :

$$
\begin{equation*}
A_{n}=\varphi_{n}, B_{n}=\frac{1}{2 \nu \sigma_{n}} \varphi_{n}+\frac{1}{\sigma_{n}} \psi_{n} . \tag{22}
\end{equation*}
$$

Here $\varphi_{n}$ and $\psi_{n}$ denote the Fourier coefficients of the functions $\varphi(x)$ and $\psi(x)$, respectively, with regard to the system $\left\{\sin \lambda_{n} x\right\}$.

Thus, formally we came to the following representation of the solution of the mixed problem (16)-(18):

$$
\begin{align*}
& \vartheta(x, t)=\sum_{n=1}^{\infty} y_{n}(t) q_{n}(x)= \\
& =\sum_{n=1}^{\infty} e^{-\frac{t}{2 \nu}}\left[\varphi_{n} \cos \sigma_{n} t+\left(\frac{1}{2 \nu \sigma_{n}} \varphi_{n}+\frac{1}{\sigma_{n}} \psi_{n}\right) \sin \sigma_{n} t\right] \sin \lambda_{n} x . \tag{23}
\end{align*}
$$

Formally expanding the desired solution $w(x, t)$ of the problem (19)-(21) and the right-hand side of the equation (19) $f(t) g(x)$ in a series of eigen functions $\left\{\sin \lambda_{n} x\right\}: w(x, t)=\sum_{n=1}^{\infty} \theta_{n}(t) \sin \lambda_{n} x$, and $f(t) g(x)=\sum_{n=1}^{\infty} f(t) g_{n} \sin \lambda_{n} x$ and taking into account these functions in (19)-(21), we obtain:

$$
\begin{gather*}
\nu \theta_{n}^{\prime \prime}(t)+\theta_{n}^{\prime}(t)+\lambda_{n}^{2} \theta_{n}(t)=f_{n}(t), \\
\theta_{n}(0)=\theta_{n}^{\prime}(0)=0, \tag{24}
\end{gather*}
$$

where $f_{n}(t)=f(t) g_{n}, n=1,2, \ldots$
The solution of the problem (24) has the following form:

$$
\theta_{n}(t)=\frac{e^{-\frac{t}{2 \nu}}}{2 \sigma_{n}} \int_{0}^{t} e^{\frac{\tau}{2 \nu}} f_{n}(\tau) \sin \sigma_{n}(t-\tau) d \tau
$$

Given that $w_{n}(x, t)=\theta_{n}(t) \sin \lambda_{n} x$, we get:

$$
\begin{align*}
u(x, t)= & e^{-\frac{t}{2 \nu}} \sum_{n=1}^{\infty}\left\{\left(\varphi_{n} \cos \sigma_{n} t+\frac{\varphi_{n}+2 \nu \psi_{n}}{2 \nu \sigma_{n}} \sin \sigma_{n} t\right) \sin \lambda_{n} x+\right. \\
& \left.+\frac{1}{2 \sigma_{n}} \int_{0}^{t} e^{\frac{\tau}{2 \nu}} f_{n}(\tau) \sin \sigma_{n}(t-\tau) \sin \lambda_{n} x d \tau\right\} \tag{25}
\end{align*}
$$

In order for the function (25) to be a solution of the problem (1)-(3), for each $f(t) \in C[0, T]$ the series (25) and the following formally composed series must converge uniformly:

$$
\begin{align*}
& u_{t}(x, t)=-\frac{1}{2 \nu} u(x, t)+e^{-\frac{t}{2 \nu}} \sum_{n=1}^{\infty}\left\{\left(-\sigma_{n} \varphi_{n} \cos \sigma_{n} t+\frac{\varphi_{n}+2 \nu \psi_{n}}{2 \nu} \cos \sigma_{n} t\right)\right. \\
&\left.\times \sin \lambda_{n} x+\frac{1}{2} \int_{0}^{t} e^{\frac{\tau}{2 \nu}} f_{n}(\tau) \cos \sigma_{n}(t-\tau) \sin \lambda_{n} x d \tau\right\}  \tag{26}\\
& u_{t t}(x, t)=-\frac{1}{2 \nu} u_{t}(x, t)-\frac{1}{2 \nu} \sum_{n=1}^{\infty}\left\{\left(-\sigma_{n} \varphi_{n} \sin \sigma_{n} t+\frac{\varphi_{n}+2 \nu \psi_{n}}{2 \nu} \cos \sigma_{n} t\right) \times\right. \\
&\left.\times \sin \lambda_{n} x+\frac{1}{2} \int_{0}^{t} e^{\frac{\tau}{2 \nu}} f_{n}(\tau) \cos \sigma_{n}(t-\tau) \sin \lambda_{n} x d \tau\right\}+ \\
&+e^{-\frac{t}{2 \nu}} \sum_{n=1}^{\infty}\left\{\left(-\sigma_{n}^{2} \varphi_{n} \cos \sigma_{n} t-\frac{\varphi_{n}+2 \nu \psi_{n}}{2 \nu} \sin \sigma_{n} t\right) \sin \lambda_{n} x+\right. \\
&\left.+\frac{1}{2}\left[e^{\frac{t}{2 \nu}} f_{n}(t)(-1)^{n}+\sigma_{n} \int_{0}^{t} e^{\frac{\tau}{2 \nu}} f_{n}(\tau) \sin \sigma_{n}(t-\tau) \sin \lambda_{n} x d \tau\right]\right\}  \tag{27}\\
& u_{x x}(x, t)= e^{-\frac{t}{2 \nu}} \sum_{n=1}^{\infty}\left\{\lambda_{n}^{2}\left(-\varphi_{n} \cos \sigma_{n} t-\frac{\varphi_{n}+2 \nu \psi_{n}}{2 \nu \sigma_{n}} \cos \sigma_{n} t\right) \sin \lambda_{n} x-\right.
\end{align*}
$$

$$
\begin{equation*}
\left.-\frac{\lambda_{n}^{2}}{2 \sigma_{n}} \int_{0}^{t} e^{\frac{\tau}{2}} f_{n}(\tau) \sin \sigma_{n}(t-\tau) \sin \lambda_{n} x d \tau\right\} \tag{28}
\end{equation*}
$$

The following series are the majorants of the series (25)-(28), respectively:

$$
\begin{gathered}
\sum_{n=1}^{\infty}\left|u_{n}(x, t)\right| \leq c_{10} \sum_{n=1}^{\infty}\left[\left|\varphi_{n}\right|+\frac{\left|\varphi_{n}+2 \nu \psi_{n}\right|}{2 \nu \sigma_{n}}\right], \\
\sum_{n=1}^{\infty}\left|u_{n t}(x, t)\right| \leq c_{11} \sum_{n=1}^{\infty}\left[\left|u_{n}(x, t)\right|+\left|\sigma_{n} \varphi_{n}\right|+\frac{\left|\varphi_{n}+2 \nu \psi_{n}\right|}{2 \nu}+\left|g_{n}\right|\right], \\
\sum_{n=1}^{\infty}\left|u_{n t t}(x, t)\right| \leq c_{12} \sum_{n=1}^{\infty}\left[\left|\sigma_{n} \varphi_{n}\right|+\left|\varphi_{n}\right|+\left|\psi_{n}\right|+\left|g_{n}\right|+\left|\sigma_{n}^{2} \varphi_{n}\right|+\left|\sigma_{n} \nu \varphi_{n}\right|\right], \\
\sum_{n=1}^{\infty}\left|u_{n x x}(x, t)\right| \leq c_{13} \sum_{n=1}^{\infty}\left[\left|\lambda_{n}^{2} \varphi_{n}\right|+\left|\frac{\lambda_{n}^{2}}{2 \nu \sigma_{n}} \varphi_{n}\right|+\left|\frac{\lambda_{n}^{2} \psi_{n}}{\sigma_{n}}\right|+\left|\frac{\lambda_{n}^{2}}{2 \nu \sigma_{n}} g_{n}\right|\right] .
\end{gathered}
$$

Under the conditions of Theorem 2, the majorant series converge [1].
Thus, for each $f(t) \in C[0, T]$, the function (25) is a solution to problem (1)-(3) in the sense of Definition 1 .

Now we will show the existence of the function $f(t) \in C[0, T]$.
Denote $Q=C[0, T]$. Write equation (5) in operator form:

$$
\begin{gathered}
M[f(t)]=f(t), M: Q \rightarrow Q \\
M[f(t)]=\left[h^{\prime}(t)+h^{\prime \prime}(t)\right] / g_{0}+e^{-\frac{t}{2 \nu}}\left\{\sum _ { m = 1 } ^ { \infty } 2 \lambda _ { 2 m - 1 } \left(\varphi_{2 m-1} \cos \sigma_{2 m-1} t+\right.\right. \\
\left.\left.+\frac{\varphi_{2 m-1}+2 \nu \psi_{2 m-1}}{2 \nu \sigma_{2 m-1}} \sin \sigma_{2 m-1} t\right)+\sum_{m=1}^{\infty} \frac{\lambda_{2 m-1}}{\sigma_{2 m-1}} \int_{0}^{t} e^{\frac{\tau}{2 \nu}} f_{2 m-1}(\tau) \sin \sigma_{2 m-1}(t-\tau) d \tau\right\} / g_{0} .
\end{gathered}
$$

Denote

$$
Q^{\prime}=\left\{f|f(t) \in C[0, T],| f(t) \leq f_{0}, t \in[0, T]\right\},
$$

where $f_{0}>0$ is some constant.
It is clear that $M\left[Q^{\prime}\right] \subset Q$. Show that the set $M\left[Q^{\prime}\right]$ is uniformly bounded and equicontinuous:

$$
|M[f(t)]|=\left[\left|h^{\prime}(t)\right|+\left|h^{\prime \prime}(t)\right|\right] /\left|g_{0}\right|+e^{-\frac{t}{2 \nu}}\left\{\sum_{m=1}^{\infty} 2 \lambda_{2 m-1}\left|\varphi_{2 m-1}\right|+\right.
$$

$$
\left.+\frac{\left|\varphi_{2 m-1}+2 \nu \psi_{2 m-1}\right|}{2 \nu \sigma_{2 m-1}}+\frac{\lambda_{2 m-1}}{\sigma_{2 m-1}}\left|g_{n}\right||f(t)| T\right\} /\left|g_{0}\right|
$$

Under the conditions of Theorem 2 and by the relation $f(t) \in Q^{\prime}$, from the last inequality we obtain the uniform boundedness of the set $M\left[Q^{\prime}\right]$.

Now let's show the equicontinuity of the set $M\left[Q^{\prime}\right]$. Estimate the difference $M\left[f\left(t_{1}\right)\right]-M\left[f\left(t_{2}\right)\right]$ for any $t_{1}, t_{2} \in[0, T]:$

$$
\begin{gathered}
\left|M\left[f\left(t_{1}\right)\right]-M\left[f\left(t_{2}\right)\right]\right| \leq\left[\left|h^{\prime}\left(t_{1}\right)-h^{\prime}\left(t_{2}\right)\right|+\left|h^{\prime \prime}\left(t_{1}\right)-h^{\prime \prime}\left(t_{2}\right)\right|\right] /\left|g_{0}\right|+ \\
+e^{-\frac{t}{2 \nu}} \sum_{m=1}^{\infty}\left\{2 \lambda _ { 2 m - 1 } \left[\left|\varphi_{2 m-1}\right|\left|\cos \sigma_{2 m-1} t_{1}-\cos \sigma_{2 m-1} t_{2}\right|+\frac{\left|\varphi_{2 m-1}+2 \nu \psi_{2 m-1}\right|}{2 \nu \sigma_{2 m-1}} \times\right.\right. \\
\left.\times\left|\sin \sigma_{2 m-1} t_{1}-\sin \sigma_{2 m-1} t_{2}\right|\right]+\frac{\lambda_{2 m-1}}{\sigma_{2 m-1}}\left[\int_{t_{2}}^{t_{1}} e^{\frac{\tau}{2 \nu}}\left|f_{2 m-1}(\tau) \sin \sigma_{2 m-1}\left(t_{1}-\tau\right)\right| d \tau+\right. \\
\left.\left.+\int_{0}^{t} e^{\frac{\tau}{2 \nu}}\left|f_{2 m-1}(\tau) \sin \sigma_{2 m-1}\left(t_{1}-\tau\right)-\sin \sigma_{2 m-1}\left(t_{2}-\tau\right)\right| d \tau\right]\right\} /\left|g_{0}\right|+ \\
++e^{\frac{t_{1}+t_{2}}{2 \nu}}\left|e^{\frac{t_{2}}{2 \nu}}-e^{\frac{t_{1}}{2 \nu}}\right| \sum_{m=1}^{\infty}\left\{2 \lambda _ { 2 m - 1 } \left(\left|\varphi_{2 m-1}\right|\left|\cos \sigma_{2 m-1} t_{2}\right|+\right.\right. \\
\left.+\frac{\left|\varphi_{2 m-1}+2 \nu \psi_{2 m-1}\right|}{2 \nu \sigma_{2 m-1}}\left|\sin \sigma_{2 m-1} t_{2}\right|\right)+ \\
\left.\left.\frac{\lambda_{2 m-1}}{\sigma_{2 m-1}} \int_{0}^{t} e^{\frac{\tau}{2 \nu}}\left|f_{2 m-1}(\tau) \sin \sigma_{2 m-1}\left(t_{1}-\tau\right)\right| d \tau\right]\right\} /\left|g_{0}\right| .
\end{gathered}
$$

Taking into account the conditions of Theorem 2, for the last inequality we have

$$
\left|M\left[f\left(t_{1}\right)\right]-M\left[f\left(t_{2}\right)\right]\right| \leq c_{14}\left|t_{1}-t_{2}\right|
$$

Thus, by the Arzela theorem, the set $M\left[Q^{\prime}\right]$ is compact in $Q$ [9]. In this case, according to the Schauder theorem, the operator $M[f(t)]$ has at least one fixed point, in other words, the operator equation $M[f(t)]=f(t)$ has a solution $f(t) \in Q=C[0, T]$.

Theorem 2 is proved.

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Received 25 July 2022
Accepted 21 September 2022


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