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Some Results on Matrix Transformation of Complex Uncertain Sequences

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Abstract. The main aim of the paper is to characterize matrix transformation of complex uncertain sequences upto some extent. We introduce the spaces $\ell_p(\Gamma_{u.a.s})$ and $[\ell_\infty(p)]_{\Gamma_{u.a.s}}$ and then derive necessary and sufficient condition for an infinite real matrix $A = (a_{nk})$ to transform sequences from $\ell_1(\Gamma_{u.a.s})$ to $\ell_p(\Gamma_{u.a.s})$ and $[\ell_\infty(p)]_{\Gamma_{u.a.s}}$ to $\ell_\infty(\Gamma_{u.a.s})$. Also, we extend the study of summable complex uncertain sequences with respect to uniformly almost surely via real infinite matrices, by introducing absolutely summable sequence in the same direction.

Key Words and Phrases: complex uncertain sequence, uncertainty space, matrix transformation, summable sequence.

2010 Mathematics Subject Classifications: 60B10, 60E05, 40A05, 40A30, 40F05

1. Introduction

The study of sequence spaces occupies a prominent position in analysis till date. Researchers applied the theory of sequence spaces to the several branches of mathematics like in the structural theory of topological vector spaces, law of large numbers, summability theory and theory of functions. Toeplitz first made a detailed study on matrix transformation on sequence spaces and then several mathematicians made progress enormously in this particular direction.

In most cases the general linear operator on one sequence space into another is actually given by an infinite matrix. But originally the study of general matrix transformation and sequence spaces theory was inspired by special results in the theory of summability. The theory of summability is concerned with the generalization of the notion of the limit of a sequence or a series which is usually affected by an auxiliary sequence of linear means of the given sequences or series.

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In the year 2007, when the theory of uncertainty was introduced by B. Liu [10], it was applied to the basics of different fields of mathematics viz. measure theory, programming, risk analysis, reliability analysis, propositional logic, entailment, set theory, inference, process, renewal process, calculus, differential equations, finance, statistics, chance theory have been studied in uncertain environment. As part of the study of uncertainty theory, Liu [10] introduced four concepts of convergence by considering sequences of uncertain variables and named these as convergence in mean, convergence in measure, convergence in distribution and convergence in almost surely. He has also shown the differences between these convergence concepts by putting the interconnections among the same. You [11] introduced a new type of convergence concept. Namely, convergence in uniformly almost surely, which is slightly different from convergence in almost surely. He proved that a convergent uncertain sequence in uniformly almost surely must converge in almost surely and provided the existence of such sequence which converges in almost surely but not uniformly. To describe the complex uncertain quantities, the notions of complex uncertain variable and complex uncertain distribution have been presented by Peng [15]. Chen et al. [14] explored convergence of sequence of complex uncertain variables due to Peng [15] and reported five convergence concepts introduced by Liu [10] and You [11], by establishing interrelationships among them. These convergence concepts of complex uncertain sequences have also been generalized by Datta and Tripathy [12], Das et al. [1, 2, 3, 5] in different aspects like almost convergence [1, 5], statistical convergence [2, 3], etc.

Very recently, researchers introduced the notion of convergence of complex uncertain series [4] and summability of complex uncertain sequences [6]. Using the same concept, authors characterized matrix transformation of convergent complex uncertain sequences with respect to uniformly almost surely [4], mean [8], measure [9] and almost surely [6, 7]. In this article, we extend the same work to some extent. New types of sequence spaces are introduced in this context and properties are studied via matrix transformation.

Before going to the main section, we would like to present some preliminary ideas which will play significant role in the whole study.

2. Preliminaries

Definition 1. *[10] Let $\mathcal{M}$ be a σ-algebra on a non-empty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following axioms:*
Axiom 1 (Normality Axiom). $\mathcal{M}\{\Gamma\}=1$;
Axiom 2 (Duality Axiom). $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\}=1$, *for any* $\Lambda \in L$;
Axiom 3 (Subadditivity Axiom). For every countable sequence of $\{\Lambda_j\} \in \mathcal{L}$, *we*

have

$$\mathcal{M}\{\bigcup_{j=1}^{\infty} \Lambda_j\} \leq \sum_{j=1}^{\infty} \mathcal{M}(\Lambda_j).$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ *is called an uncertainty space, and each element* Λ *in* L *is called an event. In order to obtain an uncertain measure of compound events, a product uncertain measure is defined as follows:*
Axiom 4 (Product Axiom). Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ *be uncertainty spaces, for* $k =$ *1,2,3..... The product uncertain measure* $\mathcal{M}$ *is an uncertain measure satisfying*

$$\mathcal{M}\{\prod_{j=1}^{\infty} \Lambda_j\} = \bigwedge_{j=1}^{\infty} \mathcal{M}(\Lambda_j),$$

where Λ_k *are arbitrarily chosen events from* Γ_k*, for* $k = 1, 2, 3, ...$ *respectively.*

Definition 2. *[15] A complex uncertain variable is a measurable function* ζ *from an uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to the set of complex numbers. i.e., for any Borel set* B *of complex numbers, the set* $\{\zeta \in B\}$=$\{\gamma \in \Gamma : \zeta(\gamma) \in B\}$ *is an event.*

Definition 3. *[14] The complex uncertain sequence* $\{\zeta_n\}$ *is said to be convergent uniformly almost surely (u.a.s.) to* ζ *if there exists a sequence of events* $\{E_{k'}\}$, $\mathcal{M}\{E_{k'}\} \to 0$ *such that* $\{\zeta_n\}$ *converges uniformly to* ζ *in* $\Gamma - E_{k'}$*, for fixed* $k' \in \mathbb{N}$.

Definition 4. *[4] An infinite complex uncertain series* $\sum\limits_{k=1}^{\infty} \zeta_k(\gamma)$ *is said to be convergent with respect to uniformly almost surely if the sequence of its partial sums* $\{S_n\}$ *is convergent with respect to uniformly almost surely. Then, there exist events* E_k*, where* $\mathcal{M}\{E_k\} \to 0$ *such that* $\{S_n\}$ *converges uniformly to a finite limit* S *in* $\Gamma - E_k$*, for each* k.

Definition 5. *[4] A complex uncertain sequence* $\{\zeta_n\}$ *is said to be uniformly almost surely A-summable to* ζ *if there exists a sequence* $\{E_x\}$ *of events with uncertain measure of each events tending to zero such that the A-limit of the complex uncertain sequence* $\{\zeta_n\}$ *is* ζ*. That is*

$$\lim_{n\to\infty} (A\zeta)_n = \lim_{n\to\infty} A(\zeta_n(\gamma)) = \zeta(\gamma),$$

for all $\gamma \in \Gamma - E_x$.

3. Main Results

Matrix transformation of sequences always play an important role in the study of sequence spaces. Nowadays, several mathematicians are exploring the characterizations of sequence spaces in the environment of uncertainty. Authors have already made the initial contribution in the direction of matrix transformation of complex uncertain sequences. It has been seen that several results regarding matrix transformation are different in nature when we consider different aspects of uncertainty (in measure, in mean, in almost surely, in distribution, in uniformly almost surely). This treatise may be considered as an extension of that work and as a reference of this research work readers may follow [4].

Definition 6. *Let $p \geq 1$ be a fixed real number. The complex uncertain sequence $\{\zeta_n\}$ is called uniformly almost surely p-absolutely summable if there exists a sequence $\{E_x\}$ of events with $\mathcal{M}\{E_x\} \to 0$ such that*

$$\sum_{j=1}^{\infty} ||\zeta_j(\gamma)||^p < \infty, \qquad \gamma \in \Gamma - E_x$$

and the norm is defined by

$$||\zeta|| = \left(\sum_{j=1}^{\infty} ||\zeta_j(\gamma)||^p \right)^{\frac{1}{p}}.$$

The class of all uniformly almost surely p-absolutely summable sequences is denoted by $\ell_p(\Gamma_{u.a.s})$.

Theorem 1. $A \in (\ell_1(\Gamma_{u.a.s}), \ell_p(\Gamma_{u.a.s}))$ *if and only if the following conditions hold:*

(i) $M = \sup_k \sum_{n=1}^{\infty} |a_{nk}|^p < \infty$, *uniformly for all* $\forall\ k$ *and* $1 \leq p < \infty$;

(ii) $\sup_{n,k} |a_{nk}| < \infty$, *when* $p = \infty$.

Proof. Let $A : \ell_1(\Gamma_{u.a.s}) \to \ell_p(\Gamma_{u.a.s})$ be a bounded linear operator such that $A\zeta = A_n(\zeta(\gamma)) = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$, $\forall\ \gamma \in \Gamma - E_x$, where E_x are the events with $\mathcal{M}\{E_x\} \to 0$.
Now, $A\zeta \in \ell_p(\Gamma_{u.a.s})$, which implies

$$\sum_{i=1}^{\infty} ||A_i(\zeta)||^p < \infty, \text{ where } \zeta \in \ell_1(\Gamma_{u.a.s}).$$

Since each $A_n(\zeta(\gamma)) = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$ converges in uniformly almost surely for all $\gamma \in \Gamma - E_x$, by Banach-Steinhaus theorem ([13]) we have $\sup_k |a_{nk}| < \infty$, uniformly for all n.
Hence, $A_n\zeta \in \ell_1^*(\Gamma_{u.a.s})$.
Next, define $q_n(\zeta)$ as follows:

$$q_n(\zeta) = q_n(\zeta(\gamma)) = \left(\sum_{k=1}^{n} ||A_k(\zeta(\gamma))||^p\right)^{1/p}, \quad \text{uniformly for all } n.$$

Each of the $q_n(\zeta)$ is subadditive, by Minkowski's inequality. Thus, each q_n is a seminorm in $\ell_1(\Gamma_{u.a.s})$, since $q_n(\lambda\zeta) = |\lambda| q_n(\zeta)$.
Again, since $A_n\zeta \in \ell_1^*(\Gamma_{u.a.s})$, A_n is a bounded linear functional on $\ell_1(\Gamma_{u.a.s})$. Then, each q_n is bounded therein.
Hence, $\{q_n\}$ is a sequence of continuous seminorms on $\ell_1(\Gamma_{u.a.s})$ with

$$\sup_n q_n(\zeta(\gamma)) = \left(\sum_{k=1}^{\infty} ||A_k(\zeta(\gamma))||^p\right)^{1/p} < \infty,$$

for each $\zeta \in \ell_1(\Gamma_{u.a.s})$.
Applying the generalized form of the Banach-Steinhaus theorem (Theorem 11 page 114 of [13]), we obtain a constant H such that

$$\sum_{k=1}^{\infty} ||A_k(\zeta(\gamma))||^p \leq H^p ||x||^p. \tag{1}$$

If we consider the uncertain sequence $\zeta = \{e_n\}$ defined by

$$e_n(\gamma_k) = \begin{cases} 1, & \text{if } n = k; \\ 0, & \text{otherwise,} \end{cases}$$

then from the inequality (1) condition (i) can be proved.
Condition (ii) is the consequence of condition (i).
For the converse part, let us consider a complex uncertain sequence $\zeta \in \ell_1(\Gamma_{u.a.s})$. We show $A\zeta = A_n(\zeta) \in \ell_p(\Gamma_{u.a.s})$, by applying Minkowski's inequality.

$$\left(\sum_{n=1}^{\infty} \left|\left|\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)\right|\right|^p\right)^{1/p} \leq \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} ||a_{nk}\zeta_k(\gamma)||^p\right)^{1/p}$$
$$= \sum_{k=1}^{\infty} ||\zeta_k(\gamma)|| \left(\sum_{n=1}^{\infty} |a_{nk}|^p\right)^{1/p}$$
$\leq ||\zeta|| M^{1/p}$, by condition (i).

Now since $\sup_{n,k} |a_{nk}| < \infty$ when $p = \infty$, $\left(\sum_{n=1}^{\infty} \left|\left|\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)\right|\right|^p\right)^{1/p}$ is finite, means bounded with respect to uniformly almost surely.
Hence, $A\zeta(\gamma) \in \ell_p(\Gamma_{u.a.s})$. ◀

Definition 7. *Let $p = \{p_k\}$ be a bounded sequence of strictly positive real numbers such that $p_k \leq \sup p_k = H$, where H is finite. The space $[\ell_\infty(p)]_{\Gamma_{u.a.s}}$ is defined as follows:*

$$[\ell_\infty(p)]_{\Gamma_{u.a.s}} = \left\{\zeta = \{\zeta_k\} : \sup_{k\in\mathbb{N}} ||\zeta_k(\gamma)||^{p_k} < \infty\right\},$$

where $\zeta = \{\zeta_k\}$ *is a complex uncertain sequence and* $\gamma \in \Gamma - E_x$*,* E_x *being the uncertain events with uncertain measure tending to zero.*

Theorem 2. $A \in \left([\ell_\infty(p)]_{\Gamma_{u.a.s}}, \ell_\infty(\Gamma_{u.a.s})\right)$ *if and only if*

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| N^{\frac{1}{p_k}} < \infty, \ \forall\ N \in \mathbb{N}\ (N > 1).$$

Proof. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $\{E_x\}$ be events with $\mathcal{M}\{E_x\} \to 0$.
Let the infinite matrix operator $A \in ([\ell_\infty(p)]_{\Gamma_{u.a.s}}, \ell_\infty(\Gamma_{u.a.s}))$.
If possible, let $N \in \mathbb{N}$ $(N > 1)$ be such that

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| N^{\frac{1}{p_k}} = \infty.$$

Consider the infinite matrix operator $B = (b_{nk})$ defined by

$$b_{nk} = a_{nk} N^{\frac{1}{p_k}}, \ \forall\ k, n \in \mathbb{N}.$$

Obviously, $B \notin (\ell_\infty(\Gamma_{u.a.s}), \ell_\infty(\Gamma_{u.a.s}))$.
Thus, there exists complex uncertain sequence $\zeta = \{\zeta_k\} \in \ell_\infty(\Gamma_{u.a.s})$, with $||\zeta|| = 1$ such that

$$\sum_{k=1}^{\infty} b_{nk}\zeta_k(\gamma) \notin \ell_\infty(\Gamma_{u.a.s}), \text{ where } \gamma \in \Gamma - E_x.$$

Then,

$$\sum_{k=1}^{\infty} a_{nk} N^{\frac{1}{p_k}} \zeta_k(\gamma) \notin \ell_\infty(\Gamma_{u.a.s}), \text{ for } \gamma \in \Gamma - E_x.$$

Let $\eta = \{\eta_k\}$ be a complex uncertain sequence such that $\eta_k(\gamma) = N^{\frac{1}{p_k}} \zeta_k(\gamma)$.
Then, $\eta \in [\ell_\infty(p)]_{\Gamma_{u.a.s}}$. But

$$(A\eta)_n = A_n\eta(\gamma) = \sum_{k=1}^{\infty} a_{nk} N^{\frac{1}{p_k}} \zeta_k(\gamma) \notin \ell_\infty(\Gamma_{u.a.s}).$$

which contradicts the hypothesis that $A \in \left([\ell_\infty(p)]_{\Gamma_{u.a.s}}, \ell_\infty(\Gamma_{u.a.s})\right)$.

Conversely, let $\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| N^{\frac{1}{p_k}}$ be finite and $\zeta = \{\zeta_k\} \in [\ell_\infty(p)]_{\Gamma_{u.a.s}}$.

Consider a natural number N such that

$$N > \max\left\{1, \sup_{k\in\mathbb{N}} ||\zeta_k(\gamma)||^{\frac{1}{p_k}}\right\}, \quad \text{for } \gamma \in \Gamma - E_x.$$

Then,

$$\sup_{k\in\mathbb{N}} |(A\zeta)_n| \le \sup_{k\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| N^{\frac{1}{p_k}} < \infty, \quad \forall\, \gamma \in \Gamma - E_x.$$

Therefore, $A\zeta \in \ell_\infty(\Gamma_{u.a.s})$. ◄

In [4], authors already introduced the notion of A-summable complex uncertain sequences in uniformly almost surely. In this context, we now initiate the concept of absolute summability in the same direction and establish inter-relationship between summable and absolutely summable sequences of complex uncertain variable.

Definition 8. *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $p \ge 1$ be a fixed real number. The sequence $\{\zeta_n\}$ of complex uncertain variables is said to be uniformly almost surely p-absolutely summable if $\sum\limits_{j=1}^{\infty} |\zeta_j(\gamma)|^p < \infty$ and the norm is defined by*

$$||\zeta|| = \left(\sum_{j=1}^{\infty} |\zeta_j(\gamma)|^p\right)^{\frac{1}{p}}, \quad \textit{for each } \gamma \in \Gamma - E_x,$$

where E_x are uncertain events whose uncertain measure tends to zero.
The class of all almost surely p-absolutely summable sequences is denoted by $\ell_p(\Gamma_{u.a.s})$.

Theorem 3. *Every uniformly almost surely absolutely A-summable complex uncertain sequence $\{\zeta_n(\gamma)\}$ with index $p < 1$ is uniformly almost surely A-summable to $\zeta \equiv 0$.*

Proof. Let the complex uncertain sequence $\{\zeta_n\}$ be almost surely absolutely A-summable to ζ with index $p < 1$.
Then, there exist events E_x each approaching zero uncertain measure such that

$$\sum_{n=1}^{\infty} ||A\zeta||^p = \sum_{n=1}^{\infty} ||A\zeta_n(\gamma)||^p = \zeta(\gamma),$$

for all $\gamma \in \Gamma - E_x$.
This implies that $(A\zeta)_n \to 0$, as $n \to \infty$. ◄

Remark 1. *The converse of the above theorem is not true in general. This is demonstrated in the example below.*

Example 1. *Let* $(\Gamma, \mathcal{L}, \mathcal{M})$ *be an uncertainty space and* $\{E_x\}$ *be a sequence of uncertain events with* $\mathcal{M}(E_x) \to 0$, *for each* x.
Define the complex uncertain sequence $\{\zeta_n\}$ *as follows:*

$$\zeta_n(\gamma) = \begin{cases} \left(\frac{n+1}{log(n+3)} - \frac{n}{log(n+2)}\right) i, & \forall\ \gamma \in \Gamma - E_x; \\ 0, & otherwise. \end{cases}$$

Then, $\lim\limits_{n\to\infty} (C_1\zeta)_n = \lim\limits_{n\to\infty} \frac{1}{n+1} \sum\limits_{k=0}^{n} \left[\frac{k+1}{log(k+3)} - \frac{k}{log(k+2)}\right] i$

$= \lim\limits_{n\to\infty} \frac{n+1}{(n+1)log(n+3)}$

$= \lim\limits_{n\to\infty} \frac{1}{log(n+3)} = 0.$

Therefore $\{\zeta_n(\gamma)\}$ *is a uniformly almost surely* C_1*-summable sequence.*

But for any $p < 1$,

$$\sum_{n=1}^{\infty} |C_1\zeta_n(\gamma)|^p = \sum_{n=1}^{\infty} \frac{1}{[log(n+3)]^p} \to \infty.$$

Hence, the complex uncertain sequence $\{\zeta_n\}$ *is not uniformly almost surely absolute* C_1*-summable with index* p $(p < 1)$.

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