The Symmetric $H_q$-Laguerre-Hahn Orthogonal Polynomials of Class Zero

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Abstract. We consider the system of Laguerre-Freud equations associated with the $H_q$-Laguerre-Hahn orthogonal polynomials of class zero. This system is solved in the symmetric case. There are essentially three canonical cases.

Key Words and Phrases: orthogonal polynomials, $q$-difference operator, $H_q$-Laguerre-Hahn forms.

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1. Introduction and preliminary results

The D-Laguerre-Hahn polynomial sequences, where $D$ is the derivative operator, have attracted the interest of researchers from many points of view (see [1, 2, 4, 5, 10] among others). They constitute a very remarkable family of orthogonal polynomials taking into consideration most of the monic orthogonal polynomial sequences (MOPS) found in literature. In particular, semiclassical orthogonal polynomials are Laguerre-Hahn MOPS [11]. The concept of D-Laguerre-Hahn polynomial sequences has been extended to discrete Laguerre-Hahn polynomials which are related to a divided difference operator. Indeed, the $H_q$-Laguerre-Hahn polynomial sequences, where $H_q$ is the Hahn’s operator, have been considered by several authors. Essentially, an algebraic theory was presented in [6, 9]. These families are extensions of discrete semiclassical polynomials [8, 7].

The D-Laguerre-Hahn polynomial sequences of class zero are completely described in [2]. For the difference operator $D_w$, the $D_w$-Laguerre-Hahn polynomial sequences of class zero has been analyzed in [12]. So, the aim of this paper is to determine the symmetric $H_q$-Laguerre-Hahn forms of class $s = 0$, through the study of the differential functional equation fulfilled by these forms and the

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resolution of a nonlinear system satisfied by the coefficients of the three-term recurrence relation of their sequences of monic orthogonal polynomials.

The structure of the manuscript is as follows. The first section contains material of preliminary and some results regarding the $H_q$-Laguerre-Hahn forms. In the second section, the system of Laguerre-Freud equations is built. In the third section, using this system, we obtain the symmetric sequences which we look for.

Let $P$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $P'$ be its dual. We denote by $\langle u, f \rangle$ the action of $u \in P'$ on $f \in P$. In particular, we denote by $(u)_n := \langle u, x^n \rangle, n \geq 0$, the moments of $u$. For instance, for any form $u$, any polynomial $g$ and any $a \in \mathbb{C}\{0\}$, we let $Du = u', gu, h_a u$ and $x^{-1}u$ be the forms defined by duality
\[
\langle u', f \rangle := -\langle u, f' \rangle, \quad \langle gu, f \rangle := \langle u, gf \rangle, \quad \langle h_a u, f \rangle := \langle u, h_a f \rangle, \quad \langle x^{-1}u, f \rangle := \langle u, \theta_0 f \rangle, \quad f \in P,
\]
where $(h_a f)(x) = f(ax)$ and $(\theta_0 f)(x) = \frac{f(x) - f(0)}{x}$.

We also define the right-multiplication of a form $u$ by a polynomial $h$ as
\[
(uh)(x) := \langle u, \frac{xh(x) - \xi h(\xi)}{x - \xi} \rangle = \sum_{i=0}^{n} \left( \sum_{j=1}^{n} a_j(v)_{j-i} \right) x^i, \quad h(x) = \sum_{i=0}^{n} a_i x^i. \quad (1)
\]
Next, it is possible to define the product of two forms through
\[
\langle uv, f \rangle := \langle u, vf \rangle, \quad u, v \in P', \quad f \in P.
\]

A form $u$ is called regular if there exists a sequence of polynomials $\{S_n\}_{n \geq 0}$ ($\deg S_n \leq n$) such that
\[
\langle u, S_n S_m \rangle = r_{n \delta_{n,m}}, \quad r_n \neq 0, \quad n \geq 0.
\]
Then, $\deg S_n = n, n \geq 0$ and we can always suppose each $S_n$ is monic. In such a case, the sequence $\{S_n\}_{n \geq 0}$ is unique. The sequence $\{S_n\}_{n \geq 0}$ is said to be the sequence of monic orthogonal polynomials with respect to $u$. In the sequel, it will be denoted as MOPS. It is a very well known fact that the sequence $\{S_n\}_{n \geq 0}$ satisfies the recurrence relation [3]
\[
S_{n+2}(x) = (x - \beta_{n+1})S_{n+1}(x) - \gamma_{n+1} S_n(x), \quad n \geq 0,
\]
\[
S_1(x) = x - \beta_0, \quad S_0(x) = 1,
\]
with $(\beta_n, \gamma_{n+1}) \in \mathbb{C} \times \mathbb{C}\{0\}, \quad n \geq 0$. By convention we set $\gamma_0 = (u)_0$. The form $u$ is called normalized if $(u)_0 = 1$. 

We recall that a form \( u \) is called symmetric if \( (u)_{2n+1} = 0, n \geq 0 \). The conditions \((u)_{2n+1} = 0, n \geq 0\) are equivalent to the fact that the corresponding MOPS \( \{S_n\}_{n \geq 0} \) satisfies the three-term recurrence relation (2) with \( \beta_n = 0, n \geq 0 \) [3].

Let us introduce the Hahn’s operator [9]

\[
(H_q f)(x) = \frac{f(qx) - f(x)}{(q - 1)x}, \quad f \in \mathcal{P}, \quad q \in \tilde{\mathbb{C}},
\]

where \( \tilde{\mathbb{C}} := \mathbb{C} - \left( \{0\} \cup \left( \bigcup_{n \geq 0} \{z \in \mathbb{C}, \quad z^n = 1\} \right) \right) \). When \( q \to 1 \), we meet again the derivative \( D \).

By duality, we can define \( H_q \) from \( \mathcal{P}' \) to \( \mathcal{P}' \) such that

\[
\langle H_q u, f \rangle = -\langle u, H_q f \rangle, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}'.
\]

In particular, this yields \( (H_q u)_n = -[n]_q (u)_{n-1} , n \geq 0 \) with \( (u)_{-1} = 0 \) and \( [n]_q := \frac{q^n - 1}{q - 1} , \quad n \geq 0 \).

**Definition 1.** [6] The regular form \( u \) is called a \( H_q \)-Laguerre-Hahn form when it is regular and there exist three polynomials \( \Phi \) (monic) , \( \Psi \) and \( B \), \( \deg(\Phi) = t \geq 0 \), \( \deg(\Psi) = p \geq 1 \), \( \deg(B) = r \geq 0 \), such that

\[
H_q(\Phi u) + \Psi u + B(x^{-1}u(h_q u)) = 0 \quad (3)
\]

The corresponding MOPS \( \{S_n\}_{n \geq 0} \) is said to be \( H_q \)-Laguerre-Hahn.

**Remark 1.** When \( B = 0 \), the form \( u \) is \( H_q \)-semiclassical.

**Proposition 1.** [6] We define \( d = \max(t, r) \). The \( H_q \)-Laguerre-Hahn form \( u \) satisfying (3) is of class \( s = \max(d - 2, p - 1) \) if and only if

\[
\prod_{c \in Z(\Phi)} \{[(H_q \Phi)(c) + q(h_q \Phi)(c)] + [q(h_q B)(c)]

+ |\langle u, (\theta_{cq} \circ \theta_{c} \Phi) + q(\theta_{cq} \Psi) + q(h_q u)(\theta_0 \circ \theta_{cq} B) \rangle|\} \neq 0,
\]

where \( Z(\Phi) := \{z \in C, \Phi(z) = 0\} \).

The \( H_q \)-Laguerre-Hahn character is invariant by shifting. Indeed, the shifted form \( \tilde{u} = h_{a-1}u, \quad a \in C - \{0\} \) satisfies

\[
H_q(\Phi \tilde{u}) + \Psi \tilde{u} + \tilde{B}(x^{-1}\tilde{u}(h_q \tilde{u})) = 0,
\]
with \( \tilde{\Phi}(x) = a^{-1}\Phi(ax), \quad \tilde{B}(x) = a^{-1}B(ax), \quad \tilde{\Psi}(x) = a^{-1}\Psi(ax) \).

The sequence \( \{\tilde{S}_n(x) = a^{-n}S_n(ax)\}_{n \geq 0} \) is orthogonal with respect to \( \tilde{u} \) and fulfills (2) with

\[
\tilde{\beta}_n = \frac{\beta_n}{a}, \quad \tilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \ n \geq 0.
\]

The next result [6] characterizes the elements of the functional equation satisfied by any symmetric \( H_q \)-Laguerre-Hahn form.

**Proposition 2.** Let \( u \) be a symmetric \( H_q \)-Laguerre-Hahn form of class \( s \) satisfying (3). The following statements hold.

1. When \( s \) is odd, then \( \Phi \) and \( B \) are odd and \( \Psi \) is even.
2. When \( s \) is even, then \( \Phi \) and \( B \) are even and \( \Psi \) is odd.

### 2. The Laguerre-Freud equations

In this section, we will establish the non-linear system satisfied by \( \beta_n \) and \( \gamma_n \), simply by using the functional equation.

In the sequel, we assume that \( \{S_n\}_{n \geq 0} \) is a \( H_q \)-Laguerre-Hahn sequence of class zero satisfying (2) and its corresponding form \( u \) satisfying (3) with

\[
\Phi(x) = c_2x^2 + c_1x + c_0, \quad \Psi(x) = a_1x + a_0, \\
B(x) = b_2x^2 + b_1x + b_0, \quad a_1 \neq 0, \quad |c_2| + |c_1| + |c_0| \neq 0.
\]  \hspace{1cm} (4)

Then, from (3) and (4), we obtain for \( n \geq 0 \)

\[
-\langle \Phi u, H_q(S_n(\xi)S_n(q^{-1}\xi))(x) \rangle + \langle \Psi u, S_n(x)S_n(q^{-1}x) \rangle \\
+ \langle u, B(x^{-1}u(h_qu)) \rangle, S_n(x)S_n(q^{-1}x) \rangle = 0, \\
-\langle \Phi u, H_q(S_{n+1}(q^{-1}\xi)S_n(\xi))(x) \rangle + \langle \Psi u, S_n(x)S_{n+1}(q^{-1}x) \rangle \\
+ \langle u, B(x^{-1}u(h_qu)) \rangle, S_n(x)S_{n+1}(q^{-1}x) \rangle = 0.
\]  \hspace{1cm} (5)

Now, let us define for \( n \geq 0 \) and \( 0 \leq k \leq 2 \)

\[
I_{n,k}(q) = \langle u, x^kS_n(x)S_{n+1}(q^{-1}x) \rangle, \ 0 \leq k \leq 1, \\
J_{n,k}(q) = \langle u, x^kS_n(x)S_{n+1}(q^{-1}x) \rangle, \ 0 \leq k \leq 1, \\
K_{n,k}(q) = \langle u, x^kH_q(S_n(\xi)S_n(q^{-1}\xi))(x) \rangle, \ 0 \leq k \leq 2, \\
L_{n,k}(q) = \langle u, x^kH_q(S_{n+1}(q^{-1}\xi)S_n(\xi))(x) \rangle, \ 0 \leq k \leq 2, \\
M_{n,k}(q) = \langle x^{-1}u(h_qu), x^kS_n(x)S_n(q^{-1}x) \rangle, \ 0 \leq k \leq 2, \\
N_{n,k}(q) = \langle x^{-1}u(h_qu), x^kS_n(x)S_{n+1}(q^{-1}x) \rangle, \ 0 \leq k \leq 2.
\]  \hspace{1cm} (6)
If we expand $\Phi$, $\Psi$ and $B$ according to (4), we get from (5) and (6)

\begin{align*}
    a_1 I_{n,1}(q) + a_0 J_{n,0}(q) + b_2 M_{n,2}(q) + b_1 M_{n,1}(q) + b_0 M_{n,0}(q) \\
    - c_2 K_{n,2}(q) - c_1 K_{n,1}(q) - c_0 K_{n,0}(q) = 0 ,
\end{align*}

(7)

\begin{align*}
    a_1 J_{n,1}(q) + a_0 J_{n,0}(q) + b_2 N_{n,2}(q) + b_1 N_{n,1}(q) + b_0 N_{n,0}(q) \\
    - c_2 L_{n,2}(q) - c_1 L_{n,1}(q) - c_0 L_{n,0}(q) = 0 .
\end{align*}

(8)

**Lemma 1.** [13] We have

\begin{equation}
    I_{n,0}(q) = q^{-n} \langle u, S_n^2 \rangle , n \geq 0 ,
\end{equation}

(9)

\begin{equation}
    I_{n,1}(q) = q^{-n} \left\{ \beta_n + (1 - q) \sum_{\nu=0}^{n-1} \beta_{\nu} \right\} \langle u, S_n^2 \rangle , n \geq 0 ,
\end{equation}

(10)

\begin{equation}
    J_{n,0}(q) = q^{-n-1} (1 - q) \left( \sum_{\nu=0}^{n} \beta_{\nu} \right) \langle u, S_n^2 \rangle , n \geq 0 ,
\end{equation}

(11)

\begin{align*}
    J_{n,1}(q) = q^{-n-1} \left\{ \gamma_{n+1} + (1 - q) \left[ (1 + q) \sum_{\nu=0}^{n-1} \gamma_{\nu+1} + \sum_{\nu=0}^{n} \beta_{\nu}^2 \right. \\
    + (1 - q) \sum_{\nu=0}^{n-1} \beta_{\nu+1} \sum_{k=0}^{\nu} \beta_{k} \right] \right\} \langle u, S_n^2 \rangle , n \geq 0 ,
\end{align*}

(12)

\begin{equation}
    K_{n,0}(q) = 0 , n \geq 0 ,
\end{equation}

(13)

\begin{equation}
    K_{n,1}(q) = q^{-n} [2n]_q \langle u, S_n^2 \rangle , n \geq 0 ,
\end{equation}

(14)

\begin{equation}
    K_{n,2}(q) = q^{-n} \left\{ [2n]_q \beta_n + (1 + q^{2n-1}) \sum_{k=0}^{n-1} \beta_{k} \right\} \langle u, S_n^2 \rangle , n \geq 0 ,
\end{equation}

(15)

\begin{equation}
    L_{n,0}(q) = q^{-n-1} [n + 1]_q \langle u, S_n^2 \rangle , n \geq 0 ,
\end{equation}

(16)

\begin{equation}
    L_{n,1}(q) = q^{-n-1} \left( \sum_{\nu=0}^{n} \beta_{\nu} \right) \langle u, S_n^2 \rangle , n \geq 0 ,
\end{equation}

(17)

\begin{equation}
    L_{0,2}(q) = q^{-1} (\beta_0^2 + \gamma_1) ,
\end{equation}

(18)

\begin{align*}
    L_{n,2}(q) = q^{-n-1} \left\{ [2n + 1]_q \gamma_{n+1} + (1 + q) \sum_{\nu=0}^{n-1} \gamma_{\nu+1} + \sum_{\nu=0}^{n} \beta_{\nu}^2 \right. \\
    + (1 - q) \sum_{\nu=0}^{n-1} \beta_{\nu+1} \sum_{k=0}^{\nu} \beta_{k} \right\} \langle u, S_n^2 \rangle , n \geq 1 , \sum_{\nu=0}^{n} \beta_{\nu} = 0 .
\end{align*}

(19)
In order to determine \( \{M_{n,k}(q)\}_{n \geq 0} \) and \( \{N_{n,k}(q)\}_{n \geq 0}, 0 \leq k \leq 2 \), we need the following results:

**Lemma 2.** [13] For \( n \geq 0 \) we have
\[
\langle u, x^{n+1} S_n \rangle = \left( \sum_{\nu=0}^{n} \beta_{\nu} \right) \langle u, S_n^2 \rangle, \tag{20}
\]
\[
\langle u, x^{n+2} S_n \rangle = \left( \sum_{\nu=0}^{n} \gamma_{\nu+1} + \sum_{\nu=0}^{n} \beta_{\nu} \sum_{k=0}^{\nu} \beta_{k} \right) \langle u, S_n^2 \rangle. \tag{21}
\]

**Lemma 3.** [3] We have
\[
S_{n+2}(x) = x^{n+2} + d_{n+1} x^{n+1} + e_n x^n + \ldots, \quad n \geq 0,
\]
\[
S_1(x) = x + d_0,
\]
with
\[
d_n = -\sum_{\nu=0}^{n} \beta_{\nu}, \tag{23}
\]
\[
e_n = -\sum_{\nu=0}^{n} \gamma_{\nu+1} - \sum_{\nu=0}^{n} \beta_{\nu} \sum_{k=0}^{\nu} \beta_{k} + \sum_{\nu=0}^{n+1} \beta_{\nu} \sum_{\nu=0}^{n} \beta_{\nu}, \tag{24}
\]
for \( n \geq 0 \).

**Lemma 4.** [12] For \( u, v \in \mathcal{P}' \) and \( f, g \in \mathcal{P} \), we have
\[
f(uv) = (fu)v + x(u \theta_0 f)v, \tag{25}
\]
\[
\langle uv, \theta_0(fg) \rangle = \langle u, f(v \theta_0 g) \rangle + \langle v, g(u \theta_0 f) \rangle. \tag{26}
\]

**Lemma 5.** We have
\[
M_{n,0}(q) = 0, \quad n \geq 0, \tag{27}
\]
\[
M_{0,1}(q) = 1, \tag{28}
\]
\[
M_{n,1}(q) = q^{-n}(1 + q^{2n})\langle u, S_n^2 \rangle, \quad n \geq 1, \tag{29}
\]
\[
M_{0,2}(q) = (q + 1)\beta_0, \tag{30}
\]
\[
M_{n,2}(q) = q^{-n}\{ (q^{2n} + q)(\beta_0 + \beta_n) + (q-1)(q^{2n} - 1) \sum_{\nu=0}^{n} \beta_{\nu} \} \langle u, S_n^2 \rangle, \quad n \geq 1, \tag{31}
\]
\[
N_{n,0}(q) = q^{-n-1}\langle u, S_n^2 \rangle, \quad n \geq 0, \tag{32}
\]
\[ N_{n,1}(q) = q^{-n-1}\{q\beta_0 + (1-q)\sum_{\nu=0}^{n}\beta_\nu\}\langle u, S_n^2 \rangle, \; n \geq 0, \] (33)

\[ N_{0,2}(q) = q^{-1}[\beta_0^2 + (q^2 + 1)\gamma_1], \] (34)

\[ N_{n,2}(q) = q^{-n-1}\{(1+q^{2n+2})\gamma_{n+1} + q(q-1)\sum_{\nu=0}^{n}\beta_\nu \sum_{\nu=1}^{n}\beta_\nu \]
\[ + (1-q^2)(\sum_{\nu=0}^{n-1}\gamma_{\nu+1} + \sum_{\nu=0}^{n-1}\beta_{\nu+1} + \sum_{k=0}^{\nu}\beta_k + \sum_{\nu=0}^{n}\beta_\nu^2) \]
\[ + q^2(\beta_0^2 + \gamma_1)\}\langle u, S_n^2 \rangle, \; n \geq 1. \] (35)

**Proof.** From (26), we have

\[ M_{n,0}(q) = \langle u, S_n(x)(h_q u \theta_0(h_q^{-1} S_n)(x)) + \langle u, S_n(x)(u \theta_0 S_n)(qx) \rangle, \; n \geq 0. \]

By the orthogonality of \(\{S_n\}_{n \geq 0}\), we obtain (27).

Taking \(f(x) = S_n(x)\) and \(g(x) = x(h_q^{-1} S_n)(x)\) in (26), we can deduce that for \(n \geq 0\)

\[ M_{n,1}(q) = \langle u, S_n(x)(h_q u \theta_0(h_q^{-1} S_n)(x)) + \langle u, qx S_n(x)(u \theta_0 S_n)(qx) \rangle. \] (36)

Making \(n = 0\) in (36), we get \(M_{0,1}(q) = \langle u \rangle_0 = 1.\)

When \(n \geq 1\), by (1) and the orthogonality of \(\{S_n\}_{n \geq 0}\), we obtain (29).

By virtue of (26), when \(f(x) = x S_n(x)\) and \(g(x) = x(h_q^{-1} S_n)(x)\), we can deduce that for \(n \geq 0\)

\[ M_{n,2}(q) = \langle u, x S_n(x)(h_q u \theta_0(h_q^{-1} S_n)(x)) + \langle u, qx S_n(x)(u \theta_0 S_n)(qx) \rangle. \] (37)

Making \(n = 0\) in (37) and taking into account that \(\langle u \rangle_1 = \beta_0\), we obtain (30).

When \(n \geq 1\), by (1), (22) and according to the orthogonality of \(\{S_n\}_{n \geq 0}\), we get

\[ M_{n,2}(q) = (q^{-n} + q^{n+1})\langle u, x^{n+1} S_n(x) \rangle + (q^{1-n} + q^n)(\beta_0 + d_{n-1})\langle u, S_n^2 \rangle. \]

Thus, from (20) and (23), we can deduce (31).

From (26), we have

\[ N_{n,0}(q) = \langle u, S_{n+1}(x)(u \theta_0 S_n)(qx) \rangle + \langle u, S_n(x)(h_q u \theta_0 h_q^{-1} S_{n+1})(x) \rangle, \; n \geq 0. \]

Hence, by the orthogonality of \(\{S_n\}_{n \geq 0}\), we obtain (32).

Taking \(f(x) = S_{n+1}(q^{-1} x)\) and \(g(x) = x S_n(x)\) in (26), we get for \(n \geq 0\)

\[ N_{n,1}(q) = \langle u, S_{n+1}(x)(u S_n)(qx) \rangle + \langle u, x S_n(x)(h_q u \theta_0 h_q^{-1} S_{n+1})(x) \rangle. \]
By (1), (22) and the orthogonality of \( \{S_n\}_{n \geq 0} \), we obtain
\[
N_{n,1}(q) = q^{-u-1} \langle u, x^{n+1}S_n(x) \rangle + q^{-n} (\beta_0 + d_n) \langle u, S_n^2 \rangle.
\]

Then, from (20) and (23), we can deduce (33).

For \( n \geq 0 \), by (26), we have
\[
N_{n,2}(q) = \langle u, qxS_{n+1}(x)(uS_n)(qx) \rangle + \langle u, xS_n(x)(h_quh_{q^{-1}}S_{n+1})(x) \rangle.
\] (38)

When \( n = 0 \), from (38) and taking into account that \( (u)_2 = \beta_0^2 + \gamma_1 \), we obtain (34).

Now, for \( n \geq 1 \), by (1), (22), (38) and taking into account the regularity of the form \( u \), we get
\[
N_{n,2}(q) = q^{-n-1} \langle u, x^{n+2}S_n(x) \rangle + q^{-n} (\beta_0 + d_n) \langle u, x^{n+1}S_n(x) \rangle
+ \{q^{n+1} \gamma_{n+1} + q^{1-n}(\beta_0^2 + \gamma_1) + q^{1-n}(\epsilon_{n-1} + \beta_0 d_n) \} \langle u, S_n^2 \rangle.
\]

Hence, from Lemma 2, (23) and (24), we can deduce (35). \( \square \)

The following is the main result of this section.

**Proposition 3.** We have the following system:
\[
\Psi(\beta_0) + (H_qB)(\beta_0) = 0,
\] (39)
\[
(1 + q^{2n-1}) \sum_{\nu=0}^{n-1} (\theta_{\beta_0} \Phi)(\beta_\nu) - \Psi(\beta_\nu) - (q + q^{2n})(\theta_{\beta_0} B)(\beta_0)
+ ((1 + q^{2n-1}) n - [2n]_q)(c_2 \beta_n + c_1) + (q - 1) \{(q^{2n} - 1) \beta_n b_2
+ [(q^{2n} - 1) b_2 - a_1] \sum_{\nu=0}^{n-1} \beta_\nu - b_1 \} = 0, n \geq 1,
\] (40)
\[
[a_1 + (1 + q^2)b_2 - c_2] \gamma_1 = \Phi(\beta_0) - B(\beta_0) - (1 - q) \beta_0 \Psi(\beta_0),
\} (41)
\[
[a_1 + (q^{2n+2} + 1)b_2 - [2n + 1]_q c_2] \gamma_{n+1} + (1 + q)(1 - q)(a_1 + b_2)
- c_2 \sum_{\nu=0}^{n-1} \beta_\nu + \sum_{\nu=0}^{n-1} \Phi(\beta_\nu) + ([n + 1]_q - n - 1)c_0 - B(q \beta_0)
- q^2 b_2 \gamma_1 + (1 - q) \{(c_2 + (q - 1)b_2) \sum_{\nu=0}^{n-1} \beta_{\nu+1} \sum_{k=0}^{\nu} \beta_k
- (a_0 + b_1) \sum_{\nu=0}^{n-1} \beta_\nu - [a_1 + (q + 1)b_2] \sum_{\nu=0}^{n-1} \beta_\nu
+ q b_2 \sum_{\nu=0}^{n-1} \beta_\nu \sum_{\nu=0}^{n-1} \beta_\nu \} = 0, n \geq 1.
\] (42)
Proof.

Making \( n = 0 \) in (7) and taking the relations (9) – (10), (13) – (15), (27) – (28) and (30) into account, we can deduce (39). \( \) Let \( n \geq 1 \). Then, by virtue of the relations (9) – (10), (13) – (15), (27), (29) and (31), the equation (7) becomes

\[
\begin{align*}
&\{2n\}q\beta_n + (1 + q^{2n-1}) \sum_{\nu=0}^{n-1} \beta_\nu \}c_2 + [2n]q c_1 - \{\beta_n + (1 - q) \sum_{\nu=0}^{n-1} \beta_\nu \}a_1 \\
&- a_0 - \{(q^{2n} + q)(\beta_0 + \beta_n) + (q - 1)(q^{2n} - 1) \sum_{\nu=0}^{n-1} \beta_\nu \}b_2 - (1 + q^{2n})b_1 = 0.
\end{align*}
\]

But, \( (\theta_\beta \Phi)(\beta_\nu) = c_2(\beta_n + \beta_\nu) + c_1 \) and \( (\theta_\beta B)(\beta_0) = b_2(\beta_n + \beta_0) + b_1 \). Then, we can deduce (40).

Let \( n = 0 \) in (8). Then, by virtue of (11) – (12), (16) – (18) and (32) – (34) we get (41).

When \( n \geq 1 \), in view of (11) – (12), (16) – (17), (19), (32) – (33) and (35), (8) becomes

\[
\begin{align*}
&\{2n + 1\}q\gamma_{n+1} + (1 + q) \sum_{\nu=0}^{n-1} \gamma_{\nu+1} + \sum_{\nu=0}^{n} \beta_\nu^2 + (1 - q) \sum_{\nu=0}^{n-1} \beta_{\nu+1} \sum_{k=0}^{n} \beta_k \}c_2 \\
&c_1 \sum_{\nu=0}^{n} \beta_\nu + [n + 1]q c_0 - \{\gamma_{n+1} + (1 - q)\left[ (1 + q) \sum_{\nu=0}^{n-1} \gamma_{\nu+1} + \sum_{\nu=0}^{n} \beta_\nu^2 \right. \\
&\left. + (1 - q) \sum_{\nu=0}^{n-1} \beta_{\nu+1} \sum_{k=0}^{n} \beta_k \right]\}a_1 - (1 - q)a_0 \sum_{\nu=0}^{n} \beta_\nu - \{(q^{2n+2} + 1)\gamma_{n+1} \\
&+ q(q - 1) \sum_{\nu=0}^{n} \beta_\nu \}b_2 - \{q\beta_0 + (1 - q) \sum_{\nu=0}^{n} \beta_\nu \}b_1 - b_0 = 0.
\end{align*}
\]

Hence, we can deduce (42). △

3. The symmetric case when \( s = 0 \)

In the sequel, we assume that \( \{S_n\}_{n \geq 0} \) is a symmetric \( H_q \)-Laguerre-Hahn orthogonal sequence of class zero.

Then, we have

\[
\begin{align*}
&S_{n+2}(x) = xS_{n+1}(x) - \gamma_{n+1} S_n(x), \quad n \geq 0, \\
&S_1(x) = x, \quad S_0(x) = 1.
\end{align*}
\]
By virtue of the Proposition 2, it follows that
\[ H_q(\Phi u) + \Psi u + B(x^{-1}u^2) = 0 , \]  
(44)
with
\[ \Phi(x) = c_2x^2 + c_0 , \Psi(x) = a_1x , B(x) = b_2x^2 + b_0 . \]  
(45)
In this case the system (39) - (42) becomes
\[ (r_2 - c_2)\gamma_1 = c_0 - b_0 , \]  
(46)
\[ \{r_{2n+2} - [2n + 1]q c_2\}\gamma_{n+1} + (1 + q)\{(1 - q)(a_1 + b_2) - c_2\} \sum_{\nu=0}^{n-1} \gamma_{\nu+1} \]  
(47)
\[ = [n + 1]q c_0 - b_0 - q^2 b_2 \gamma_1 , \quad n \geq 1 , \]
with
\[ r_n = a_1 + (1 + q^n)b_2 , \quad n \geq 0 . \]  
(48)
Let
\[ T_n = \sum_{\nu=0}^{n} \gamma_{\nu+1} , \quad n \geq 0 . \]  
(49)
Then,
\[ T_n - T_{n-1} = \gamma_{n+1} , \quad n \geq 0 , \quad T_{-1} = 0 . \]  
(50)
Taking the relations (49) and (50) into account, the system (46) - (47) becomes
\[ (r_2 - c_2)T_0 = c_0 - b_0 , \]  
(51)
\[ \{r_{2n+2} - [2n + 1]q c_2\}T_n - (1 + q)\{(1 - q)(a_1 + b_2) - c_2\} \sum_{\nu=0}^{n-1} \gamma_{\nu+1} \]  
(52)
\[ = [n + 1]q c_0 - b_0 - q^2 b_2 T_0 , \quad n \geq 1 . \]

**Proposition 4.** We have for \( n \geq 1 \)
\[ T_n = \frac{q[n]q[n+1]q c_0 + [(a_1 - c_2)[2n + 2]q + [2n + 4]q b_2]\gamma_1}{(q + 1)\{r_{2n+2} - [2n + 1]q c_2\}} . \]  
(53)

**Proof.** The equation (52) can be written as
\[ \{r_{2n+2} - [2n + 1]q c_2\}T_n - q^2 \{r_{2n} - [2n - 1]q c_2\}T_{n-1} \]  
\[ = [n + 1]q c_0 - b_0 - q^2 b_2 T_0 , \quad n \geq 1 . \]
So, we obtain
\[ \{r_{2n+2} - [2n + 1]q c_2\}T_n - q^{2n} \{r_2 - c_2\}T_0 \]  
\[ = q^{2n} \sum_{k=1}^{n} \left\{ q^{-2k} \left( [k + 1]q c_0 - b_0 - q^2 b_2 T_0 \right) \right\} , \quad n \geq 1 . \]
Taking the relation (51) into account, we obtain (53). ▶
Corollary 1. The sequence \( \{\gamma_{n+2}\}_{n \geq 0} \) is defined by

\[
\gamma_{n+2} = \frac{q^{n+1}}{(r_{n+4} - [2n + 3]q^2)(r_{n+2} - [2n + 1]q^2)} \left\{ [n + 1]q(r_{n+2} - [n + 1]q^2)c_0 \\
+ [(a_1 + b_2)q^{n+1}r_2 - (q(a_1 + b_2) + c_2 - r_2)q^n c_2] \gamma_1 \right\}.
\] (54)

Proof. From (50) and Proposition 4, we get for \( n \geq 1 \)

\[
(q + 1)\{r_{2n+4} - [2n + 3]q^2\}\{r_{2n+2} - [2n + 1]q^2\} \gamma_{n+2}
= \{r_{2n+2} - [2n + 1]q^2\}\{q[n + 1]q[n + 2]c_0 + [(a_1 - c_2)[2n + 4]q + [2n + 6]q^2]c_2\} \\
- \{r_{2n+4} - [2n + 3]q^2\}\{q[n]q[n + 1]q^2c_0 + [(a_1 - c_2)[2n + 2]q + [2n + 4]q^2]c_2\} \gamma_1.
\]

Then, we can deduce (54) after some straightforward calculations. \( \blacksquare \)

4. The canonical cases

Before considering different canonical situations, let us proceed to the general transformation

\[
\tilde{S}_n(x) = a^{-n}S_n(ax), \quad n \geq 0,
\]

\[
\tilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \quad n \geq 0.
\]

The form \( \tilde{u} = h_{a-1}u \) fulfills

\[
H_q(a^{-t}\Phi(ax)\tilde{u}) + a^{1-t}\Psi(ax)\tilde{u} + a^{-t}B(ax)(x^{-1}(\tilde{u}(h_q\tilde{u}))) = 0.
\]

Any so-called canonical case will be denoted by \( \tilde{\gamma}_{n+1}, \tilde{u} \).

By (54) and Corollary 1, we get the general situation

\[
\left\{
\begin{aligned}
\gamma_{n+2} &= \frac{q^{n+1}}{(r_{2n+4} - [2n + 3]q^2)(r_{2n+2} - [2n + 1]q^2)} \left\{ [n + 1]q(r_{n+2} - [n + 1]q^2)c_0 \\
&+ [(a_1 + b_2)q^{n+1}r_2 - (q(a_1 + b_2) + c_2 - r_2)q^n c_2] \gamma_1 \right\}, \quad n \geq 0, \\
H_q((c_2x^2 + c_0)u) + a_1 xu + (b_2x^2 + b_0)(x^{-1}u^2) &= 0.
\end{aligned}
\right.
\] (55)

Theorem 1. The following canonical cases arise:

1. When \( \Phi(x) = 1 \), we have the following subcases:

   (i) \( a_1 + b_2 \neq 0 \)
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2. The case where $\Phi(x) = x^2$, we obtain the canonical case below:

$$
\begin{align*}
\gamma_1 &= \rho q^{\tau+1} \frac{1}{2}, \\
\gamma_{n+2} &= \frac{2}{\rho} \big( (\tau + \frac{1}{2}) q^{n+1} + (\tau + \frac{1}{2}) q^n \big) \frac{1}{n+1} \{ [n+\tau+2]_q \\
&+ \big( \frac{1}{\rho} - 1 \big) (n+3)_q - q^{n+1} [n+1]_q - (q+1) q^{n+\tau+2} \big), \\
H_q(\tilde{u}) &= \frac{2}{\rho} (2 - \rho) q^{-\tau} x \tilde{u} + \frac{2}{\rho} (\rho - 1) q^{-\tau} x^2 \\
&+ 1 - (1 + q^2 (\rho - 1)) [\tau + 1]_q (x^{-1} \tilde{u}^2) = 0.
\end{align*}
$$

(ii) $a_1 = -b_2$

$$
\begin{align*}
\gamma_1 &= \frac{\rho q}{\rho}, \\
\gamma_{n+2} &= \frac{[n+1]_q}{2 q^{n+1}}, \\
H_q(\tilde{u}) &= 2 x \tilde{u} + (2 x^2 + 1 - q \rho) (x^{-1} \tilde{u}^2) = 0.
\end{align*}
$$

3. When $\Phi(x) = x^2 + c_0$, $c_0 \neq 0$, we have the following subcases:

(i) $q(a_1 + b_2) + 1 = 0$

$$
\begin{align*}
\gamma_1 &= q \rho, \\
\gamma_{n+2} &= q^{n+2} \frac{(q^{n+1} - 1)(q^{n+1} - 1)}{(q^{n+1} - 1)(q^{n+1} - 1)} [n+\tau+1]_q, \\
H_q((x^2 - 1) \tilde{u}) &= (q - 1)^{-1} (q^{n+1} - 1) x \tilde{u} + \{ -q^{-1} (q-1)^{-1} (q^{n+1} - 1) x^2 \\
&+ \rho (q - 1 + q(q-1)^{-1} (q^{n+1} - 1)) - 1 \} (x^{-1} \tilde{u}^2) = 0.
\end{align*}
$$
\(\text{(ii)} \ q(a_1 + b_2) + 1 \neq 0\)

\[
\begin{align*}
\gamma_1 &= \rho q^{r+2} + 2(1 - (q^{r+1}) - (q^{2r+2} - 1)) \\
\gamma_{n+2} &= q^{n+r+2} + 3(q^{n+r+2} - 1) + \Lambda_2 \Delta_2 n, \quad n \geq 0, \\
H_q(x^2 - q^{-2})\dot{u} &= q^{-2r} - a_2(q-1)^{-1}(q^{2r+2} - 1) \times \\
&\quad + (1 + q^2)(\frac{1}{\rho} - 1)(q^{2r+2} - 1)x\dot{u} \\
&\quad + \{q^{-2r} - a_1(q-1)^{-1}(q^{2r+2} - 1)x^2 \\
&\quad + \rho q^{-2r} x + 1\} = 0 \\
\text{with (for } n \geq 0\text{)} \\
\Delta_n &= q^{n+2r+2} + 3 - (q^n - 1)(q^{2r+2} - 1)(\frac{1}{\rho} - 1), \\
\Lambda_n &= (q^n - 1)(q^{2r+2} - 1)(1 - \frac{1}{\rho}).
\end{align*}
\]

\textbf{Proof.}

1. In this case, (55) becomes

\[
\begin{align*}
\gamma_{n+2} &= \frac{q^{n+2}}{r_2 n_4 + 4 r_2 n_2} \{[n+1]q^{n+2} + q^{n+2}(a_1 + b_2)r_2\gamma_1\}, \quad n \geq 0, \\
H_q(u) + a_1 xu + (b_2x^2 + b_0)(x^{-1}u^2) &= 0.
\end{align*}
\]

We need to discuss the following situations:

(i) \(a_1 + b_2 \neq 0\). Choosing \(a^2(a_1 + b_2) = \frac{2}{\rho} q^{-2}\) and putting \(a^2 \gamma_1 = \rho^2 [r+1] x\), \(a^2 b_2 = \frac{2}{\rho} (q - 1) q^{-2}\), we get (56) from (51).

(ii) If \(a_1 + b_2 = 0\), then (61) becomes

\[
\begin{align*}
\gamma_{n+2} &= \frac{(n+1)q}{q^{n+2} + b_2}, \quad n \geq 0, \\
H_q(u) + a_1 xu + (b_2x^2 + b_0)(x^{-1}u^2) &= 0.
\end{align*}
\]

With the choice \(a^2 b_2 = 2\) and putting \(\gamma_1 = \frac{q}{q^{2r}}\), we get (57) from (51).

2. In this case, (55) can be written as

\[
\begin{align*}
\gamma_{n+2} &= \frac{q^{2n+2}[r+1]q^2(a_1 + b_2) + 1]}{r_2 n_4 + 2 q^{n+2} + 1}, \quad n \geq 0, \\
H_q(x^2 u) + a_1 xu + (b_2x^2 + b_0)(x^{-1}u^2) &= 0.
\end{align*}
\]

We choose \(a\) such that \(a^2[r+1]q^2(a_1 + b_2) + 1\) \(\gamma_1 = q^{-2r-2}\), and putting \(q^{-2r-2} \{2r + 3\} = 1 - r_2\), \(\gamma_1 = -\rho \frac{q^{2r+1}}{2r+3\rho}, \) we obtain (58) from (51).
3. In this case, (55) becomes

\[
\begin{align*}
\gamma_{n+2} &= \frac{q^{n+1}}{r_{2n+4}-[2n+3]q_{2}}\{[n+1]_{q}(r_{n+2} - [n+1]_{q})c_{0} \\
&\quad + q^{n}(1 + q(a_{1} + b_{2}))(r_{2} - 1)\gamma_{1}\}, \quad n \geq 0, \\
H_{q}(x^{2} + c_{0})u + a_{1}xu + (b_{2}x^{2} + b_{0})(x^{-1}u^{2}) &= 0. \\
\end{align*}
\]

We can consider two subcases:

(i) If \(1 + q(a_{1} + b_{2}) = 0\), then (62) becomes

\[
\begin{align*}
\gamma_{n+2} &= \frac{q^{n+1}}{r_{2n+4}-[2n+3]q_{2}}\{[n+1]_{q}(r_{n+2} - [n+1]_{q})c_{0} \\
&\quad + q^{n}(1 + q(a_{1} + b_{2}))(r_{2} - 1)\gamma_{1}\}, \quad n \geq 0, \\
H_{q}(x^{2} + c_{0})u + a_{1}xu + (b_{2}x^{2} + b_{0})(x^{-1}u^{2}) &= 0. \\
\end{align*}
\]

With the choice \(a^{-2}c_{0} = -1\) and putting \(\gamma_{1} = \rho, a_{1} = (q - 1)^{-1}(q^{a-1} - 1)\), we obtain the desired result (59) from (51).

(ii) When \(1 + q(a_{1} + b_{2}) \neq 0\), we choose \(a\) such that \(a^{-2}c_{0} = -q^{-\tau}\) and putting

\[
\begin{align*}
a^{-2}(1 + q(a_{1} + b_{2}))(r_{2} - 1)\gamma_{1} &= q^{-3\tau-2a-1}(q - 1)^{-1}(q^{r+2a+1} - 1)[\tau + 1]_{q}, \\
1 + q(a_{1} + b_{2}) &= -\frac{q^{-2\tau-2a-1}}{\rho}(q - 1)^{-1}(q^{2\tau+2a+1} - 1), \\
1 - r_{2} &= q^{-2\tau-2a-2}(q - 1)^{-1}(q^{2\tau+2a+3} - 1),
\end{align*}
\]

we get (60) from (51).

\[\square\]

Remark 2. 1. If \(q \to 1\) in (56), we obtain

\[
\begin{align*}
\tilde{\gamma}_{1} &= \rho \frac{\tau+1}{2}, \quad \tilde{\gamma}_{n+2} = \frac{n+\tau+2}{2}, \quad n \geq 0, \\
(\tilde{u}^{\prime})^{2} + \frac{2}{\rho}(2 - \rho)x\tilde{u} + \left(\frac{2}{\rho}(\rho - 1)x^{2} + 1 - \rho(\tau + 1)\right)(x^{-1}\tilde{u}^{2}) &= 0.
\end{align*}
\]

In this case \(\tilde{u}\) is the co-dilated of the associated form of order \(\tau\) of Hermite [2].

2. When \(q \neq 1\), from (56) we obtain

\[
\begin{align*}
\tilde{\gamma}_{1} &= \rho\left[1 - q^{-\tau+1}\right], \\
\tilde{\gamma}_{n+2} &= \frac{q^{n+\tau+1}}{\left\{1 + q^{n+\tau+1}\right\}\{1 - q^{n+\tau+2}\}}\left\{(1 - q^{n+\tau+2}) \\
&\quad + (\frac{1}{\rho} - 1)(1 - q^{n+3}) - q^{n+2}(1 - q^{n+1}) + (q^{2} - 1)q^{n+\tau+2}\right\}, \quad n \geq 0, \\
H_{q}(\tilde{u}) + \frac{1}{\rho}(2 - \rho)(1 - q)^{-1}q^{-\tau}x\tilde{u} + \left\{\frac{1}{\rho}(\rho - 1)(1 - q)^{-1}q^{-\tau}x^{2} \\
&\quad + 1 - (1 + q^{2}(\rho - 1))\right\}\{\tau + 1\}_{q}\right\}(x^{-1}\tilde{u}^{2}) &= 0,
\end{align*}
\]

(63)
with \( \hat{u} = h \frac{1}{\sqrt{2(1-q)}} \tilde{u} \). When \( \rho = 1 \) and \( \tau = 0 \), \( \hat{u} \) is a particular case of \( q \)-polynomials of Al-Salam Carlitz [9].

3. If \( q \to 1 \) in (57), we get

\[
\left\{ \begin{array}{l}
\tilde{\gamma}_1 = \frac{\rho}{2}, \\
\tilde{\gamma}_{n+2} = \frac{n+1}{2}, n \geq 0, \\
(\hat{u})' - 2 \hat{u} + (2 \hat{u}^2 + 1 - \rho)(x^{-1} \hat{u}^2) = 0.
\end{array} \right.
\]

In this case \( \hat{u} \) is the Laguerre-Hahn form of class zero, analogous to the classical Hermite’s one [2].

4. When \( q \to 1 \) in (58), we obtain

\[
\left\{ \begin{array}{l}
\tilde{\gamma}_1 = -\frac{\rho}{(2\tau+1)(2\tau+3)}, \\
\tilde{\gamma}_{n+2} = -\frac{1}{(2n+2\tau+5)(2n+2\tau+3)}, n \geq 0, \\
(x^2 \hat{u})' + 2[(1 - \frac{1}{\rho})(2\tau + 1) - \tau - 1]x \hat{u} \\
+ [(\frac{1}{\rho} - 1)(2\tau + 1)x^2 - \frac{1}{2\tau+1}](x^{-1} \hat{u}^2) = 0.
\end{array} \right.
\]

In this case \( \hat{u} \) is the co-dilates of the associated form of order \( \tau \) of Bessel of parameter 1 [2].

5. If \( q \to 1 \) in (59), we get

\[
\left\{ \begin{array}{l}
\tilde{\gamma}_1 = \rho, \\
\tilde{\gamma}_{n+2} = \frac{(n+1)(n+\alpha+1)}{(2n+\alpha+3)(2n+\alpha+1)}, n \geq 0, \\
((x^2 - 1) \hat{u})' + (\alpha - 2)x \hat{u} + ((\alpha - 1)x^2 + \rho(\alpha - 1) - 1)(x^{-1} \hat{u}^2) = 0.
\end{array} \right.
\]

In this case \( \hat{u} \) is the Laguerre-Hahn form of class zero, analogous to the classical Jacobi’s one [2].

6. When \( q \to 1 \) in (60), we have

\[
\left\{ \begin{array}{l}
\tilde{\gamma}_1 = \rho \frac{(\tau+1)(\tau+2\alpha+1)}{(2\tau+2\alpha+3)(2\tau+2\alpha+1)}, \\
\tilde{\gamma}_{n+2} = \frac{(n+\tau+2)(n+\tau+2\alpha+2)}{(2n+2\tau+2\alpha+5)(2n+2\tau+2\alpha+3)}, n \geq 0, \\
((x^2 - 1) \hat{u})' - 2[\tau + \alpha + 1 + (\frac{1}{\rho} - 1)(2\tau + 2\alpha + 1)]x \hat{u} \\
+ [(\frac{1}{\rho} - 1)(2\tau + 2\alpha + 1)x^2 + \rho \frac{(\tau+1)(\tau+2\alpha+1)}{(2\tau+2\alpha+3)(2\tau+2\alpha+1)} - 1](x^{-1} \hat{u}^2) = 0.
\end{array} \right.
\]

In this case \( \hat{u} \) is the co-dilates of the associated form of order \( \tau \) of Gegenbauer [2].
7. If we put \( b = q^\alpha \), (60) reduces to

\[
\begin{align*}
\tilde{\gamma}_1 &= \rho b^2 q^{r+1} \frac{(q^{r+1}+1)(b^2q^{r+1}+1)}{(b^2q^{r+1}+1)(b^2q^{r+1}+1)} , \\
\tilde{\gamma}_{n+2} &= b^2 q^{n+r+2} \frac{(q^{n+r+2}+1)(b^2q^{n+r+2}+1)+\Lambda_n}{\Delta_{2n} \Delta_{2n+2}} , \quad n \geq 0 , \\
H_q((x^2 - q^{-\tau}) \tilde{u}) &= q^{-2r-4}(b^2(q - 1))^{-1}(b^2q^{2r+1} - 1)x \tilde{u} \\
&+ (1 + q^2)(\frac{1}{\rho} - 1)(b^2q^{2r+1} - 1) \tilde{u} \\
&+ \{q^{-2r-4}(b^2(q - 1))^{-1}(\frac{1}{\rho} - 1)(b^2q^{2r+1} - 1)\} x^2 \\
&+ \rho q^{-r}(\tau + 1) b^2q^{r+1} - 1 - q^{-r} \xi^2 = 0 ,
\end{align*}
\]

with (for \( n \geq 0 \))

\[
\begin{align*}
\Delta_n &= b^2 q^{n+2r+3} - 1 - (q^n - 1)(b^2q^{2r+1} - 1)(\frac{1}{\rho} - 1) , \\
\Lambda_n &= (q^{n+1} - 1)(q^n - 1)(q^{2r+1}b^2 - 1)(1 - \frac{1}{\rho}) .
\end{align*}
\]

When \( \rho = 1 \) and \( \tau = 0 \), \( \tilde{u} \) is a particular case of Big \( q \)-Jacobi polynomials [9].

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