Azerbaijan Journal of Mathematics
V. 13 , No 1, 2023, January
ISSN 2218-6816

The Symmetric H_q-Laguerre-Hahn Orthogonal Polynomials of Class Zero

M. Sghaier*, M. Zaatra, M. Mechri

Abstract. We consider the system of Laguerre-Freud equations associated with the H_q-Laguerre-Hahn orthogonal polynomials of class zero. This system is solved in the symmetric case. There are essentially three canonical cases.

Key Words and Phrases: orthogonal polynomials, q-difference operator, H_q-Laguerre-Hahn forms.

2010 Mathematics Subject Classifications: 33C45, 42C05

1. Introduction and preliminary results

The D-Laguerre-Hahn polynomial sequences, where D is the derivative operator, have attracted the interest of researchers from many points of view (see [1, 2, 4, 5, 10] among others). They constitute a very remarkable family of orthogonal polynomials taking into consideration most of the monic orthogonal polynomial sequences (MOPS) found in literature. In particular, semiclassical orthogonal polynomials are Laguerre-Hahn MOPS [11]. The concept of D-Laguerre-Hahn polynomial sequences has been extended to discrete Laguerre-Hahn polynomials which are related to a divided difference operator. Indeed , the H_q-Laguerre-Hahn polynomial sequences, where H_q is the Hahn's operator, have been considered by several authors. Essentially, an algebraic theory was presented in [6, 9]. These families are extensions of discrete semiclassical polynomials [8, 7].

The D-Laguerre-Hahn polynomial sequences of class zero are completely described in [2]. For the difference operator D_w, the D_w-Laguerre-Hahn polynomial sequences of class zero has been analyzed in [12]. So, the aim of this paper is to determine the symmetric H_q-Laguerre-Hahn forms of class $s = 0$, through the study of the differential functional equation fulfilled by these forms and the

*Corresponding author.

 © 2010 AZJM All rights reserved.

resolution of a nonlinear system satisfied by the coefficients of the three-term recurrence relation of their sequences of monic orthogonal polynomials.

The structure of the manuscript is as follows. The first section contains material of preliminary and some results regarding the H_q-Laguerre-Hahn forms. In the second section, the system of Laguerre-Freud equations is built. In the third section, using this system, we obtain the symmetric sequences which we look for.

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle\,, n \geq 0$, the moments of u. For instance, for any form u, any polynomial g and any $a \in \mathbb{C}\backslash\{0\}$, we let $Du = u'$, gu, $h_a u$ and $x^{-1}u$ be the forms defined by duality

$$\big\langle u', f\big\rangle := -\big\langle u, f'\big\rangle\,,\ \langle gu, f\rangle := \langle u, gf\rangle\,,\ \langle h_a u, f\rangle := \langle u, h_a f\rangle\,,$$
$$\big\langle x^{-1}u, f\big\rangle := \langle u, \theta_0 f\rangle\,,\ f \in \mathcal{P},$$

where $(h_a f)(x) = f(ax)$ and $\big(\theta_0 f\big)(x) = \dfrac{f(x)-f(0)}{x}$.

We also define the right-multiplication of a form u by a polynomial h as

$$(uh)(x) := \Big\langle u, \frac{xh(x)-\xi h(\xi)}{x-\xi}\Big\rangle = \sum_{i=0}^{n}\Big(\sum_{j=i}^{n} a_j (v)_{j-i}\Big)x^i\,,\ h(x) = \sum_{i=0}^{n} a_i x^i\,. \quad (1)$$

Next, it is possible to define the product of two forms through

$$\langle uv, f\rangle := \langle u, vf\rangle\,,\ \ u, v \in \mathcal{P}',\ f \in \mathcal{P}.$$

A form u is called regular if there exists a sequence of polynomials $\{S_n\}_{n\geq 0}$ $(\deg S_n \leq n)$ such that

$$\langle u, S_n S_m\rangle = r_n \delta_{n,m} \quad , \quad r_n \neq 0, \quad n \geq 0\,.$$

Then, $\deg S_n = n, n \geq 0$ and we can always suppose each S_n is monic. In such a case, the sequence $\{S_n\}_{n\geq 0}$ is unique. The sequence $\{S_n\}_{n\geq 0}$ is said to be the sequence of monic orthogonal polynomials with respect to u. In the sequel, it will be denoted as MOPS. It is a very well known fact that the sequence $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation [3]

$$\begin{aligned} &S_{n+2}(x) = (x-\beta_{n+1})S_{n+1}(x) - \gamma_{n+1}S_n(x)\,, \quad n \geq 0\,,\\ &S_1(x) = x - \beta_0\,, \qquad S_0(x) = 1\,, \end{aligned} \quad (2)$$

with $\big(\beta_n, \gamma_{n+1}\big) \in \mathbb{C}\times\mathbb{C}\backslash\{0\}\,, \quad n \geq 0\,.$ By convention we set $\gamma_0 = (u)_0$. The form u is called normalized if $(u)_0 = 1$.

We recall that a form u is called symmetric if $(u)_{2n+1} = 0, n \geq 0$. The conditions $(u)_{2n+1} = 0, n \geq 0$ are equivalent to the fact that the corresponding MOPS $\{S_n\}_{n\geq 0}$ satisfies the three-term recurrence relation (2) with $\beta_n = 0, n \geq 0$ [3].

Let us introduce the Hahn's operator [9]

$$(H_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x} \; , f \in \mathcal{P} \; , \quad q \in \tilde{\mathbb{C}} \; ,$$

where $\tilde{\mathbb{C}} := \mathbb{C} - \left(\{0\} \bigcup \left(\bigcup_{n\geq 0} \{z \in \mathbb{C} \, , \quad z^n = 1\}\right)\right)$. When $q \longrightarrow 1$, we meet again the derivative D.

By duality, we can define H_q from $\mathcal{P}'$ to $\mathcal{P}'$ such that

$$\langle H_q u, f \rangle = - \langle u, H_q f \rangle \; , f \in \mathcal{P} \; , u \in \mathcal{P}' \; .$$

In particular, this yields $(H_q u)_n = -[n]_q (u)_{n-1} \; , n \geq 0$ with $(u)_{-1} = 0$ and $[n]_q := \frac{q^n - 1}{q-1} \; , \;\; n \geq 0$.

Definition 1. *[6] The regular form u is called a H_q-Laguerre-Hahn form when it is regular and there exist three polynomials Φ (monic) ,Ψ and B, $\deg(\Phi) = t \geq 0$, $\deg(\Psi) = p \geq 1$, $\deg(B) = r \geq 0$, such that*

$$H_q(\Phi u) + \Psi u + B\big(x^{-1} u (h_q u)\big) = 0 \; . \tag{3}$$

The corresponding MOPS $\{S_n\}_{n\geq 0}$ is said to be H_q-Laguerre-Hahn.

Remark 1. *When $B = 0$, the form u is H_q-semiclassical.*

Proposition 1. *[6] We define $d = \max(t, r)$. The H_q-Laguerre-Hahn form u satisfying (3) is of class $s = \max(d-2, p-1)$ if and only if*

$$\prod_{c \in \mathcal{Z}_{(\Phi)}} \{ |(H_q \Phi)(c) + q(h_q \Psi)(c)| + |q(h_q B)(c)| \\ + |\langle u, (\theta_{cq} \circ \theta_c \Phi) + q(\theta_{cq} \Psi) + q(h_q u)(\theta_0 \circ \theta_{cq} B) \rangle| \} \neq 0 \; ,$$

where $Z_{(\Phi)} := \{z \in C, \Phi(z) = 0\}$.

The H_q-Laguerre-Hahn character is invariant by shifting. Indeed, the shifted form $\tilde{u} = h_{a^{-1}} u, \; a \in C - \{0\}$ satisfies

$$H_q(\tilde{\Phi} \tilde{u}) + \tilde{\Psi} \tilde{u} + \tilde{B}\big(x^{-1} \tilde{u} (h_q \tilde{u})\big) = 0 \; ,$$

with

$$\tilde{\Phi}(x) = a^{-t}\Phi(ax) , \quad \tilde{B}(x) = a^{-t}B(ax) , \quad \tilde{\Psi}(x) = a^{1-t}\Psi(ax) .$$

The sequence $\{\tilde{S}_n(x) = a^{-n}S_n(ax)\}_{n\geq 0}$ is orthogonal with respect to $\tilde{u}$ and fulfils (2) with

$$\tilde{\beta}_n = \frac{\beta_n}{a} , \quad \tilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2} , \ n \geq 0 .$$

The next result [6] characterizes the elements of the functional equation satisfied by any symmetric H_q-Laguerre-Hahn form.

Proposition 2. *Let u be a symmetric H_q-Laguerre-Hahn form of class s satisfying* (3)*. The following statements hold.*

1. *When s is odd, then Φ and B are odd and Ψ is even.*
2. *When s is even, then Φ and B are even and Ψ is odd.*

2. The Laguerre-Freud equations

In this section, we will establish the non-linear system satisfied by β_n and γ_n, simply by using the functional equation.

In the sequel, we assume that $\{S_n\}_{n\geq 0}$ is a H_q-Laguerre-Hahn sequence of class zero satisfying (2) and its corresponding form u satisfying (3) with

$$\begin{aligned} &\Phi(x) = c_2x^2 + c_1x + c_0 , \quad \Psi(x) = a_1x + a_0 , \\ &B(x) = b_2x^2 + b_1x + b_0 , \quad a_1 \neq 0 , \quad |c_2| + |c_1| + |c_0| \neq 0 . \end{aligned} \tag{4}$$

Then, from (3) and (4), we obtain for $n \geq 0$

$$\begin{aligned} &-\langle \Phi u, H_q\big(S_n(\xi)S_n(q^{-1}\xi)\big)(x)\rangle + \langle \Psi u, S_n(x)S_n(q^{-1}x)\rangle \\ &\qquad +\langle u, B(x^{-1}u(h_qu)), S_n(x)S_n(q^{-1}x)\rangle = 0 , \\ &-\langle \Phi u, H_q\big(S_{n+1}(q^{-1}\xi)S_n(\xi)\big)(x)\rangle + \langle \Psi u, S_n(x)S_{n+1}(q^{-1}x)\rangle \\ &\qquad +\langle u, B(x^{-1}u(h_qu)), S_n(x)S_{n+1}(q^{-1}x)\rangle = 0 . \end{aligned} \tag{5}$$

Now, let us define for $n \geq 0$ and $0 \leq k \leq 2$

$$\begin{aligned} &I_{n,k}(q) = \langle u, x^kS_n(x)S_n(q^{-1}x)\rangle , \ 0 \leq k \leq 1 , \\ &J_{n,k}(q) = \langle u, x^kS_n(x)S_{n+1}(q^{-1}x)\rangle , \ 0 \leq k \leq 1, \\ &K_{n,k}(q) = \langle u, x^kH_q\big(S_n(\xi)S_n(q^{-1}\xi)\big)(x)\rangle , \ 0 \leq k \leq 2 , \\ &L_{n,k}(q) = \langle u, x^kH_q\big(S_{n+1}(q^{-1}\xi)S_n(\xi)\big)(x)\rangle , \ 0 \leq k \leq 2 , \\ &M_{n,k}(q) = \langle x^{-1}u(h_qu), x^kS_n(x)S_n(q^{-1}x)\rangle , \ 0 \leq k \leq 2 , \\ &N_{n,k}(q) = \langle x^{-1}u(h_qu), x^kS_n(x)S_{n+1}(q^{-1}x)\rangle , \ 0 \leq k \leq 2 . \end{aligned} \tag{6}$$

If we expand Φ, Ψ and B according to (4), we get from (5) and (6)

$$\begin{aligned} a_1 I_{n,1}(q) + a_0 I_{n,0}(q) + b_2 M_{n,2}(q) + b_1 M_{n,1}(q) + b_0 M_{n,0}(q) \\ -c_2 K_{n,2}(q) - c_1 K_{n,1}(q) - c_0 K_{n,0}(q) = 0 \ , \end{aligned} \tag{7}$$

$$\begin{aligned} a_1 J_{n,1}(q) + a_0 J_{n,0}(q) + b_2 N_{n,2}(q) + b_1 N_{n,1}(q) + b_0 N_{n,0}(q) \\ -c_2 L_{n,2}(q) - c_1 L_{n,1}(q) - c_0 L_{n,0}(q) = 0 \ . \end{aligned} \tag{8}$$

Lemma 1. *[13] We have*

$$I_{n,0}(q) = q^{-n}\langle u, S_n^2\rangle \ , \ n \geq 0 \ , \tag{9}$$

$$I_{n,1}(q) = q^{-n}\Big\{\beta_n + (1-q)\sum_{\nu=0}^{n-1}\beta_\nu\Big\}\langle u, S_n^2\rangle \ , \ n \geq 0 \ , \tag{10}$$

$$J_{n,0}(q) = q^{-n-1}(1-q)\Big(\sum_{\nu=0}^{n}\beta_\nu\Big)\langle u, S_n^2\rangle \ , \ n \geq 0 \ , \tag{11}$$

$$\begin{aligned} J_{n,1}(q) = q^{-n-1}\Big\{\gamma_{n+1} + (1-q)\Big[(1+q)\sum_{\nu=0}^{n-1}\gamma_{\nu+1} + \sum_{\nu=0}^{n}\beta_\nu^2 \\ +(1-q)\sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu}\beta_k\Big]\Big\}\langle u, S_n^2\rangle \ , \ n \geq 0 \ , \end{aligned} \tag{12}$$

$$K_{n,0}(q) = 0 \ , \ n \geq 0 \ , \tag{13}$$

$$K_{n,1}(q) = q^{-n}[2n]_q\langle u, S_n^2\rangle \ , \ n \geq 0 \ , \tag{14}$$

$$K_{n,2}(q) = q^{-n}\Big\{[2n]_q\beta_n + (1+q^{2n-1})\sum_{k=0}^{n-1}\beta_k\Big\}\langle u, S_n^2\rangle \ , \ n \geq 0 \ , \tag{15}$$

$$L_{n,0}(q) = q^{-n-1}[n+1]_q\langle u, S_n^2\rangle \ , \ n \geq 0 \ , \tag{16}$$

$$L_{n,1}(q) = q^{-n-1}\Big(\sum_{\nu=0}^{n}\beta_\nu\Big)\langle u, S_n^2\rangle \ , \ n \geq 0 \ , \tag{17}$$

$$L_{0,2}(q) = q^{-1}(\beta_0^2 + \gamma_1) \ , \tag{18}$$

$$\begin{aligned} L_{n,2}(q) = q^{-n-1}\Big\{[2n+1]_q\gamma_{n+1} + (1+q)\sum_{\nu=0}^{n-1}\gamma_{\nu+1} + \sum_{\nu=0}^{n}\beta_\nu^2 \\ +(1-q)\sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu}\beta_k\Big\}\langle u, S_n^2\rangle \ , \ n \geq 1 \ , \ \sum_{\nu=0}^{-1} = 0 \ . \end{aligned} \tag{19}$$

In order to determine $\{M_{n,k}(q)\}_{n\geq 0}$ and $\{N_{n,k}(q)\}_{n\geq 0}$, $0 \leq k \leq 2$, we need the following results:

Lemma 2. *[13] For $n \geq 0$ we have*

$$\langle u, x^{n+1} S_n \rangle = \left(\sum_{\nu=0}^{n} \beta_\nu \right) \langle u, S_n^2 \rangle \ , \tag{20}$$

$$\langle u, x^{n+2} S_n \rangle = \left(\sum_{\nu=0}^{n} \gamma_{\nu+1} + \sum_{\nu=0}^{n} \beta_\nu \sum_{k=0}^{\nu} \beta_k \right) \langle u, S_n^2 \rangle \ . \tag{21}$$

Lemma 3. *[3] We have*

$$\begin{aligned} &S_{n+2}(x) = x^{n+2} + d_{n+1} x^{n+1} + e_n x^n + \ldots \ , \quad n \geq 0 \ , \\ &S_1(x) = x + d_0 \ , \end{aligned} \tag{22}$$

with

$$d_n = - \sum_{\nu=0}^{n} \beta_\nu \ , \tag{23}$$

$$e_n = - \sum_{\nu=0}^{n} \gamma_{\nu+1} - \sum_{\nu=0}^{n} \beta_\nu \sum_{k=0}^{\nu} \beta_k + \sum_{\nu=0}^{n+1} \beta_\nu \sum_{\nu=0}^{n} \beta_\nu \tag{24}$$

for $n \geq 0$.

Lemma 4. *[12] For u, $v \in \mathcal{P}'$ and f, $g \in \mathcal{P}$, we have*

$$f(uv) = (fu)v + x(u\theta_0 f)v \ , \tag{25}$$

$$\langle uv, \theta_0(fg) \rangle = \langle u, f(v\theta_0 g) \rangle + \langle v, g(u\theta_0 f) \rangle \ . \tag{26}$$

Lemma 5. *We have*

$$M_{n,0}(q) = 0 \ , \ n \geq 0 \ , \tag{27}$$

$$M_{0,1}(q) = 1 \ , \tag{28}$$

$$M_{n,1}(q) = q^{-n}(1 + q^{2n}) \langle u, S_n^2 \rangle \ , \ n \geq 1 \ , \tag{29}$$

$$M_{0,2}(q) = (q+1)\beta_0 \ , \tag{30}$$

$$M_{n,2}(q) = q^{-n}\{(q^{2n}+q)(\beta_0+\beta_n)+(q-1)(q^{2n}-1)\sum_{\nu=0}^{n} \beta_\nu\} \langle u, S_n^2 \rangle \ , \ n \geq 1 \ , \tag{31}$$

$$N_{n,0}(q) = q^{-n-1} \langle u, S_n^2 \rangle \ , \ n \geq 0 \ , \tag{32}$$

$$N_{n,1}(q) = q^{-n-1}\{q\beta_0 + (1-q)\sum_{\nu=0}^{n}\beta_\nu\}\langle u, S_n^2\rangle\ ,\ n \geq 0\ , \tag{33}$$

$$N_{0,2}(q) = q^{-1}[\beta_0^2 + (q^2+1)\gamma_1]\ , \tag{34}$$

$$\begin{aligned} N_{n,2}(q) = q^{-n-1}\{&(1+q^{2n+2})\gamma_{n+1} + q(q-1)\sum_{\nu=0}^{n}\beta_\nu\sum_{\nu=1}^{n}\beta_\nu \\ &+(1-q^2)\big(\sum_{\nu=0}^{n-1}\gamma_{\nu+1} + \sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu}\beta_k + \sum_{\nu=0}^{n}\beta_\nu^2\big) \\ &+q^2(\beta_0^2+\gamma_1)\}\langle u, S_n^2\rangle\ ,\ n \geq 1\ . \end{aligned} \tag{35}$$

Proof. From (26), we have

$$M_{n,0}(q) = \langle u, S_n(x)\big(h_q u\theta_0(h_{q^{-1}}S_n)(x)\rangle + \langle u, S_n(x)(u\theta_0 S_n)(qx)\rangle\ ,\ n \geq 0\ .$$

By the orthogonality of $\{S_n\}_{n\geq 0}$, we obtain (27).

Taking $f(x) = S_n(x)$ and $g(x) = x(h_{q^{-1}}S_n)(x)$ in (26), we can deduce that for $n \geq 0$

$$M_{n,1}(q) = \langle u, S_n(x)\big(h_q u(h_{q^{-1}}S_n)\big)(x)\rangle + \langle u, qxS_n(x)(u\theta_0 S_n)(qx)\rangle\ . \tag{36}$$

Making $n = 0$ in (36), we get $M_{0,1}(q) = (u)_0 = 1$.

When $n \geq 1$, by (1) and the orthogonality of $\{S_n\}_{n\geq 0}$, we obtain (29).

By virtue of (26), when $f(x) = xS_n(x)$ and $g(x) = x(h_{q^{-1}}S_n)(x)$, we can deduce that for $n \geq 0$

$$M_{n,2}(q) = \langle u, xS_n(x)\big(h_q u(h_{q^{-1}}S_n\big)(x)\rangle + \langle u, qx)S_n(x)(uS_n)(qx)\rangle\ . \tag{37}$$

Making $n = 0$ in (37) and taking into account that $(u)_1 = \beta_0$, we obtain (30).

When $n \geq 1$, by (1), (22) and according to the orthogonality of $\{S_n\}_{n\geq 0}$, we get

$$M_{n,2}(q) = (q^{-n} + q^{n+1})\langle u, x^{n+1}S_n(x)\rangle + (q^{1-n} + q^n)(\beta_0 + d_{n-1}\big)\langle u, S_n^2\rangle\ .$$

Thus, from (20) and (23), we can deduce (31).

From (26), we have

$$N_{n,0}(q) = \langle u, S_{n+1}(x)\big(u\theta_0 S_n\big)(qx)\rangle + \langle u, S_n(x)(h_q u\theta_0 h_{q^{-1}}S_{n+1})(x)\rangle\ ,\ n \geq 0\ .$$

Hence, by the orthogonality of $\{S_n\}_{n\geq 0}$, we obtain (32).

Taking $f(x) = S_{n+1}(q^{-1}x)$ and $g(x) = xS_n(x)$ in (26), we get for $n \geq 0$

$$N_{n,1}(q) = \langle u, S_{n+1}(x)\big(uS_n\big)(qx)\rangle + \langle u, xS_n(x)(h_q u\theta_0 h_{q^{-1}}S_{n+1})(x)\rangle\ .$$

By (1), (22) and the orthogonality of $\{S_n\}_{n\geq 0}$, we obtain

$$N_{n,1}(q) = q^{-n-1}\langle u, x^{n+1}S_n(x)\rangle + q^{-n}\big(\beta_0 + d_n\big)\langle u, S_n^2\rangle \ .$$

Then, from (20) and (23), we can deduce (33).

For $n \geq 0$, by (26), we have

$$N_{n,2}(q) = \langle u, qxS_{n+1}(x)\big(uS_n\big)(qx)\rangle + \langle u, xS_n(x)(h_q u h_{q^{-1}}S_{n+1})(x)\rangle \ . \tag{38}$$

When $n = 0$, from (38) and taking into account that $(u)_2 = \beta_0^2 + \gamma_1$, we obtain (34).

Now, for $n \geq 1$, by (1), (22), (38) and taking into account the regularity of the form u, we get

$$\begin{aligned}N_{n,2}(q) &= q^{-n-1}\langle u, x^{n+2}S_n(x)\rangle + q^{-n}\big(\beta_0 + d_n\big)\langle u, x^{n+1}S_n(x)\rangle \\ &+\{q^{n+1}\gamma_{n+1} + q^{1-n}(\beta_0^2 + \gamma_1) + q^{1-n}(e_{n-1} + \beta_0 d_n)\}\langle u, S_n^2\rangle \ .\end{aligned}$$

Hence, from Lemma 2, (23) and (24), we can deduce (35). ◄

The following is the main result of this section.

Proposition 3. *We have the following system:*

$$\Psi(\beta_0) + (H_q B)(\beta_0) = 0 \ , \tag{39}$$

$$\begin{aligned}&(1+q^{2n-1})\sum_{\nu=0}^{n-1}(\theta_{\beta_n}\Phi)(\beta_\nu) - \Psi(\beta_n) - (q+q^{2n})(\theta_{\beta_n}B)(\beta_0) = \\ &\quad +\big((1+q^{2n-1})n - [2n]_q\big)(c_2\beta_n + c_1) + (q-1)\{(q^{2n}-1)\beta_n b_2 \\ &\quad +[(q^{2n}-1)b_2 - a_1]\sum_{\nu=0}^{n-1}\beta_\nu - b_1\} = 0 \ , \ n \geq 1 \ ,\end{aligned} \tag{40}$$

$$[a_1 + (1+q^2)b_2 - c_2]\gamma_1 = \Phi(\beta_0) - B(\beta_0) - (1-q)\beta_0\Psi(\beta_0) \ , \tag{41}$$

$$\begin{aligned}&\{a_1 + (q^{2n+2}+1)b_2 - [2n+1]_q c_2\}\gamma_{n+1} + (1+q)\{(1-q)(a_1+b_2) \\ &-c_2\}\sum_{\nu=0}^{n-1}\gamma_{\nu+1} = \sum_{\nu=0}^{n}\Phi(\beta_\nu) + \big([n+1]_q - n - 1\big)c_0 - B(q\beta_0) \\ &\quad -q^2b_2\gamma_1 + (1-q)\{[c_2 + (q-1)a_1 - (q+1)b_2]\sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu}\beta_k \\ &\quad -(a_0+b_1)\sum_{\nu=0}^{n}\beta_\nu - [a_1 + (q+1)b_2]\sum_{\nu=0}^{n}\beta_\nu^2 \\ &\quad +qb_2\sum_{\nu=0}^{n}\beta_\nu\sum_{\nu=1}^{n}\beta_\nu\} \ , \ n \geq 1 \ .\end{aligned} \tag{42}$$

Proof. Making $n=0$ in (7) and taking the relations (9) – (10), (13) – (15), (27) – (28) and (30) into account, we can deduce (39). Let $n \geq 1$. Then, by virtue of the relations (9) – (10), (13) – (15), (27), (29) and (31), the equation (7) becomes

$$\{[2n]_q\beta_n + (1+q^{2n-1})\sum_{\nu=0}^{n-1}\beta_\nu\}c_2 + [2n]_q c_1 - \{\beta_n + (1-q)\sum_{\nu=0}^{n-1}\beta_\nu\}a_1$$
$$-a_0 - \{(q^{2n}+q)(\beta_0+\beta_n) + (q-1)(q^{2n}-1)\sum_{\nu=0}^{n}\beta_\nu\}b_2 - (1+q^{2n})b_1 = 0\ .$$

But, $(\theta_{\beta_n}\Phi)(\beta_\nu) = c_2(\beta_n+\beta_\nu)+c_1$ and $(\theta_{\beta_n}B)(\beta_0) = b_2(\beta_n+\beta_0)+b_1$. Then, we can deduce (40).

Let $n=0$ in (8). Then, by virtue of (11) – (12), (16) – (18) and (32) – (34) we get (41).

When $n \geq 1$, in view of (11) – (12), (16) – (17), (19), (32) – (33) and (35), (8) becomes

$$\{[2n+1]_q\gamma_{n+1} + (1+q)\sum_{\nu=0}^{n-1}\gamma_{\nu+1} + \sum_{\nu=0}^{n}\beta_\nu^2 + (1-q)\sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu}\beta_k\}c_2$$
$$c_1\sum_{\nu=0}^{n}\beta_\nu + [n+1]_q c_0 - \{\gamma_{n+1} + (1-q)\Big[(1+q)\sum_{\nu=0}^{n-1}\gamma_{\nu+1} + \sum_{\nu=0}^{n}\beta_\nu^2$$
$$+(1-q)\sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu}\beta_k\Big]\}a_1 - (1-q)a_0\sum_{\nu=0}^{n}\beta_\nu - \{(q^{2n+2}+1)\gamma_{n+1}$$
$$+q(q-1)\sum_{\nu=0}^{n}\beta_\nu\sum_{\nu=1}^{n}\beta_\nu + (1-q^2)\big(\sum_{\nu=0}^{n-1}\gamma_{\nu+1} + \sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu}\beta_k + \sum_{\nu=0}^{n}\beta_\nu^2\big)$$
$$+q^2(\beta_0^2+\gamma_1)\}b_2 - \{q\beta_0 + (1-q)\sum_{\nu=0}^{n}\beta_\nu\}b_1 - b_0 = 0\ .$$

Hence, we can deduce (42). ◄

3. The symmetric case when $s=0$

In the sequel, we assume that $\{S_n\}_{n\geq 0}$ is a symmetric H_q-Laguerre-Hahn orthogonal sequence of class zero.

Then, we have

$$\begin{aligned} &S_{n+2}(x) = xS_{n+1}(x) - \gamma_{n+1}S_n(x)\ , \quad n \geq 0\ ,\\ &S_1(x) = x\ , \qquad S_0(x) = 1\ . \end{aligned} \tag{43}$$

By virtue of the Proposition 2, it follows that

$$H_q(\Phi u) + \Psi u + B\big(x^{-1}u^2\big) = 0 \;, \tag{44}$$

with

$$\Phi(x) = c_2 x^2 + c_0 \;,\; \Psi(x) = a_1 x \;,\; B(x) = b_2 x^2 + b_0 \;. \tag{45}$$

In this case the system (39) – (42) becomes

$$(r_2 - c_2)\gamma_1 = c_0 - b_0 \;, \tag{46}$$

$$\begin{aligned} &\{r_{2n+2} - [2n+1]_q c_2\}\gamma_{n+1} + (1+q)\{(1-q)(a_1+b_2) - c_2\}\sum_{\nu=0}^{n-1}\gamma_{\nu+1} \\ &= [n+1]_q c_0 - b_0 - q^2 b_2 \gamma_1 \;,\; n \geq 1 \;, \end{aligned} \tag{47}$$

with

$$r_n = a_1 + (1+q^n) b_2 \;,\; n \geq 0. \tag{48}$$

Let

$$T_n = \sum_{\nu=0}^{n} \gamma_{\nu+1} \;,\; n \geq 0 \;. \tag{49}$$

Then,

$$T_n - T_{n-1} = \gamma_{n+1} \;,\; n \geq 0 \;,\; T_{-1} = 0 \;. \tag{50}$$

Taking the relations (49) and (50) into account, the system (46) – (47) becomes

$$(r_2 - c_2)T_0 = c_0 - b_0 \;, \tag{51}$$

$$\begin{aligned} &\{r_{2n+2} - [2n+1]_q c_2\}(T_n - T_{n-1}) + (1+q)\{(1-q)(a_1+b_2) \\ &-c_2\}T_{n-1} = [n+1]_q c_0 - b_0 - q^2 b_2 T_0 \;,\; n \geq 1 \;. \end{aligned} \tag{52}$$

Proposition 4. *We have for $n \geq 1$*

$$T_n = \frac{q[n]_q[n+1]_q c_0 + [(a_1 - c_2)[2n+2]_q + [2n+4]_q b_2]\gamma_1}{(q+1)\{r_{2n+2} - [2n+1]_q c_2\}} \;. \tag{53}$$

Proof. The equation (52) can be written as

$$\begin{aligned} &\{r_{2n+2} - [2n+1]_q c_2\}T_n - q^2\{r_{2n} - [2n-1]_q c_2\}T_{n-1} \\ &= [n+1]_q c_0 - b_0 - q^2 b_2 T_0 \;,\; n \geq 1 \;. \end{aligned}$$

So, we obtain

$$\begin{aligned} &\{r_{2n+2} - [2n+1]_q c_2\}T_n - q^{2n}\{r_2 - c_2\}T_0 \\ &= q^{2n}\sum_{k=1}^{n}\Big\{q^{-2k}\Big([k+1]_q c_0 - b_0 - q^2 b_2 T_0\Big)\Big\} \;,\; n \geq 1 \;. \end{aligned}$$

Taking the relation (51) into account, we obtain (53). ◄

Corollary 1. *The sequence $\{\gamma_{n+2}\}_{n\geq 0}$ is defined by*

$$\begin{aligned}\gamma_{n+2} = \tfrac{q^{n+1}}{\{r_{2n+4}-[2n+3]_q c_2\}\{r_{2n+2}-[2n+1]_q c_2\}}&\{[n+1]_q\big(r_{n+2}-[n+1]_q c_2\big)c_0 \\ &+[(a_1+b_2)q^{n+1}r_2 - \big(q(a_1+b_2)+c_2-r_2\big)q^n c_2]\gamma_1\}\ .\end{aligned} \tag{54}$$

Proof. From (50) and Proposition 4, we get for $n \geq 1$

$$\begin{aligned}&(q+1)\{r_{2n+4}-[2n+3]_q c_2\}\{r_{2n+2}-[2n+1]_q c_2\}\gamma_{n+2} \\ &= \{r_{2n+2}-[2n+1]_q c_2\}\{q[n+1]_q[n+2]_q c_0 + [(a_1-c_2)[2n+4]_q + [2n+6]_q b_2]\gamma_1\} \\ &-\{r_{2n+4}-[2n+3]_q c_2\}\{q[n]_q[n+1]_q c_0 + [(a_1-c_2)[2n+2]_q + [2n+4]_q b_2]\gamma_1\}\ .\end{aligned}$$

Then, we can deduce (54) after some straightforward calculations. ◄

4. The canonical cases

Before considering different canonical situations, let us proceed to the general transformation

$$\tilde{S}_n(x) = a^{-n}S_n(ax)\ , \quad n \geq 0\ ,$$

$$\tilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}\ ,\ n \geq 0\ .$$

The form $\tilde{u} = h_{a^{-1}}u$ fulfils

$$H_q(a^{-t}\Phi(ax)\tilde{u}) + a^{1-t}\Psi(ax)\tilde{u} + a^{-t}B(ax)\big(x^{-1}(\tilde{u}(h_q\tilde{u}))\big) = 0\ .$$

Any so-called canonical case will be denoted by $\tilde{\gamma}_{n+1}$, $\tilde{u}$.

By (54) and Corollary 1, we get the general situation

$$\begin{cases}\gamma_{n+2} = \tfrac{q^{n+1}}{\{r_{2n+4}-[2n+3]_q c_2\}\{r_{2n+2}-[2n+1]_q c_2\}}\{[n+1]_q\big(r_{n+2}-[n+1]_q c_2\big)c_0 \\ \qquad +[(a_1+b_2)q^{n+1}r_2 - \big(q(a_1+b_2)+c_2-r_2\big)q^n c_2]\gamma_1\}\ ,\ n\geq 0\ , \\ H_q\big((c_2x^2+c_0)u\big) + a_1 xu + (b_2x^2+b_0)(x^{-1}u^2) = 0\ .\end{cases} \tag{55}$$

Theorem 1. *The following canonical cases arise:*

1. *When $\Phi(x) = 1$, we have the following subcases:*
   (i) $a_1 + b_2 \neq 0$

$$\begin{cases} \tilde{\gamma}_1 = \rho\frac{q^\tau[\tau+1]_q}{2} \;, \\ \tilde{\gamma}_{n+2} = \frac{q^{n+\tau+1}}{2\{\frac{1}{\rho}+(1-\frac{1}{\rho})q^{2n+4}\}\{\frac{1}{\rho}+(1-\frac{1}{\rho})q^{2n+2}\}}\{[n+\tau+2]_q \\ \qquad +(\frac{1}{\rho}-1)\big([n+3]_q - q^{n+2}[n+1]_q - (q+1)q^{n+\tau+2}\big)\} \;,\; n \geq 0 \;, \\ H_q(\tilde{u}) + \frac{2}{\rho}(2-\rho)q^{-\tau}x\tilde{u} + \{\frac{2}{\rho}(\rho-1)q^{-\tau}x^2 \\ \qquad +1 - \big(1+q^2(\rho-1)\big)[\tau+1]_q\}(x^{-1}\tilde{u}^2) = 0 \;. \end{cases} \tag{56}$$

(ii) $a_1 = -b_2$

$$\begin{cases} \tilde{\gamma}_1 = \frac{\rho}{2q} \;,\; \tilde{\gamma}_{n+2} = \frac{[n+1]_q}{2q^{2n+3}} \;,\; n \geq 0 \;, \\ H_q(\tilde{u}) - 2x\tilde{u} + (2x^2 + 1 - q\rho)(x^{-1}\tilde{u}^2) = 0 \;. \end{cases} \tag{57}$$

2. The case where $\Phi(x) = x^2$, *we obtain the canonical case below:*

$$\begin{cases} \tilde{\gamma}_1 = -\rho\frac{q^{2\tau+1}}{[2\tau+1][2\tau+3]} \;, \\ \tilde{\gamma}_{n+2} = -\frac{q^{2n+2\tau+3}}{\Omega_{2n+2}\Omega_{2n}} \;,\; n \geq 0 \;, \\ H_q(x^2\tilde{u}) + q^{-2\tau-4}\{(1+q^2)(1-\frac{1}{\rho})[2\tau+1]_q - q^2[2\tau+2]_q\}x\tilde{u} \\ \qquad +\{q^{-2\tau-4}(\frac{1}{\rho}-1)[2\tau+1]_q x^2 - \rho\frac{q^{-1}}{[2\tau+1]_q}\}(x^{-1}\tilde{u}^2) = 0 \;, \end{cases} \tag{58}$$

with (for $n \geq 0$*)*

$$\Omega_n = [n+2\tau+3]_q + (q^n - 1)(1-\frac{1}{\rho})[2\tau+1]_q \;.$$

3. When $\Phi(x) = x^2 + c_0,\; c_0 \neq 0$, *we have the following subcases:*

(i) $q(a_1+b_2) + 1 = 0$

$$\begin{cases} \tilde{\gamma}_1 = q\rho \;,\; \tilde{\gamma}_{n+2} = q^{n+2}\frac{(q^{n+1}-1)(q^{n+\alpha+1}-1)}{(q^{2n+\alpha+3}-1)(q^{2n+\alpha+1}-1)} \;,\; n \geq 0 \;, \\ H_q\big((x^2-1)\tilde{u}\big) + (q-1)^{-1}(q^{\alpha-2}-1)x\tilde{u} + \{-q^{-1}(q-1)^{-1}(q^{\alpha-1}-1)x^2 \\ \qquad +\rho\big(q-1+q(q-1)^{-1}(q^{\alpha-1}-1)\big) - 1\}(x^{-1}\tilde{u}^2) = 0 \;. \end{cases} \tag{59}$$

(ii) $q(a_1+b_2)+1\neq 0$

$$\begin{cases} \tilde{\gamma}_1 = \rho q^{\tau+2\alpha+2}\frac{(q^{\tau+1}-1)(q^{\tau+2\alpha+1}-1)}{(q^{2\tau+2\alpha+3}-1)(q^{2\tau+2\alpha+1}-1)}\ , \\ \tilde{\gamma}_{n+2} = q^{n+\tau+2\alpha+3}\frac{(q^{n+\tau+2}-1)(q^{n+\tau+2\alpha+2}-1)+\Lambda_n}{\Delta_{2n}\Delta_{2n+2}}\ ,\ n\geq 0\ , \\ H_q\big((x^2-q^{-\tau})\tilde{u}\big) - q^{-2\tau-2\alpha-4}(q-1)^{-1}\{q^2(q^{2\tau+2\alpha+2}-1) \\ \qquad +(1+q^2)(\frac{1}{\rho}-1)(q^{2\tau+2\alpha+1}-1)\}x\tilde{u} \\ \qquad +\{q^{-2\tau-2\alpha-4}(q-1)^{-1}(\frac{1}{\rho}-1)(q^{2\tau+2\alpha+1}-1)x^2 \\ \qquad +\rho q^{-\tau}[\tau+1]_q\frac{q^{\tau+2\alpha+1}-1}{q^{2\tau+2\alpha+1}-1} - q^{-\tau}\}(x^{-1}\tilde{u}^2) = 0\ , \end{cases} \tag{60}$$

with (for $n\geq 0$)

$$\begin{cases} \Delta_n = q^{n+2\tau+2\alpha+3}-1-(q^n-1)(q^{2\tau+2\alpha+1}-1)(\frac{1}{\rho}-1)\ , \\ \Lambda_n = (q^{n+1}-1)(q^n-1)(q^{2\tau+2\alpha+1}-1)(1-\frac{1}{\rho})\ . \end{cases}$$

Proof.

1. In this case, (55) becomes

$$\begin{cases} \gamma_{n+2} = \frac{q^{n+1}}{r_{2n+4}r_{2n+2}}\{[n+1]_q r_{n+2} + q^{n+1}(a_1+b_2)r_2\gamma_1\}\ ,\ n\geq 0\ , \\ H_q\big(u\big) + a_1xu + (b_2x^2+b_0)(x^{-1}u^2) = 0\ . \end{cases} \tag{61}$$

We need to discuss the following situations:
(i) $a_1+b_2\neq 0$. Choosing $a^2(a_1+b_2)=\frac{2}{\rho}q^{-\tau}$ and putting $a^{-2}\gamma_1 = \rho\frac{q^{\tau}[\tau+1]}{2}$, $a^2b_2 = \frac{2}{\rho}(\rho-1)q^{-\tau}$, we get (56) from (51).
(ii) If $a_1+b_2=0$, then (61) becomes

$$\begin{cases} \gamma_{n+2} = \frac{[n+1]_q}{q^{2n+3}b_2}\ ,\ n\geq 0\ , \\ H_q\big(u\big) + a_1xu + (b_2x^2+b_0)(x^{-1}u^2) = 0\ . \end{cases}$$

With the choice $a^2b_2=2$ and putting $\tilde{\gamma}_1 = \frac{\rho}{2q}$, we get (57) from (51).

2. In this case, (55) can be written as

$$\begin{cases} \gamma_{n+2} = \frac{q^{2n+1}(r_2-1)\{q(a_1+b_2)+1\}\gamma_1}{\{r_{2n+4}-[2n+3]_q\}\{r_{2n+2}-[2n+1]_q\}}\ ,\ n\geq 0\ , \\ H_q\big(x^2u\big) + a_1xu + (b_2x^2+b_0)(x^{-1}u^2) = 0\ . \end{cases}$$

We choose a such that $a^{-2}(r_2-1)\{q(a_1+b_2)+1\}\gamma_1 = q^{-2\tau-2}$, and putting $q^{-2\tau-2}[2\tau+3] = 1-r_2$, $\tilde{\gamma}_1 = -\rho\frac{q^{2\tau+1}}{[2\tau+3]_q[2\tau+1]_q}$, we obtain (58) from (51).

3. In this case, (55) becomes

$$\begin{cases} \gamma_{n+2} = \frac{q^{n+1}}{\{r_{2n+4}-[2n+3]_q\}\{r_{2n+2}-[2n+1]_q\}}\{[n+1]_q\big(r_{n+2}-[n+1]_q\big)c_0 \\ \qquad +q^n\big(1+q(a_1+b_2)\big)(r_2-1)\gamma_1\}\ ,\ n\geq 0\ , \\ H_q\big(x^2+c_0)u\big)+a_1xu+(b_2x^2+b_0)(x^{-1}u^2)=0\ . \end{cases} \tag{62}$$

We can consider two subcases:

(i) If $1+q(a_1+b_2)=0$, then (62) becomes

$$\begin{cases} \gamma_{n+2} = \frac{q^{n+1}}{\{r_{2n+4}-[2n+3]_q\}\{r_{2n+2}-[2n+1]_q\}}\{[n+1]_q\big(r_{n+2}-[n+1]_q\big)c_0\}\ ,\ n\geq 0\ , \\ H_q\big(x^2+c_0)u\big)+a_1xu+(b_2x^2+b_0)(x^{-1}u^2)=0\ . \end{cases}$$

With the choice $a^{-2}c_0=-1$ and putting $\tilde{\gamma}_1=\rho$, $a_1=(q-1)^{-1}(q^{\alpha-1}-1)$, we obtain the desired result (59) from (51).

(ii) When $1+q(a_1+b_2)\neq 0$, we choose a such that $a^{-2}c_0=-q^{-\tau}$ and putting

$$\begin{cases} a^{-2}\big(1+q(a_1+b_2)\big)(r_2-1)\gamma_1 = q^{-3\tau-2\alpha-1}(q-1)^{-1}(q^{\tau+2\alpha+1}-1)[\tau+1]_q\ , \\ 1+q(a_1+b_2) = -\frac{q^{-2\tau-2\alpha-1}}{\rho}(q-1)^{-1}(q^{2\tau+2\alpha+1}-1)\ , \\ 1-r_2 = q^{-2\tau-2\alpha-2}(q-1)^{-1}(q^{2\tau+2\alpha+3}-1)\ , \end{cases}$$

we get (60) from (51). ◄

Remark 2. *1. If $q\to 1$ in (56), we obtain*

$$\begin{cases} \tilde{\gamma}_1=\rho\frac{\tau+1}{2}\ ,\quad \tilde{\gamma}_{n+2}=\frac{n+\tau+2}{2}\ ,\ n\geq 0\ , \\ (\tilde{u})'+\frac{2}{\rho}(2-\rho)x\tilde{u}+\big(\frac{2}{\rho}(\rho-1)x^2+1-\rho(\tau+1)\big)(x^{-1}\tilde{u}^2)=0. \end{cases}$$

In this case $\tilde{u}$ is the co-dilated of the associated form of order τ of Hermite [2].

2. When $q\neq 1$, from (56) we obtain

$$\begin{cases} \widehat{\gamma}_1=\rho q^{\tau}(1-q^{\tau+1})\ , \\ \widehat{\gamma}_{n+2}=\frac{q^{n+\tau+1}}{\{\frac{1}{\rho}+(1-\frac{1}{\rho})q^{2n+4}\}\{\frac{1}{\rho}+(1-\frac{1}{\rho})q^{2n+2}\}}\{(1-q^{n+\tau+2}) \\ +(\frac{1}{\rho}-1)\big((1-q^{n+3})-q^{n+2}(1-q^{n+1})+(q^2-1)q^{n+\tau+2}\big)\}\ ,\ n\geq 0\ , \\ H_q(\widehat{u})+\frac{1}{\rho}(2-\rho)(1-q)^{-1}q^{-\tau}x\widehat{u}+\{\frac{1}{\rho}(\rho-1)(1-q)^{-1}q^{-\tau}x^2 \\ \qquad +1-\big(1+q^2(\rho-1)\big)[\tau+1]_q\}(x^{-1}\widehat{u}^2)=0\ , \end{cases} \tag{63}$$

with $\widehat{u} = h_{\sqrt{2(1-q)}}\tilde{u}$. *When* $\rho = 1$ *and* $\tau = 0$, $\widehat{u}$ *is a particular case of q-polynomials of Al-Salam Carlitz [9].*

3. *If* $q \to 1$ *in* (57), *we get*

$$\begin{cases} \tilde{\gamma}_1 = \frac{\rho}{2}\,, \quad \tilde{\gamma}_{n+2} = \frac{n+1}{2}\,,\; n \geq 0\,, \\ (\tilde{u})' - 2x\tilde{u} + \big(2x^2 + 1 - \rho\big)\big)(x^{-1}\tilde{u}^2) = 0\,. \end{cases}$$

In this case $\tilde{u}$ *is the Laguerre-Hahn form of class zero, analogous to the classical Hermite's one [2].*

4. *When* $q \to 1$ *in* (58), *we obtain*

$$\begin{cases} \tilde{\gamma}_1 = -\frac{\rho}{(2\tau+1)(2\tau+3)}\,, \quad \tilde{\gamma}_{n+2} = -\frac{1}{(2n+2\tau+5)(2n+2\tau+3)}\,,\; n \geq 0\,, \\ (x^2\tilde{u})' + 2[(1-\frac{1}{\rho})(2\tau+1) - \tau - 1]x\tilde{u} \\ \qquad +[(\frac{1}{\rho} - 1)(2\tau+1)x^2 - \frac{\rho}{2\tau+1}](x^{-1}\tilde{u}^2) = 0\,. \end{cases}$$

In this case $\tilde{u}$ *is the co-dilates of the associated form of order* τ *of Bessel of parameter* 1 *[2].*

5. *If* $q \to 1$ *in* (59), *we get*

$$\begin{cases} \tilde{\gamma}_1 = \rho\,, \quad \tilde{\gamma}_{n+2} = \frac{(n+1)(n+\alpha+1)}{(2n+\alpha+3)(2n+\alpha+1)}\,,\; n \geq 0\,, \\ \big((x^2-1)\tilde{u}\big)' + (\alpha-2)x\tilde{u} + \big((\alpha-1)x^2 + \rho(\alpha-1) - 1\big)(x^{-1}\tilde{u}^2) = 0\,. \end{cases}$$

In this case $\tilde{u}$ *is the Laguerre-Hahn form of class zero, analogous to the classical Jacobi's one [2].*

6. *When* $q \to 1$ *in* (60), *we have*

$$\begin{cases} \tilde{\gamma}_1 = \rho\frac{(\tau+1)(\tau+2\alpha+1)}{(2\tau+2\alpha+3)(2\tau+2\alpha+1)}\,, \quad \tilde{\gamma}_{n+2} = \frac{(n+\tau+2)(n+\tau+2\alpha+2)}{(2n+2\tau+2\alpha+5)(2n+2\tau+2\alpha+3)}\,,\; n \geq 0\,, \\ \big((x^2-1)\tilde{u}\big)' - 2[\tau+\alpha+1+(\frac{1}{\rho}-1)(2\tau+2\alpha+1)]x\tilde{u} \\ \qquad +[(\frac{1}{\rho}-1)(2\tau+2\alpha+1)x^2 + \rho\frac{(\tau+1)(\tau+2\alpha+1)}{(2\tau+2\alpha+3)(2\tau+2\alpha+1)} - 1](x^{-1}\tilde{u}^2) = 0\,. \end{cases}$$

In this case $\tilde{u}$ *is the co-dilates of the associated form of order* τ *of Gegenbauer [2].*

7. If we put $b = q^{\alpha}$, (60) reduces to

$$\begin{cases} \tilde{\gamma}_1 = \rho b^2 q^{\tau+1} \frac{(q^{\tau+1}-1)(b^2q^{\tau+1}-1)}{(b^2q^{2\tau+3}-1)(b^2q^{2\tau+1}-1)} \ , \\ \tilde{\gamma}_{n+2} = b^2 q^{n+\tau+2} \frac{(q^{n+\tau+2}-1)(b^2q^{n+\tau+2}-1)+\Lambda_n}{\Delta_{2n}\Delta_{2n+2}} \ , \ n \geq 0 \ , \\ H_q\big((x^2 - q^{-\tau})\tilde{u}\big) - q^{-2\tau-4}(b^2(q-1))^{-1}\{b^2q^2(q^{2\tau+2}-1) \\ \qquad +(1+q^2)(\frac{1}{\rho}-1)(b^2q^{2\tau+1}-1)\}x\tilde{u} \\ \qquad +\{q^{-2\tau-4}(b^2(q-1))^{-1}(\frac{1}{\rho}-1)(b^2q^{2\tau+1}-1)x^2 \\ \qquad +\rho q^{-\tau}[\tau+1]_q \frac{b^2q^{\tau+1}-1}{b^2q^{2\tau+1}-1} - q^{-\tau}\}(x^{-1}\tilde{u}^2) = 0 \ , \end{cases} \tag{64}$$

with (for $n \geq 0$)

$$\begin{cases} \Delta_n = b^2q^{n+2\tau+3} - 1 - (q^n - 1)(b^2q^{2\tau+1}-1)(\frac{1}{\rho}-1) \ , \\ \Lambda_n = (q^{n+1}-1)(q^n-1)(q^{2\tau+1}b^2-1)(1-\frac{1}{\rho}) \ . \end{cases}$$

When $\rho = 1$ and $\tau = 0$, $\tilde{u}$ is a particular case of Big q-Jacobi polynomials [9].

References

[1] J. Alaya, P. Maroni, *Symmetric Laguerre-Hahn forms of class $s = 1$,* Integral Transforms Spec. Funct. **4**, 1996, 301-320.

[2] H. Bouakkaz, P. Maroni, *Description des polynômes orthogonaux de Laguerre-Hahn de classe zéro, in Orthoggonals Polynomials and Their Applications,* C. Brezinski, L. Gori, and A. Ronveaux, eds, Annals of Computation and Applied Mathematics, **9**, J.C.Baltzer AGasel, 1991, 189-194.

[3] T.S. Chihara, *An introduction to orthogonal polynomials,* Gordon and Breach, New York, 1978.

[4] J. Dini, *Sur les formes linires et polynes orthogonaux de Laguerre-Hahn,* These de Doctorat, Universit Pierre et Marie Curie, Paris VI, 1988.

[5] J. Dzoumba, *Sur les polynes de Laguerre-Hahn, These de troisiéme cycle,* Universit Pierre et Marie Curie, Paris VI, 1985.

[6] A. Ghressi, L. Kheriji, I. Tounsi, *An Introduction to the q-Laguerre-Hahn Orthogonal q-Polynomials,* Sym. Int. Geo. Meth. App., **7**, 2011, 1-20.

[7] A. Ghressi, L. Khériji, *The Symmetrical H_q-Semiclassical Orthogonal Polynomials of Class one,* SIGMA, **5**, 2009, 076, 22 pages, arXiv:0907.3851.

[8] M. Guerfi, *Les polynomes de Laguerre-Hahn affines discrets,* These de troisime cycle, Univ. P. et M. Curie, Paris, 1988

[9] L. Kheriji, P. Maroni, *The H_q-classical orthogonal polynomials,* Acta. Appl. Math., **71** 2002, 49-115.

[10] A. Magnus, *Riccati acceleration of Jacobi continued fractions and Laguerre-Hahn orthogonal polynomials,* in Pade Approximation and its Applications (Bad Honnef, 1983), Lecture Notes in Math., Vol. 1071, Springer, Berlin, 1984, 213-230.

[11] P. Maroni, *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques, in: Orthogonal Polynomials and their applications.* (C. Brezinski et al Editors.) IMACS, Ann. Comput. Appl. Math., **9**, (Baltzer, Basel), 1991, 95-130.

[12] M. Sghaier, M. Zaatra, *The D_w-Laguerre-Hahn Forms of Class Zero,* Mediterr. J. Math., **17(185)**, 2020, 1-28.

[13] M. Zaatra, *Laguerre-Freud equations associated with the H_q-semi-classical forms of class one,* Filomat, **32(19)**, 2018, 6769-6787.

Mabrouk Sghaier
Institut Superieur d'Informatique de Medenine. Medenine4119, Tunisia
E-mail: sghaier.mabrookbelhedi580@gmail.com or mabrouk.sghaier@isim.rnu.tn

Mohamed Zaatra
Institut Superieur des Sciences et Techniques des Eaux de Gabes. Gabes 6072, Tunisia
E-mail: medzaatra@yahoo.fr

Mehdi Mechri
Faculte des Sciences de Gabes, 6029 route de Medenine Gabes, Tunisia
E-mail: mehdi_ mechri@yahoo.fr

Received 14 June 2021
Accepted 25 February 2022