Azerbaijan Journal of Mathematics
V. 13 , No 1, 2023, January
ISSN 2218-6816

Existence Result for Nonlocal Boundary Value Problem of Fractional Order at Resonance with p-Laplacian Operator

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Abstract. The goal of this paper is to prove the existence of solutions to multi-point boundary value problem for a nonlinear fractional differential equation with p-Laplacian operator at resonance case. Our methodology are based on the famous coincidence degree theory due to the Belgian mathematician J.Mawhin. An example is included to show the importance and the applicability of this result.

Key Words and Phrases: multi-point boundary value problem, p-Laplacian operator, resonance case, nonlinear fractional.

2010 Mathematics Subject Classifications: 34B65, 34A63, 35J96

1. Introduction

Since recently, fractional differential equations play an essential role in numerous fields such as physics, chemistry, biology, electronic, control theory (see [3],[6], [7] and references therein where the authors examined the existence of solutions for fractional boundary value problems at resonance). See also [2], [5], [8], [9], [10], [11], [12], [13], and [14]. In this paper, we investigate the multi-point boundary value problem (BVP for short) for a nonlinear fractional differential equation with a p-Laplacian operator:

$$\left(\phi_p\left(D_{0^+}^{\alpha}x(t)\right)\right)' = f\left(t, x(t), D_{0^+}^{\alpha-1}x(t)\right), \quad t \in [0,1] \tag{1}$$

$$x(0) = D_{0^+}^{\alpha}x(1) = 0 \tag{2}$$

$$D_{0^+}^{\alpha-1}x(1) = \sum_{i=1}^{i=m-2} \beta_i D_{0^+}^{\alpha-1}x(\eta_i), \tag{3}$$

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where $1 < \alpha < 2$, $0 < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < 1$, $\beta_i \in \mathbb{R}_+$, for $i = 1, 2, 3, \ldots, m-2$, $(m \geq 3)$, D_{0+}^{α} is the standard Riemann-Liouville derivative, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and $\phi_p(s) = |s|^{p-2} s$ is the p-Laplacian $(p > 1)$. Recall that $\phi_p : \mathbb{R} \to \mathbb{R}$ is an odd continuous, strictly increasing operator with $\phi_p^{-1} = \phi_q$ $(\frac{1}{p} + \frac{1}{q} = 1)$. Assume also that

$$\sum_{i=1}^{i=m-2} \beta_i = 1. \tag{4}$$

There are only few results for boundary value problems at resonance when the p-Laplacian differentiation operator is involved. Indeed, the basic difficulty in solving such problems resides in the fact that since ϕ_p (in case $p \neq 2$) is a nonlinear operator, the coincidence degree method cannot be applied in a direct way. For example, in the recent work [4], the authors proved the existence of at least one solution of the same problem at the resonance case by using an alternative approach based on the decomposition $u - Lu = Nu$ and some fixed point arguments. Motivated by [4], we will establish an existence result of solution for the BVP 1 - 2 - 3 by means of the coincidence degree theory applied to an equivalent semi linear problem. It's easy to check that our problem is equivalent to the following boundary value problem:

$$D_{0+}^{\alpha} x(t) = \phi_q \left(\int_1^t f\left(s, x(s), D_{0+}^{\alpha-1} x(s)\right) ds \right) \tag{5}$$

$$x(0) = 0, \tag{6}$$

$$D_{0+}^{\alpha-1} x(1) = \sum_{i=1}^{i=m-2} \beta_i D_{0+}^{\alpha-1} x(\eta_i). \tag{7}$$

Now, we briefly give some notations and some existence results due to Mawhin (see [5]).

Let X, Y be two real Banach spaces, Ω be an open bounded subset of X, $L : dom(L) \subset X \to Y$ be a linear operator, and $N : X \to Y$ be nonlinear mapping. If $\operatorname{Im} L$ is a closed set of Y and $\dim \ker(L) = co\dim \operatorname{Im}(L) < +\infty$, then L is called a Fredholm operator of index zero. In this case there exist two linear continuous projectors $P : X \to X$, $Q : Y \to Y$ such that $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$ and we can write $X = \ker(L) \oplus \ker(P)$, $Y = \operatorname{Im}(L) \oplus \operatorname{Im}(Q)$. It follows that $L_P = L_{|dom(L) \cap \ker P} : dom(L) \cap \ker P \to \operatorname{Im} L$ is invertible. We denote the inverse of L_P by K_P. If $dom(L) \cap \bar{\Omega} \neq \emptyset$, N will be called L-compact on Ω if $QN(\bar{\Omega})$ is bounded and $K(I - Q)N : \bar{\Omega} \to X$ is compact.

Theorem 1. *Let X, Y be two real Banach spaces, $L : dom(L) \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ be an L-compact mapping on Ω. Assume that the following conditions are satisfied:*

(1) $Lx \neq \lambda Nx$ for all $(x, \lambda) \in [dom(L) \setminus \ker L \cap \partial\Omega] \times (0; 1)$;

(2) $QNx \neq 0$ for all $x \in \ker L \cap \partial\Omega$,

(3) $\deg(QN_{|\ker L}, \Omega \cap \ker L, 0) \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in dom $L \cap \bar{\Omega}$.

This paper is organized as follows: in the second section, we will state some definitions and lemmas which will be useful in proving existence of solutions to BVP 5 -6 - 7, the third section is dedicated to the study of the linear and the nonlinear part by proving a series of lemmas, while in the fourth section, we will prove a main existence result. Finally, an example is given and discussed.

For the convenience of reader, we introduce some definitions and lemmas related to the fractional calculus and which will be used in the proof of an existence theorem. For more details see [1] and [4]. We denote by $C[0,1]$ the Banach space of all continuous functions on $[0,1]$ with the sup-norm $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$.

Definition 1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (0, +\infty) \to \mathbb{R}$ is given by*

$$I_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds,$$

where $\Gamma(\alpha)$ represents the gamma function, provided that the right-hand side is pointwisely defined on $(0, +\infty)$.

Definition 2. *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $g : (0, +\infty) \to \mathbb{R}$ is given by*

$$D_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, provided that the right-hand side is defined pointwise on $(0, +\infty)$. Here $[\alpha]$ denotes the integer part of the real number α.

For $\alpha < 0$, we set by convention $D_{0^+}^{\alpha} g(t) = I_{0^+}^{-\alpha} g(t)$. If $0 \leq \beta \leq \alpha$, we get $D_{0^+}^{\beta} I_{0^+}^{\alpha} g(t) = I_{0^+}^{\alpha-\beta} g(t)$.

Given these definitions, it can be checked that the Riemann-Liouville fractional integration and fractional differentiation operators of the power functions t^{λ} yield power functions of the same form

$$I_{0^+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)} t^{\lambda+\alpha}.$$

Also

$$D_{0^+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}.$$

Notice that $D_{0^+}^{\alpha} t^{\lambda} = 0$, for all $\lambda = \alpha - i$ with $i = 1, 2, 3, ..., n$ and n is the smallest integer greater than or equal to α. We can also prove the following auxiliary lemmas.

Lemma 1. *Suppose that* $g \in L^1(0, +\infty)$ *and* α, β *are positive real numbers. Then* $I_{0^+}^{\alpha} I_{0^+}^{\beta} g(t) = I_{0^+}^{\alpha+\beta} g(t)$, $D_{0^+}^{\alpha} I_{0^+}^{\alpha} g(t) = g(t)$. *If further,* $D_{0^+}^{\alpha} g(t) \in L^1(0, +\infty)$, *then* $I_{0^+}^{\alpha} D_{0^+}^{\alpha} g(t) = g(t) + \sum\limits_{i=1}^{i=n} c_i t^{\alpha-i}$, *for some* $c_i \in \mathbb{R}$.

Lemma 2. *For a given* $\mu > 0$, *the linear space defined by*

$$C^{\mu}[0,1] = \left\{ x;\quad x(t) = I_{0^+}^{\mu} z(t) + \sum_{i=1}^{i=[\mu]} c_i t^{\mu-i},\ z \in C[0,1] \right\},$$

where $[\mu]$ *is the integer part of* μ *and* $c_i \in \mathbb{R}$ *with the norm*

$$\|x\|_{C^{\mu}} = \|x\|_{\infty} + \sum_{i=0}^{i=[\mu]} \left\| D_{0^+}^{\mu-i} x \right\|_{\infty},$$

is a Banach space.

Lemma 3. $M \subset C^{\mu}[0,1]$ *is a relatively compact set if and only if*

1. M *is uniformly bounded, i.e. there exists* $m > 0$ *such that* $\|x\|_{C^{\mu}} \leq m$, *for every* $x \in M$.
2. M *is equicontinuous, i.e. for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for all* $t_1, t_2 \in [0,1]$, $|t_2 - t_1| < \delta$, *we get*

$$|x(t_2) - x(t_1)| < \varepsilon \quad \text{and} \quad \left| D_{0^+}^{\mu-i} x(t_2) - D_{0^+}^{\mu-i} x(t_1) \right| < \varepsilon,$$

for all $x \in M$ *with* $i = 0, 1, 2, \ldots, [\mu]$.

2. The functional framework and auxiliary lemmas

Let $X = C^{\alpha-1}[0,1] = \{x;\ x(t) = I^{\alpha-1}z(t), z \in C[0,1]\}$ with the norm $||x||_{C^{\alpha-1}} = ||x||_{\infty} + \left|\left|D_{0_+}^{\alpha-1}x\right|\right|_{\infty}$ and $Y = \{y \in C[0,1] \ / \ y(1) = 0\}$ with the norm $||y||_Y = ||y||_{\infty} = \max_{t\in[0;1]}|y(t)|$.

By functional analysis theory we can prove that $(X, ||.||_X)$ and $(Y, ||.||_Y)$ are both Banach spaces (see also [12]).

Define the operators $L : dom(L) \subset X \to Y$ and $N : dom(L) \subset X \to Y$ as follows:

$$Lx(.) = D_{0_+}^{\alpha}x(.)\ , \quad x \in dom(L), \tag{8}$$

$$N : X \to Y;\ Nx(t) = \phi_q\left(\int_1^t f\left(s, x(s), D_{0^+}^{\alpha-1}x(s)\right)ds\right), \tag{9}$$

where

$$dom(L) = \left\{x \in X\ ;\ \ D_{0_+}^{\alpha}x \in Y,\ x(0) = 0,\ D_{0_+}^{\alpha-1}x(1) = \sum_{i=1}^{m-2}\beta_i D_{0_+}^{\alpha-1}x(\eta_i)\right\}.$$

Note that the problem (5)-(6)-(7) can be converted to the abstract operator equation $Lx = Nx$, $x \in dom(L)$.

2.1. Linear part

In this subsection, we present some auxiliary lemmas which illustrate linear part of this problem.

Lemma 4. *Let L be the operator defined by (8). Then*

$\ker L = \{at^{\alpha-1}; a \in, t \in [0;1]\}$ *and* $\operatorname{Im} L = \left\{y \in Y; \sum\limits_{i=1}^{i=m-2}\beta_i\int\limits_{\eta_i}^{1}y(s)\,ds = 0\right\}$

Proof. We have for each $x \in \ker L$, $Lx(t) = D_{0^+}^{\alpha}x(t) = 0$ with $t \in [0;1]$. By applying Lemma (1), we get

$$x(t) = at^{\alpha-1} + bt^{\alpha-2}.$$

As $x(0) = 0$, we have $b = 0$. On the other hand, for $x(t) = at^{\alpha-1}$

$$Lx(t) = D_{0^+}^{\alpha}x(t) = a.D_{0^+}^{\alpha}t^{\alpha-1} = a.0 = 0.$$

Now for all $y \in \operatorname{Im} L$, there exits $x \in dom\,(L)$ such that

$$D^{\alpha}_{0^+} x\,(t) = y\,(t)\,.$$

Again Lemma (1) leads us to

$$x\,(t) = I^{\alpha}_{0^+} y\,(t) + at^{\alpha-1} + bt^{\alpha-2}$$

with $(a, b) \in^2$. By the boundary conditions (6), (7) and the resonance condition (4), we obtain $b = 0$ and

$$\int_0^1 y\,(s)\,ds + a\Gamma\,(\alpha) = a\Gamma\,(\alpha) + \sum_{i=1}^{i=m-2} \beta_i \int_0^{\eta_i} y\,(s)\,ds,$$

which is equivalent to $\sum\limits_{i=1}^{i=m-2} \beta_i \int_0^1 y\,(s)\,ds = \sum\limits_{i=1}^{i=m-2} \beta_i \int_0^{\eta_i} y\,(s)\,ds$,

so $\sum\limits_{i=1}^{i=m-2} \beta_i \left(\int_0^1 y\,(s)\,ds - \int_0^{\eta_i} y\,(s)\,ds \right) = 0$ and finally $\sum\limits_{i=1}^{i=m-2} \beta_i \int_{\eta_i}^1 y\,(s)\,ds = 0$.

Conversely, if y satisfies this last condition, then $y(t) = D^{\alpha}_{0^+} x\,(t) = (Lx)\,(t)$ and $x\,(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y\,(s)\,ds$, because

$x\,(t) = I^{\alpha-1}_{0^+} \left(I^1_{0^+} y \right)$ with $I^1_{0^+} y \in C\,[0;1]$ and $x\,(0) = 0$, $D^{\alpha}_{0^+} x\,(1) = y\,(1) = 0$. Also we have

$$\sum_{i=1}^{i=m-2} \beta_i D^{\alpha-1}_{0^+} x\,(\eta_i) = \sum_{i=1}^{i=m-2} \beta_i \int_0^{\eta_i} y\,(s)\,ds = \int_0^1 y\,(s)\,ds = D^{\alpha-1}_{0^+} x\,(1)\,,$$

which completes the proof. ◄

Remark 1. *It is easy to show that* $1 - \sum\limits_{i=1}^{i=m-2} \beta_i \eta_i > 0.$

In fact, for all $i = 1, 2, ..., m-2,\ \beta_i \geq 0,\ \eta_i \in \,]0;1[$ *we have* $\beta_i\,(1 - \eta_i) \geq 0$. *By the condition (4), there exists at least* $i_0 \in \{1, 2, ..., m-2\}$ *such that* $\beta_{i_0} \neq 0$ *and hence* $\beta_{i_0} \left(1 - \eta_{i_0}\right) > 0$, *which proves that*

$$\sum_{i=1}^{i=m-2} \beta_i\,(1 - \eta_i) = \sum_{i=1}^{i=m-2} \beta_i - \sum_{i=1}^{i=m-2} \beta_i \eta_i = 1 - \sum_{i=1}^{i=m-2} \beta_i \eta_i > 0.$$

Lemma 5. *We can define two linear continuous projectors* P *and* Q *as follows:*

$$P : X \to X; Px\,(t) = \frac{1}{\Gamma\,(\alpha)} D^{\alpha-1}_{0^+} x\,(0)\,t^{\alpha-1},$$

$$Q : Y \to Y;\ Qy(t) = \frac{1}{1 - \sum\limits_{i=1}^{i=m-2} \beta_i \eta_i} \sum_{i=1}^{i=m-2} \beta_i \int_{\eta_i}^{1} y(s)\, ds.$$

The inverse of the operator $L_P = L_{|dom(L) \cap \ker P}$ *is the operator* $K_P : \operatorname{Im} L \to dom(L) \cap \ker P$ *defined by* $K_P y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds$ *and*

$$\|K_P y\|_X \le \left(1 + \frac{1}{\Gamma(\alpha+1)}\right) \|y\|_\infty . \tag{10}$$

Proof. For each $x \in X$, putting $Px(t) = \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} x(0)\, t^{\alpha-1} = u(t)$ we get

$$P^2 x(t) = P(Px)(t) = Pu(t) = \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} u(0)\, t^{\alpha-1} = Px(t),$$

because $D_{0^+}^{\alpha-1} u(t) = \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} x(0)\, \Gamma(\alpha) = D_{0^+}^{\alpha-1} x(0)$. Thus $P^2 = P$. It is easy to see that $\operatorname{Im} P = \ker L$, $\ker P = \left\{x \in X; D_{0^+}^{\alpha-1} x(0) = 0\right\}$. On the other hand,

$$\begin{aligned} \|Px\|_X &= \|Px\|_\infty + \left\|D_{0^+}^{\alpha-1} Px\right\|_\infty = \frac{1}{\Gamma(\alpha)} \left|D_{0^+}^{\alpha-1} x(0)\right| + \left|D_{0^+}^{\alpha-1} x(0)\right| \\ &= \left(1 + \frac{1}{\Gamma(\alpha)}\right) \left|D_{0^+}^{\alpha-1} x(0)\right| \le \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|x\|_X . \end{aligned}$$

Let $\frac{1}{1 - \sum\limits_{i=1}^{i=m-2} \beta_i \eta_i} \sum\limits_{i=1}^{i=m-2} \beta_i \int_{\eta_i}^1 y(s)\, ds = r$. Then, for all $y \in Y$, $Q^2 y = Q(Qy) = \frac{1}{1 - \sum\limits_{i=1}^{i=m-2} \beta_i \eta_i} \sum\limits_{i=1}^{i=m-2} \beta_i \int_{\eta_i}^1 r ds = \frac{r}{1 - \sum\limits_{i=1}^{i=m-2} \beta_i \eta_i} \sum\limits_{i=1}^{i=m-2} \beta_i (1 - \eta_i) = r$, so $Q^2 = Q$. Furthermore, we have

$$\|Qy\|_Y \le \|y\|_Y .$$

For any $x \in dom(L) \cap \ker P$, by Lemma 1, we can write

$$K_P L x(t) = I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) = x(t) + a t^{\alpha-1} + b t^{\alpha-2},\ t \in [0;1],$$

where a, b are two real constants. As $K_P L x \in dom(L) \cap \ker P$, we have $b = 0$ and $D_{0^+}^{\alpha-1} \left(x(t) + a t^{\alpha-1}\right) |_{t=0} = D_{0^+}^{\alpha-1} x(0) + a\Gamma(\alpha) = a\Gamma(\alpha) = 0$, which implies that $a = 0$, therefore $K_P L x = x$. If $y \in \operatorname{Im} L$, we get $L K_P y(t) = D_{0^+}^{\alpha} I_{0^+}^{\alpha} y(t) = y(t)$, which shows that $K_P = (L_P)^{-1}$. We also have

$$\|K_P y\|_X = \|I_{0^+}^{\alpha} y\|_\infty + \left\|I_{0^+}^{1} y\right\|_\infty \le \frac{1}{\Gamma(\alpha+1)} \|y\|_\infty + \|y\|_\infty ,$$

which completes the proof. ◄

Lemma 6. *L defined in (8) is a Fredholm operator of index* 0.

Proof. For any $y \in Y$, we can write $y = (I - Q)\, y + Qy$, where $(I - Q)\, y \in \ker Q = \operatorname{Im} L$ and $Qy \in \operatorname{Im} Q$. Then $y \in \operatorname{Im} L + \operatorname{Im} Q$. Assume that $y \in \operatorname{Im} L \cap \operatorname{Im} Q$. Thus $y = c \in$ and $\sum\limits_{i=1}^{i=m-2} \beta_i \int_{\eta_i}^{1} cds = c \left(1 - \sum\limits_{i=1}^{i=m-2} \beta_i \eta_i\right) = 0$ i.e $c = 0$, because $1 - \sum\limits_{i=1}^{i=m-2} \beta_i \eta_i \neq 0$. Therefore $\operatorname{Im} L \cap \operatorname{Im} Q = \{0\}$, which proves that $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. Finally, as $\dim \operatorname{Im} Q = \dim \ker L = 1$, L is a Fredholm operator of index 0. ◄

2.2. Nonlinear Part

Lemma 7. *Assume that B is an open bounded subset in X such that dom $(L) \cap B \neq \phi$. The operator N defined by (9) is L-compact on $\overline{B}$.*

Proof. We will apply Lemma 3. The boundness of B implies that there exists $R > 0$ such that for all $x \in B$, we have $\|x\|_X = \|x\|_\infty + \left\|D_{0^+}^{\alpha-1} x\right\|_\infty \leq R$. By the continuity of ϕ_q and f, there exists $d > 0$ such that for all $x \in B$, we get $\left|f\left(s, x(s), D_{0^+}^{\alpha-1} x(s)\right)\right| \leq d$. Then

$$\|QNx\|_\infty \leq \|Nx\|_\infty \leq d^{q-1},$$

because

$$\left|\phi_q\left(\int_1^t f\left(s, x(s), D_{0^+}^{\alpha-1} x(s)\right) ds\right)\right| = \left|\int_t^1 f\left(s, x(s), D_{0^+}^{\alpha-1} x(s)\right) ds\right|^{q-1} \leq d^{q-1}.$$

Also we have $\|(I - Q)\, Nx\|_\infty \leq \|Nx\|_\infty + \|QNx\|_\infty \leq 2d^{q-1}$. Then from (10), we conclude that

$$\|K_P (I - Q)\, Nx\|_x \leq \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|(I - Q)\, Nx\|_\infty \leq \left(1 + \frac{1}{\Gamma(\alpha)}\right) 2d^{q-1}.$$

Then $QN(B)$ and $K_{P,Q}N(B)$ are bounded.

It remains to prove that $K_{P,Q}N(B)$ is equicontinuous. Putting $0 \leq t_1 \leq t_2 \leq 1$, we have

$$\begin{aligned} & |K_P (I - Q)\, Nx(t_2) - K_P (I - Q)\, Nx(t_1)| \\ = \; & \frac{1}{\Gamma(\alpha)} \left|\int_0^{t_2} (t_2 - s)^{\alpha-1} (I - Q)\, Nx(s)\, ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} (I - Q)\, Nx(s)\, ds\right| \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)}\left|\begin{array}{c}\int_0^{t_1}(t_2-s)^{\alpha-1}Nx(s)\,ds-\int_0^{t_1}(t_1-s)^{\alpha-1}Nx(s)\,ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\times\\ \times Nx(s)\,ds+QNx\left(-\int_0^{t_2}(t_2-s)^{\alpha-1}\,ds+\int_0^{t_1}(t_1-s)^{\alpha-1}\,ds\right)\end{array}\right|\\
&= \frac{1}{\Gamma(\alpha)}\left|\begin{array}{c}\int_0^{t_1}(t_2-s)^{\alpha-1}Nx(s)\,ds-\int_0^{t_1}(t_1-s)^{\alpha-1}Nx(s)\,ds\\ +\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}Nx(s)\,ds-QNx\left(\frac{t_2^\alpha-t_1^\alpha}{\alpha}\right)\end{array}\right|\\
&\le \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\left((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right)Nx(s)\,ds\right|+\\
&\quad\left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}Nx(s)\,ds\right|+|QNx|\left|\frac{t_2^\alpha-t_1^\alpha}{\alpha}\right|\\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right)|Nx(s)|\,ds+\\
&\quad\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|Nx(s)|\,ds+|QNx|\frac{t_2^\alpha-t_1^\alpha}{\alpha}\\
&\le \frac{1}{\Gamma(\alpha)}d^{q-1}\left(\int_0^{t_1}\left((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right)ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds+\frac{t_2^\alpha-t_1^\alpha}{\alpha}\right)\\
&= \frac{2d^{q-1}}{\Gamma(\alpha+1)}\left(t_2^\alpha-t_1^\alpha\right)\ \ \to 0\ \ \text{as}\ \ \ t_1\to t_2\ \ \text{uniformly.}
\end{aligned}
$$

Similarly, we have $D_{0^+}^{\alpha-1}K_P(I-Q)Nx(t)=I_{0^+}^1(I-Q)Nx=\int_0^t(I-Q)Nx(s)\,ds$. Then

$$
\begin{aligned}
&\left|D_{0^+}^{\alpha-1}K_P(I-Q)Nx(t_2)-D_{0^+}^{\alpha-1}K_P(I-Q)Nx(t_1)\right|\\
&= \left|\int_0^{t_2}(I-Q)Nx(s)\,ds-\int_0^{t_1}(I-Q)Nx(s)\,ds\right|\\
&= \left|\int_{t_1}^{t_2}Nx(s)\,ds-(t_2-t_1)QNx\right|
\end{aligned}
$$

$$\leq \int_{t_1}^{t_2} |Nx(s)|\, ds + |QNx| (t_2 - t_1)$$
$$\leq 2d^{q-1} (t_2 - t_1) \to 0 \text{ as } t_1 \to t_2.$$

Therefore, $K_P (I - Q) N (B)$ is compact, which shows that N is L-compact on B. ◄

3. Existence result

Theorem 2. *Suppose that there exist*

$(H1)$ *a function* $\Psi : [0;1] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, *which is* L^1-*Caratheodory and non decreasing with respect to the last two variables such that*

$$|f(t, x, y)| \leq \Psi(t, |x|, |y|),$$

for all $(x; y) \in \mathbb{R}^2$ *and a.e.* $t \in [0; 1]$,

$(H2)$ *a real* $M_0 > 0$, *such that if we have* $\left|D_{0^+}^{\alpha-1} x(t)\right| > M_0$ *for all* $t \in [0; 1]$, *then*

$$\sum_{i=1}^{i=m-2} \beta_i \int_{\eta_i}^{1} \phi_q \left(\int_{1}^{t} f\left(s, x(s), D_{0^+}^{\alpha-1} x(s)\right) ds \right) dt \neq 0,$$

and

$(H3)$ *a real* $M_1 > 0$, *such that for* $|a| > M_1$, *we get*

either

$$af\left(s, as^{\alpha-1}, a\Gamma(\alpha)\right) ds < 0 \tag{11}$$

or

$$af\left(s, as^{\alpha-1}, a\Gamma(\alpha)\right) ds > 0. \tag{12}$$

Then the fractional B.V.Ps (5)- (6)-(7) has at least one solution in dom $(L) \subset X$, *provided that*

$$\int_0^1 \Psi(t, m, m)\, dt \leq \left(\frac{\Gamma(\alpha)}{8} m + \beta\right)^{\frac{1}{q-1}}, \text{.for all } m \in \mathbb{R}_+ \tag{13}$$

where β *is a real positive constant.*

Proof. **Step 1**: Let

$$\Omega_1 = \{x \in dom\,(L) - \ker L : Lx = \lambda Nx, \lambda \in [0;1]\}.$$

We will show that it is a bounded set. Note that if $x \in \Omega_1$ then $\lambda \neq 0$, because $\Omega_1 \cap \ker L = \phi$, which allows us to write $Nx = L\frac{1}{\lambda}x \in \operatorname{Im} L = \ker Q$. Then

$$QNx = \sum_{i=1}^{i=m-2} \beta_i \int_{\eta_i}^{1} \phi_q \left(\int_1^t f\left(s, x(s), D_{0^+}^{\alpha-1} x(s)\right) ds \right) dt = 0.$$

By the condition (H_2), there exists $t_0 \in [0;1]$ such that $\left|D_{0^+}^{\alpha-1} x(t_0)\right| \leq M_0$. It's clear that $\frac{d}{dt}\left(D_{0^+}^{\alpha-1} x(t)\right) = D_{0^+}^{\alpha} x(t)$, so

$$\begin{aligned} D_{0^+}^{\alpha-1} x(t) &= D_{0^+}^{\alpha-1} x(t_0) + \int_{t_0}^{t} D_{0^+}^{\alpha} x(s)\, ds \\ &= D_{0^+}^{\alpha-1} x(t_0) + \lambda \int_{t_0}^{t} \phi_q \left(\int_1^r f\left(s, x(s), D_{0^+}^{\alpha-1} x(s)\right) ds \right) dr. \end{aligned}$$

Then

$$\left|D_{0^+}^{\alpha-1} x(t)\right| \leq \left|D_{0^+}^{\alpha-1} x(t_0)\right| +$$

$$+\left| \int_{t_0}^{t} \phi_q \left(\int_1^r f\left(s, x(s), D_{0^+}^{\alpha-1} x(s)\right) ds \right) dr \right| \leq M_0 + \|Nx\|_\infty. \qquad (14)$$

Furthermore, we can write

$$\begin{aligned} x = (I-P)\,x + Px \quad &= K_P L\,(I-P)\,x + Px \\ &= K_P Lx + Px. \end{aligned}$$

Then

$$\|x\|_X \leq \|K_P Lx\|_X + \|Px\|_X.$$

By using (14), we obtain $|Px(t)| = \frac{1}{\Gamma(\alpha)}\left|D_{0^+}^{\alpha-1} x(0)\right| t^{\alpha-1} \leq \frac{1}{\Gamma(\alpha)}\left|D_{0^+}^{\alpha-1} x(0)\right| \leq \frac{1}{\Gamma(\alpha)}\left(M_0 + \|Nx\|_\infty\right)$ and $\left|D_{0^+}^{\alpha-1} Px(t)\right| = \left|D_{0^+}^{\alpha-1} x(0)\right| \leq M_0 + \|Nx\|_\infty$. Then

$$\|Px\|_X \leq \left(1 + \frac{1}{\Gamma(\alpha)}\right)\left(M_0 + \|Nx\|_\infty\right).$$

In view of (10), we have $\|K_P Lx\|_X \leq \left(1 + \frac{1}{\Gamma(\alpha+1)}\right)\|Lx\|_\infty \leq \left(1 + \frac{1}{\Gamma(\alpha+1)}\right)\|Nx\|_\infty$, which gives

$$\|x\|_X \leq \left(1 + \frac{1}{\Gamma(\alpha)}\right) M_0 + \left(2 + \frac{\alpha+1}{\Gamma(\alpha+1)}\right)\|Nx\|_\infty. \qquad (15)$$

It is easy to see that

$$
\begin{aligned}
|Nx(t)| &= \left| \phi_q \left(\int_1^t f\left(s, x(s), D_{0^+}^{\alpha-1} x(s)\right) ds \right) \right| \\
&= \left| \int_1^t f\left(s, x(s), D_{0^+}^{\alpha-1} x(s)\right) ds \right|^{q-1} \\
&\leq \left(\int_0^1 \left| f\left(s, x(s), D_{0^+}^{\alpha-1} x(s)\right) \right| ds \right)^{q-1}.
\end{aligned} \tag{16}
$$

According to conditions (H_1) and (13), we obtain

$$
\begin{aligned}
\int_0^1 \left| f\left(s, x(s), D_{0^+}^{\alpha-1} x(s)\right) \right| ds &\leq \int_0^1 \Psi\left(s, |x(s)|, \left|D_{0^+}^{\alpha-1} x(s)\right|\right) ds \\
&\leq \int_0^1 \Psi\left(s, \|x\|_X, \|x\|_X\right) ds \\
&\leq \left(\frac{\Gamma(\alpha)}{8} \|x\|_X + \beta \right)^{\frac{1}{q-1}}.
\end{aligned} \tag{17}
$$

By using (16) and (17) we get

$$
\|Nx\|_\infty \leq \left(\int_0^1 \left| f\left(s, x(s), D_{0^+}^{\alpha-1} x(s)\right) \right| ds \right)^{q-1} \leq \frac{\Gamma(\alpha)}{8} \|x\|_X + \beta.
$$

Substituting this result in (15) , we conclude that
$\|x\|_X \leq \left(1 + \frac{1}{\Gamma(\alpha)}\right) M_0 + \left(2 + \frac{\alpha+1}{\Gamma(\alpha+1)}\right) \left(\frac{\Gamma(\alpha)}{8} \|x\|_X + \beta\right)$,
hence

$$
\|x\|_X \leq \frac{\left(1 + \frac{1}{\Gamma(\alpha)}\right) M_0 + \left(2 + \frac{\alpha+1}{\Gamma(\alpha+1)}\right) \beta}{1 - \left(2 + \frac{\alpha+1}{\Gamma(\alpha+1)}\right) \frac{\Gamma(\alpha)}{8}},
$$

where $1 - \left(2 + \frac{\alpha+1}{\Gamma(\alpha+1)}\right) \frac{\Gamma(\alpha)}{8} = 1 - \left(\frac{2\alpha\Gamma(\alpha)+\alpha+1}{8\alpha}\right) > 0$. As $1 < \alpha < 2$, we have $0 < \Gamma(\alpha) < 1$. Consequently $0 < 2\alpha\Gamma(\alpha) + \alpha + 1 < 7$ and $8\alpha > 8$. Thus Ω_1 is bounded.

Step 2: Let

$$\Omega_2 = \{x \in \ker L : Nx \in \operatorname{Im} L\}.$$

For all $x \in \Omega_2$ there exists a real constant a such that $x(t) = at^{\alpha-1}$, $t \in [0;1]$ and as $Nx \in \operatorname{Im} L$, we have

$$QNx = 0.$$

In view of (H_2), there exists $t_1 \in [0;1]$ satisfying $\left|D_{0^+}^{\alpha-1} x(t_1)\right| = |a\Gamma(\alpha)| \leq M_0$, i.e $|a| \leq \frac{M_0}{\Gamma(\alpha)}$, which yields

$$\|x\|_X = |a| + |a\Gamma(\alpha)| = |a|(1+\Gamma(\alpha)) \leq \frac{M_0}{\Gamma(\alpha)}(1+\Gamma(\alpha)).$$

Then Ω_2 is bounded.

Step 3: Assume that the condition $(H_3) - (11)$ holds. Let

$$\Omega_3 = \{x \in \ker L : \lambda Jx + (1-\lambda) QNx = 0, \lambda \in [0;1]\},$$

where J is the isomorphism defined by $J : KerL \to \operatorname{Im} Q$ such that $J\left(at^{\alpha-1}\right) = a$.

For $x = at^{\alpha-1} \in \Omega_3$, we have

$$\lambda Jx + (1-\lambda) QNx =$$

$$= \lambda a + \frac{1-\lambda}{1 - \sum\limits_{i=1}^{i=m-2} \beta_i \eta_i} \sum_{i=1}^{i=m-2} \beta_i \int_{\eta_i}^{1} \phi_q \left(\int_{1}^{t} f\left(s, as^{\alpha-1}, a\Gamma(\alpha)\right) ds \right) dt = 0. \quad (18)$$

If $\lambda = 0$, we get $QNx = 0$. So by the condition (H_2) there exists $t_2 \in [0;1]$ such that $\left|D_{0^+}^{\alpha-1} x(t_2)\right| = |a\Gamma(\alpha)| \leq M_0$. So $|a| \leq \frac{M_0}{\Gamma(\alpha)}$ and hence

$$\|x\|_X = |a| + |a\Gamma(\alpha)| = |a|(1+\Gamma(\alpha)) \leq \frac{M_0}{\Gamma(\alpha)}(1+\Gamma(\alpha)).$$

In the case $\lambda \neq 0$, multiplying both sides of (18) by $\phi_q(a)$ and in view of the condition $(H_3) - (11)$, we get

$$-\lambda a^2 |a|^{q-2} = \frac{1-\lambda}{1 - \sum\limits_{i=1}^{i=m-2} \beta_i \eta_i} \sum_{i=1}^{i=m-2} \beta_i \int_{\eta_i}^{1} \phi_q \left(\int_{1}^{t} af\left(s, as^{\alpha-1}, a\Gamma(\alpha)\right) ds \right) dt > 0,$$

which contradicts (11). Then $|a| \leq M_1$, which shows that Ω_3 is bounded.

If $(H_3) - 12$ holds, we prove by the same method that

$$\Omega_3 = \{x \in \ker L : -\lambda Jx + (1-\lambda)\, QNx = 0, \lambda \in [0;1]\}$$

is a bounded set. It remains to check that all conditions of Theorem 1 are fulfilled. Let Ω be a bounded open set containing $\Omega_1 \cup \Omega_2 \cup \Omega_3$. As $\Omega_1, \Omega_2, \Omega_3$ are bounded sets, we have

1) $Lx \neq \lambda Nx$ for all $(x,\lambda) \in [dom(L) \setminus \ker L \cap \partial\Omega] \times (0;1)$;

2) $QNx \neq 0$ for all $x \in \ker L \cap \partial\Omega$..

3) Without loss of generality, assume that $(H_3) - (11\ )$ holds and define the operator

$$F(x,\lambda) = \lambda Jx + (1-\lambda) QNx.$$

As Ω_3 is bounded, $F(\lambda, x) \neq 0$ for all $(x,\lambda) \in (\ker L \cap \partial\Omega) \times (0;1)$. Thus, by the homotopy property of degree, we have

$$\begin{aligned}\deg(QN_{|\ker L}, \Omega \cap \ker L, 0) &= \deg(F(.,0), \Omega \cap \ker L, 0)\\ &= \deg(F(.,1), \Omega \cap \ker L, 0)\\ &= \deg(J, \Omega \cap \ker L, 0)\\ &\neq 0.\end{aligned}$$

Consequently, the equation $Lx = Nx$ has at least one solution in $dom(L) \subset X$. Namely, BVPs (5)-(6)-(7) has at least one solution in the space X. ◀

4. Example

Consider the following fractional differential boundary value problem with p-Laplacian

$$\begin{cases} \left[\phi_p\left(D_{0^+}^{\frac{5}{4}} u(t)\right)\right]' = \frac{p}{2^p(2^p-1)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}\left(\sin u(t) + D_{0^+}^{\frac{1}{4}} u(t) - 2\right)(t+1)\right], t \in [0;1]\,, \\ u(0) = 0 = D_{0^+}^{\frac{5}{4}} u(1)\,; \\ D_{0^+}^{\frac{1}{4}} u(1) = \frac{1}{2} D_{0^+}^{\frac{1}{4}} u\left(\frac{2}{5}\right) + \frac{1}{3} D_{0^+}^{\frac{1}{4}} u\left(\frac{3}{7}\right) + \frac{1}{6} D_{0^+}^{\frac{1}{4}} u\left(\frac{4}{9}\right), \end{cases} \tag{19}$$

where $\alpha = \frac{5}{4}; \beta_1 = \frac{1}{2}, \beta_2 = \frac{1}{3}, \beta_3 = \frac{1}{6}; \eta_1 = \frac{2}{5}, \eta_2 = \frac{3}{7}, \eta_3 = \frac{4}{9}$ and $f(t.u.v) = \frac{p}{2^p(2^p-1)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}(\sin u + v)(t+1)\right]$. Then, we have

$$\begin{aligned}|f(t.u.v)| &= \left|\frac{p}{2^p(2^p-1)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}(\sin u + v - 2)(t+1)\right]\right|\\ &= \frac{p}{2^p(2^p-1)}\left(\frac{\Gamma\left(\frac{5}{4}\right)}{8}(t+1)\right)^{p-1} |(\sin u + v - 2)|^{p-1}\end{aligned}$$

$$\leq \tfrac{p}{2^p(2^p-1)}\left(\tfrac{\Gamma\left(\frac{5}{4}\right)}{8}(t+1)\right)^{p-1}(|\sin u|+|v|+2)^{p-1}$$
$$\leq \tfrac{p}{2^p(2^p-1)}\left(\tfrac{\Gamma\left(\frac{5}{4}\right)}{8}(t+1)\right)^{p-1}(|u|+|v|+2)^{p-1}=\Psi(t,|u|,|v|).$$

It's clear that $\Psi:[0;1]\times_{+}\times_{+}\rightarrow_{+}$,

$$\Psi(t,x,y)=\frac{p}{2^p(2^p-1)}\left(\frac{\Gamma\left(\frac{5}{4}\right)}{8}(t+1)\right)^{p-1}(x+y+2)^{p-1}$$

is an L^1-Caratheodory function, non decreasing with respect to each of the variables x and y, and

$$\int_0^1\Psi(t,m,m)\,dt=\tfrac{p}{2^p(2^p-1)}\left(\tfrac{\Gamma\left(\frac{5}{4}\right)}{8}\right)^{p-1}(2m+2)^{p-1}\int_0^1(t+1)^{p-1}\,dt$$
$$=\tfrac{p}{2^p(2^p-1)}\left(\tfrac{\Gamma\left(\frac{5}{4}\right)}{8}\right)^{p-1}(2m+2)^{p-1}\tfrac{2^p-1}{p}$$
$$=\tfrac{1}{2}\left(\tfrac{\Gamma\left(\frac{5}{4}\right)}{8}(m+1)\right)^{\frac{1}{q-1}}\leq\left(\tfrac{\Gamma\left(\frac{5}{4}\right)}{8}(m+1)\right)^{\frac{1}{q-1}}.$$

Let $M_0=5$. If we have $D_{0^+}^{\frac{1}{4}}u(t)>5$ for all $s\in[0;1]$, then $\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s)-2>2$, so $\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s)-2\right)>\frac{\Gamma\left(\frac{1}{4}\right)}{16}(s+1)$. As ϕ_p,ϕ_q are nondecreasing, $p,q>1$, $0<\frac{s+1}{2}\leq 1$ for all $s\in[0;1]$ and $\eta_i\in\,]0;1[\ (i=1,2,3)$, we have $\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s)-2\right)\right]>\left(\frac{\Gamma\left(\frac{1}{4}\right)}{16}(s+1)\right)^{p-1}$, which leads us to

$$\int_t^1\frac{p}{2^p(2^p-1)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s)-2\right)\right]ds>$$
$$>\int_t^1\frac{p}{2^p(2^p-1)}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{16}(s+1)\right)^{p-1}ds=\frac{1}{2^p-1}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{16}\right)^{p-1}\left(1-\left(\frac{t+1}{2}\right)^p\right).$$

Then

$$\int_1^t\frac{p}{2^p(2^p-1)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s)+D_{0^+}^{\frac{1}{4}}u(s)-2\right)\right]ds<$$
$$<\frac{1}{2^p-1}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{16}\right)^{p-1}\left(\left(\frac{t+1}{2}\right)^p-1\right)$$

$$\leq \frac{1}{2^p - 1}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{16}\right)^{p-1}\left(\frac{t+1}{2} - 1\right).$$

Thus

$$\phi_q\left(\int_1^t \frac{p}{2^p\left(2^p-1\right)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s) + D_{0+}^{\frac{1}{4}}u(s) - 2\right)\right]ds\right) \leq$$

$$\leq \frac{1}{\left(2^p-1\right)^{q-1}}\frac{\Gamma\left(\frac{1}{4}\right)}{16}\left(-\left(\frac{1-t}{2}\right)^{q-1}\right),$$

so

$$\int_{\eta_i}^1 \phi_q\left(\int_1^t \frac{p}{2^p}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s) + D_{0+}^{\frac{1}{4}}u(s) - 2\right)\right]ds\right)dt \leq$$

$$\leq \frac{1}{\left(2^p-1\right)^{q-1}}\frac{\Gamma\left(\frac{1}{4}\right)}{16}\left(-\frac{2}{q}\left(\frac{1-\eta_i}{2}\right)^{q}\right) < 0,$$

which proves that

$$\sum_{i=1}^{i=3}\beta_i\int_{\eta_i}^1 \phi_q\left(\int_1^t \frac{p}{2^p}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}s+1\left(\sin u(s) + D_{0+}^{\frac{1}{4}}u(s)\right)\right]\right)ds < 0 \quad (\text{i.e} \neq 0),$$

because $\beta_i > 0$ for $i = 1, 2, 3$.

Now, if we assume that $D_{0+}^{\frac{1}{4}}u(t) < -5$ for all $t \in [0;1]$, we similarly get $\sin u(s) + D_{0+}^{\frac{1}{4}}u(s) - 2 < -6$. Thus, $\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s) + D_{0+}^{\frac{1}{4}}u(s) - 2\right) < -3\frac{\Gamma\left(\frac{1}{4}\right)}{16}(s+1)$.

Consequently

$$\frac{p}{2^p(2^p-1)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}s\left(\sin u(s) + D_{0+}^{\frac{1}{4}}u(s) - 2\right)\right] < -\frac{p}{2^p(2^p-1)}\left(3.\frac{\Gamma\left(\frac{1}{4}\right)}{16}(s+1)\right)^{p-1}$$

and

$$\int_t^1 \frac{p}{2^p\left(2^p-1\right)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s) + D_{0+}^{\frac{1}{4}}u(s) - 2\right)\right]ds <$$

$$< -\int_t^1 \frac{p}{2^p\left(2^p-1\right)}\left(3\frac{\Gamma\left(\frac{1}{4}\right)}{16}(s+1)\right)^{p-1}ds =$$

$$= \frac{1}{2^p(2^p-1)}\left(3\frac{\Gamma\left(\frac{1}{4}\right)}{16}\right)^{p-1}\left((t+1)^p - 2^p\right).$$

Multiplying by (-1) we find

$$\int_1^t \frac{p}{2^p(2^p-1)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}s\left(\sin u(s) + D_{0^+}^{\frac{1}{4}}u(s) - 2\right)\right]ds >$$

$$> \frac{1}{2^p(2^p-1)}\left(3\frac{\Gamma\left(\frac{1}{4}\right)}{16}\right)^{p-1}\left(2^p - (t+1)^p\right)$$

$$= \frac{1}{2^p-1}\left(3\frac{\Gamma\left(\frac{1}{4}\right)}{16}\right)^{p-1}\left(1-\left(\frac{t+1}{2}\right)^p\right)$$

$$\geq \frac{1}{2^p-1}\left(3\frac{\Gamma\left(\frac{1}{4}\right)}{16}\right)^{p-1}\left(1-\frac{t+1}{2}\right).$$

Then

$$\phi_q\left(\int_1^t \frac{p}{2^p(2^p-1)}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s) + D_{0^+}^{\frac{1}{4}}u(s) - 2\right)\right]ds\right) \geq$$

$$\geq \frac{3}{(2^p-1)^{q-1}}\frac{\Gamma\left(\frac{1}{4}\right)}{16}\left(\frac{1-t}{2}\right)^{q-1},$$

so,

$$\int_{\eta_i}^1 \phi_q\left(\int_1^t \frac{p}{2^p}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\left(\sin u(s) + D_{0^+}^{\frac{1}{4}}u(s) - 2\right)\right]ds\right)dt \geq$$

$$\geq \frac{3}{(2^p-1)^{q-1}}\frac{\Gamma\left(\frac{1}{4}\right)}{16}\frac{2}{q}\left(\frac{1-\eta_i}{2}\right)^q > 0.$$

Therefore

$$\sum_{i=1}^{i=3}\beta_i\int_{\eta_i}^1 \phi_q\left(\int_1^t \frac{p}{2^p}\phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}s+1\left(\sin u(s) + D_{0^+}^{\frac{1}{4}}u(s)\right)\right]\right)ds > 0 \text{ (i.e. } \neq 0).$$

Set $M_1 = \frac{2}{\Gamma\left(\frac{5}{4}\right)}$. For all $a \in$ such that $|a| > M_1$

$$af\left(s, as^{\frac{1}{4}}, a\Gamma\left(\frac{5}{4}\right)\right) =$$

$$= \frac{\phi_p(a)}{|a|^{p-2}} \frac{p}{2^p(2^p-1)} \phi_p\left[\frac{\Gamma\left(\frac{1}{4}\right)}{32}\left(\sin as^{\frac{1}{4}} + a\Gamma\left(\frac{5}{4}\right) - 2\right)(s+1)\right] =$$

$$= \frac{1}{|a|^{p-2}} \frac{p}{2^p(2^p-1)} \left(\frac{\Gamma\left(\frac{1}{4}\right)}{32}(s+1)\right)^{p-1} \phi_p\left[a\left(\sin as^{\frac{1}{4}} + a\Gamma\left(\frac{5}{4}\right) - 2\right)\right].$$

If $a > \frac{4}{\Gamma\left(\frac{5}{4}\right)}$, we have $a > 0$ and $\sin as^{\frac{1}{4}} + a\Gamma\left(\frac{5}{4}\right) - 2 \geq -1 + a\Gamma\left(\frac{5}{4}\right) - 2 > -1 + 4 - 2 = 1 > 0$. Then $af\left(s, as^{\frac{1}{4}}, a\Gamma\left(\frac{5}{4}\right)\right) > 0$. If $a < -\frac{4}{\Gamma\left(\frac{5}{4}\right)}$, we get $a < 0$ and $\sin as^{\frac{1}{4}} + a\Gamma\left(\frac{5}{4}\right) - 2 \leq 1 + a\Gamma\left(\frac{5}{4}\right) < 1 - 4 - 2 = -5 < 0$, thus $af\left(s, as^{\frac{1}{4}}, a\Gamma\left(\frac{5}{4}\right)\right) > 0$. The problem (19) satisfies all conditions of theorem 2, therefore it admits at least one solution in $C^{\frac{1}{4}}[0,1]$.

5. Conclusion

In the present work, we considered a class of fractional differential equations with multi-point boundary conditions at resonance with p-Laplacian operator. Using the J.Mawhin's aid of coincidence degree theory, we obtained existence results for solutions of BVP (1). An illustrative example was presented to test the applicability of the main results. BVPs of fractional differential equations at resonance with p-Laplacian are widely discussed in recent years but by using other methodology. However, there is still more work to be done in the future on this important problem. For example, establishing the uniqueness of solutions.

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Received 14 June 2021

Accepted 02 July 20221