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Some Approximation Theorems in Weighted Smirnov-Orlicz Spaces

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Abstract. In this study, we investigate the degree of approximation of matrix transforms obtained by Faber series and prove some approximation theorems in weighted Smirnov-Orlicz spaces.

Key Words and Phrases: Smirnov-Orlicz spaces, Faber series, matrix transforms, Muckenhoupt weights.

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1. Introduction

In the present study, we will focus on approximation problems in Smirnov-Orlicz spaces, which we will mention later.

The Smirnov-Orlicz space, which is the more general form of the Smirnov space, has been firstly introduced by V. Kokilashvili in [26], where some inverse theorems of approximation theory have been investigated in Smirnov-Orlicz spaces on domains with Dini-smooth boundary. In these spaces, some approximation problems have been investigated by several researchers using different modulus of smoothness and different domains. Some direct theorems of trigonometric approximation in Smirnov-Orlicz space were proved for domains bounded by Carleson curves and for domains bounded by Dini smooth curves in [18] and in [21], respectively. The modulus of continuity used in [21] is simpler than the one considered in [18]. Some direct and inverse theorems of the polynomial approximation in weighted Smirnov-Orlicz spaces were proved on domains with Dini-smooth boundary in [2, 20]. In [3], the convergence property of the interpolating polynomial based on the zeros of Faber polynomials in Smirnov-Orlicz space were investigated on domain with bounded rotation curve without cusps. In

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[4], some approximation problems in the Smirnov-Orlicz spaces were investigated in terms of the fractional modulus of smoothness. Thus, the direct and inverse theorems of the approximation theory were proved by considering the fractional modulus of smoothness. In [5], the direct and converse theorems of approximation by algebraic polynomials and rational functions in weighted Smirnov-Orlicz spaces were proved. In [22], the approximation properties of the Faber–Laurent rational series expansions in Smirnov-Orlicz spaces were studied with doubly-connected domain bounded by Dini-smooth curves. In [23], the direct theorem of polynomial approximation was proved in weighted Smirnov-Orlicz spaces.

We extend the results obtained in the Lebesgue spaces in [35] to the weighted Orlicz and Smirnov-Orlicz spaces. Moreover, similar approximation theorems in weighted Orlicz space were proved in [24]. We note that, since we use the matrix method $\tau_n^{\lambda}(x, f)$ based on Riesz submethod instead of the matrix method based on Nörlund submethod, our main results are different from the results given in [24] even if classical methods are used.

Considering these studies in the literature, it is seen that approximation problems in terms of matrix transforms were not examined in the weighted Smirnov-Orlicz spaces. In this study, approximation properties of matrix transforms obtained by Faber series are examined and some approximation theorems in the weighted Smirnov-Orlicz spaces are proved. In Section 2, we obtain trigonometric approximation results by matrix transforms via Fourier series in weighted Orlicz spaces with Muckenhoupt weights. Based on these approximation theorems, we prove two theorems of approximation by matrix transforms of Faber series in weighted Smirnov-Orlicz spaces in Section 3.

Also, we note that the Orlicz spaces have been used for applications in mechanics, fluid dynamics, applied mathematics, regularity theory and statistical physics in [1, 12, 42]. Hence, results related to trigonometric approximation using submethods of Fourier series of functions in the Orlicz spaces are also important in these research areas.

2. Some Approximation Problems In Weighted Orlicz Spaces

Let $\mathbb{T}:=[0,2\pi]$. Let $L^p(\mathbb{T})$ be the Lebesgue space of 2π periodic real valued functions defined on $\mathbb{T}$ such that

$$\|f\|_p := \left(\int_{\mathbb{T}} |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty, \ \ (1 < p < \infty, \ f \in L^p(\mathbb{T})) .$$

Let $M : [0, \infty) \to [0, \infty)$ be a convex and continuous function with $M(0) = 0$,

$M(u) > 0$ for $u > 0$. If $M(u)$ is an even function and it satisfies the conditions

$$\lim_{u\to 0}\frac{M(u)}{u} = 0 \text{ and } \lim_{u\to\infty}\frac{M(u)}{u} = \infty,$$

then $M(u)$ is called a Young function [28, p. 9].

Let M be a Young function. The function N defined as

$$N(\nu) := \max\{u\nu - M(u) : u \geq 0\}$$

is called the complementary Young function for $\nu \geq 0$ [28, p. 13].

The class of all measurable functions $f : \mathbb{T} \to \mathbb{R}$ satisfying the condition

$$\int_0^{2\pi} M(|f(x)|)\,dx < \infty$$

will be denoted by $\tilde{L}_M(\mathbb{T})$.

The linear span of $\tilde{L}_M(\mathbb{T})$ equipped with the Orlicz norm

$$\|f\|_M := \sup\left\{\int_0^{2\pi} |f(x)\,g(x)|\,dx : g \in \tilde{L}_N(\mathbb{T}),\ \int_0^{2\pi} N(|f(x)|)\,dx \leq 1\right\},$$

or with the Luxemburg norm

$$\|f\|_M^* := \inf\left\{k > 0 : \int_0^{2\pi} M\left(\frac{|f(x)|}{k}\right)dx \leq 1\right\}$$

becomes a Banach space. This Banach space is called the Orlicz space generated by M [30, p. 60-69].

Note that, the Orlicz space $L_M(\mathbb{T})$ is known as one of the generalizations of the Lebesgue space $L^p(\mathbb{T})$, $1 < p < \infty$. If we take $M_p(u) := \frac{u^p}{p}$, $1 < p < \infty$, then the space $L_M(\mathbb{T})$ turns into the Lebesgue space $L^p(\mathbb{T})$. Every function in $L_M(\mathbb{T})$ is integrable on $\mathbb{T}$ [39, p. 50], i.e.

$$L_M(\mathbb{T}) \subset L_1(\mathbb{T}).$$

For detailed information about the Orlicz spaces, we refer to [11, 27, 28, 31, 39].

The Orlicz and Luxemburg norms satisfy the inequalities

$$\|f\|_M^* \leq \|f\|_M \leq 2\|f\|_M^*,\ f \in L_M(\mathbb{T}),$$

and hence they are equivalent. Furthermore, the Orlicz norm can be determined by means of the Luxemburg norm [28, p. 79-80] as

$$\|f\|_M := \sup\left\{ \int_0^{2\pi} |f(x)\, g(x)|\, dx : \quad \|g\|_N^* \le 1 \right\}.$$

The Hölder inequalities for every $f \in L_M(\mathbb{T})$ and $g \in L_N(\mathbb{T})$ are given [28, p. 80] as

$$\int_0^{2\pi} |f(x)\, g(x)|\, dx \le \|f\|_M \, \|g\|_N^* ,$$

$$\int_0^{2\pi} |f(x)\, g(x)|\, dx \le \|f\|_M^* \, \|g\|_N .$$

Let $M^{-1} : [0,\infty) \to [0,\infty)$ be the inverse of the Young function M and

$$h(t) := \limsup_{x\to\infty} \frac{M^{-1}(x)}{M^{-1}(t/x)}, \; t > 0.$$

The numbers α_M and β_M defined by

$$\alpha_M := \lim_{t\to\infty} \frac{-\log h(t)}{\log t} \text{ and } \beta_M := \lim_{t\to 0^+} \frac{-\log h(t)}{\log t}$$

are called the lower and upper Boyd indices [10]. It is known that these indices satisfy the conditions

$$0 \le \alpha_M \le \beta_M \le 1$$

and

$$\alpha_M + \beta_N = 1 \text{ and } \alpha_N + \beta_M = 1.$$

These indices were firstly considered by Matuszewska and Orlicz [32].

The Orlicz space $L_M(\mathbb{T})$ is reflexive if and only if $0 < \alpha_M \le \beta_M < 1$, i.e. if the Boyd indices are nontrivial.

If $1 \le q < \frac{1}{\beta_M} \le \frac{1}{\alpha_M} < p \le \infty$, then $L_p(\mathbb{T}) \subset L_M(\mathbb{T}) \subset L_q(\mathbb{T})$, the inclusions being continuous and hence $L_\infty(\mathbb{T}) \subset L_M(\mathbb{T}) \subset L_1(\mathbb{T})$ [11, p. 340].

For detailed information about the Boyd indices, we refer to [9, 10, 25, 31].

A measurable 2π-periodic function $\omega : \mathbb{T} \to [0,\infty]$ is called a weight function if the set $\omega^{-1}(\{0,\infty\})$ has the Lebesgue measure zero.

The class of measurable functions f defined on $\mathbb{T}$ and satisfying the condition $\omega f \in L_M(\mathbb{T})$ is called weighted Orlicz space $L_M(\mathbb{T},\omega)$ with the norm

$$\|f\|_{M,\omega} := \|\omega f\|_M .$$

From Hölder inequality it follows that if $\omega \in L_M(\mathbb{T})$ and $\frac{1}{\omega} \in L_N(\mathbb{T})$, then $L_\infty(\mathbb{T}) \subset L_M(\mathbb{T},\omega) \subset L_1(\mathbb{T})$.

Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. The weight functions ω used in this section belong to the Muckenhoupt class $A_p(\mathbb{T})$ which is defined by

$$\left(\frac{1}{|I|}\int_I \omega^p(x)dx\right)^{\frac{1}{p}} \left(\frac{1}{|I|}\int_I \omega^{-q}(x)dx\right)^{\frac{1}{q}} \leq C$$

with a finite constant C independent of I, where I is any subinterval of $\mathbb{T}$ and $|I|$ denotes the length of I.

For detailed information about the Muckenhoupt class, we refer to [11, 36].

The best approximation of $f \in L_M(\mathbb{T},\omega)$ by polynomials in $\mathcal{T}_n$ is defined by

$$E_n(f)_{L_M(\mathbb{T},\omega)} := \inf_{T_n \in \mathcal{T}_n} \|f - T_n\|_{M,\omega}$$

where $\mathcal{T}_n$ is the set of trigonometric polynomials of degree $\leq n$.

Let $L_M(\mathbb{T},\omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \leq \beta_M < 1$ and $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$. The k-modulus of smoothness of the function $f \in L_M(\mathbb{T},\omega)$ is defined as

$$\Omega^k_{M,\omega}(f,\delta) = \sup_{\substack{0 < h_i \leq \delta \\ 1 \leq i \leq k}} \left\| \prod_{i=1}^{k} (I - \sigma_{h_i}) f \right\|_{M,\omega}, \qquad \delta > 0,$$

where I is the identity operator and

$$(\sigma_h f)(x) := \frac{1}{2h}\int_{-h}^{h} f(x+t)dt, \; 0 < h < \pi, \; x \in \mathbb{T}.$$

The modulus of smoothness $\Omega^k_{M,\omega}(f,\delta)$ is well defined because σ_h is a bounded linear operator on $L_M(\mathbb{T},\omega)$ [19, Lemma 1] under the conditions $0 < \alpha_M \leq \beta_M < 1$ and $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$.

The modulus of continuity $\Omega^k_{M,\omega}(f,\delta)$ is defined in this way, since the space $L_M(\mathbb{T},\omega)$ is non-invariant, in general, under the usual shift $f(x) \to f(x+h)$.

If $k=0$ we write $\Omega^{0}_{M,\omega}(f,\delta)=\|f\|_{M,\omega}$ and if $k=1$ we write $\Omega_{M,\omega}(f,\delta)=\Omega^{1}_{M,\omega}(f,\delta)$. The modulus of smoothness $\Omega^{k}_{M,\omega}(f,\cdot)$ is a nondecreasing, nonnegative, continuous function and

$$\Omega^{k}_{M,\omega}(f+g,\delta)\le\Omega^{k}_{M,\omega}(f,\delta)+\Omega^{k}_{M,\omega}(g,\delta)$$

for $f,g\in L_M(\mathbb{T},\omega)$.

For $0<\gamma$, let $k=[\frac{\gamma}{2}]+1$. The generalized Lipschitz class $Lip\,(\gamma,L_M(\mathbb{T},\omega))$ is defined as

$$Lip\,(\gamma,L_M(\mathbb{T},\omega))=\left\{f\in L_M(\mathbb{T},\omega):\Omega^{k}_{M,\omega}(f,\delta)=O\left(\delta^{\gamma}\right),\ \delta>0\right\}.$$

Since $L_M(\mathbb{T},\omega)\subset L_1$, we can define the Fourier series of $f\in L_M(\mathbb{T},\omega)$ as

$$f(x)\backsim\frac{a_0(f)}{2}+\sum_{k=1}^{\infty}\left(a_k(f)\cos kx+b_k(f)\sin kx\right)\tag{1}$$

and the conjugate Fourier series as

$$\tilde{f}(x)\backsim\frac{a_0(f)}{2}+\sum_{k=1}^{\infty}\left(a_k(f)\sin kx-b_k(f)\cos kx\right).$$

Here $a_0(f),a_k(f),b_k(f)$, $k=1,...,$ are Fourier coefficients of f. Let $S_n(x,f)$ $(n=0,1,2,...)$ be the nth partial sum of the series (1) at the point x, that is,

$$S_n(x,f):=\sum_{k=0}^{n}U_k(x,f),$$

where

$$U_0(x,f)=\frac{a_0}{2},\ U_k(x,f)=a_k(f)\cos kx+b_k(f)\sin kx,\qquad k=1,2,....$$

Let $(\lambda_n)_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers. For a sequence (x_k) of real or complex numbers, the Cesàro submethod C_λ is defined by

$$(C_\lambda x)_n:=\frac{1}{\lambda_n}\sum_{k=1}^{\lambda_n}x_k,\ (n=1,2,...).$$

In particular, when $\lambda_n=n$, we note that $(C_\lambda x)_n$ is the classical Cesàro method C_1. Therefore, the Cesàro submethod C_λ yields a subsequence of the Cesàro method C_1. The basic properties of the method C_λ were investigated firstly by Armitage and Maddox [6] and Osikiewicz [37]. In these works, the relations between the classical Cesàro method and Cesàro submethod were obtained. Further information about the method C_λ can be found in [6, 37].

We denote by $A \equiv (a_{n,k})$ a lower triangular regular matrix with nonnegative entries and let $S_n^{(A)}$ $(n = 0, 1, ...)$ be the row sums of this matrix, that is

$$S_n^{(A)} = \sum_{k=0}^{n} a_{n,k}.$$

We define

$$\tau_n^{\lambda}(x, f) := \sum_{k=0}^{\lambda_n} a_{\lambda_n,k} S_k(x, f), \; n = 0, 1, 2, ... \tag{1.5}$$

A nonnegative sequence $u := (u_n)$ is called almost monotone decreasing (increasing), if there exists a constant $K := K(u)$, depending on the sequence u only, such that $u_n \leq K u_m$ $(K u_n \geq u_m)$ for all $n \geq m$. Such sequences will be denoted by $u \in AMDS$ $(u \in AMIS)$.

Let

$$A_{n,k} := \frac{1}{k+1} \sum_{i=1}^{k} a_{n,i.}$$

If $\{A_{n,k}\} \in AMDS$ $(\{A_{n,k}\} \in AMIS)$, then we will say that $\{a_{n,k}\}$ is an almost monotone decreasing (increasing) mean sequence with respect to $k = 1, 2, ..., n$ for all n. Briefly, we will write $\{a_{n,k}\} \in AMDMS$ $(\{a_{n,k}\} \in AMIMS)$. For detailed information see [34].

The relation $\preceq$ is defined as "$A \preceq B \Leftrightarrow$ there exists a positive constant C, independent of essential parameters, such that $A \leq CB$". The operator Δ_k is defined by $\Delta_k \varsigma_{n,k} = \varsigma_{n,k} - \varsigma_{n,k+1}$.

We shall use the following lemmas for proving our main theorems.

Lemma 1. *[24] Let $f \in Lip\,(\gamma, L_M(\mathbb{T},\omega))$, $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$, $0 < \gamma \leq 1$, and $0 < \alpha_M \leq \beta_M < 1$. Then, the estimate*

$$\|f - S_n(f)\|_{L_M(\mathbb{T},\omega)} = O\left(n^{-\gamma}\right) \tag{4.2}$$

holds for $n = 1, 2, ...$

Lemma 2. *[24] Let $f \in Lip\,(1, L_M(\mathbb{T},\omega))$, $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$, and $0 < \alpha_M \leq \beta_M < 1$. Then, the estimate*

$$\|S_n(f) - \sigma_n(f)\|_{L_M(\mathbb{T},\omega)} = O\left(n^{-1}\right) \tag{4.3}$$

holds for $n = 1, 2, ...$

Lemma 3. *[35] Let $(a_{n,k})$ be a lower triangular matrix with nonnegative entries and row sums 1. If $(a_{n,k}) \in AMIMS$ or $(a_{n,k}) \in AMDMS$ and $(n+1)\,a_{n,0} = O(1)$, then*

$$\sum_{k=0}^{n} (k+1)^{-\gamma}\, a_{n,k} = O((n+1)^{-\gamma})$$

holds for $0 < \gamma < 1$.

Trigonometric approximation results obtained by Cesàro submethod C_λ have been studied by many researchers in [7, 8, 13, 14, 15, 29, 30, 33, 40].

But in these papers degree of approximation using matrix methods obtained by Cesàro submethod were not investigated in the weighted Orlicz spaces.

In this section, we examine trigonometric approximation by the matrix method $\tau_n^\lambda(x,f)$ to the functions in weighted Orlicz spaces with the degree of $O(\lambda_n^{-\alpha})$ $(0 < \alpha \leq 1)$.

Our new results in weighted Orlicz spaces are the following:

Theorem 1. *Let $f \in Lip\,(\gamma, L_M(\mathbb{T},\omega))$, $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$ and $A = (a_{\lambda_n,k})$ be a lower triangular matrix with nonnegative entries and row sums 1. If one of the following conditions*

(i) $0 < \gamma < 1$ *and* $\{a_{\lambda_n,k}\} \in AMIMS$,

(ii) $0 < \gamma < 1$, $\{a_{\lambda_n,k}\} \in AMDMS$ *and* $(\lambda_n + 1)\, a_{\lambda_n,0} = O(1)$,

holds, then

$$\left\| f(\cdot) - \tau_n^\lambda(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} = O\left(\lambda_n^{-\gamma}\right).$$

Proof. Since

$$\tau_n^\lambda(x,f) - f(x) = \sum_{k=0}^{\lambda_n} a_{\lambda_n,k} S_k(x,f) - f(x),$$

using Lemma 1 and Lemma 3 we have

$$\begin{aligned} \left\| \tau_n^\lambda(\cdot, f) - f(\cdot) \right\|_{L_M(\mathbb{T},\omega)} &\leq \sum_{k=0}^{\lambda_n} a_{\lambda_n,k} \left\| S_k(f) - f \right\|_{L_M(\mathbb{T},\omega)} \\ &= O\left(\sum_{k=0}^{\lambda_n} a_{\lambda_n,k} (k+1)^{-\gamma} \right) \\ &= O(\lambda_n^{-\gamma}). \end{aligned}$$

◄

Theorem 2. *Let $f \in Lip\left(1, L_M(\mathbb{T},\omega)\right)$, $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$ and $A = (a_{\lambda_n,k})$ be a lower triangular matrix with nonnegative entries and row sums 1. If the following condition*

$$\sum_{k=0}^{\lambda_n-2} |\Delta_k A_{\lambda_n,k}| = O\left(\lambda_n^{-1}\right)$$

holds, then

$$\left\| f(\cdot) - \tau_n^{\lambda}(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} = O\left(\lambda_n^{-1}\right).$$

Proof. For $\gamma = 1$, applying Abel's transformation two times, we get

$$\begin{aligned}
\tau_n^{\lambda}(x,f) - f(x) &= \sum_{k=0}^{\lambda_n} a_{\lambda_n,k} \left[S_k(x,f) - f(x)\right] \\
&= \sum_{k=0}^{\lambda_n-1} \left[S_k(x,f) - S_{k+1}(x,f)\right] \sum_{i=0}^{k} a_{\lambda_n,i} \\
&\quad + \left[S_n(x,f) - f(x)\right] \\
&= S_n(x,f) - f(x) - \sum_{k=0}^{\lambda_n-2} \left(A_{\lambda_n,k} - A_{\lambda_n,k+1}\right) \sum_{i=0}^{k} (i+1)\, U_{i+1}(x,f) \\
&\quad - A_{\lambda_n,\lambda_n-1} \sum_{k=0}^{\lambda_n-1} (k+1)\, U_{k+1}(x,f) \\
&= S_n(x,f) - f(x) - \sum_{k=0}^{\lambda_n-2} \left(A_{\lambda_n,k} - A_{\lambda_n,k+1}\right) \times \\
&\quad \sum_{i=0}^{k} (i+1)\, U_{i+1}(x,f) - \frac{1}{\lambda_n} \sum_{i=0}^{\lambda_n-1} a_{\lambda_n,i} \sum_{k=0}^{\lambda_n-1} (k+1)\, U_{k+1}(x,f).
\end{aligned}$$

Therefore by Minkowski inequality, we get

$$\begin{aligned}
\left\| \tau_n^{\lambda}(\cdot,f) - f(\cdot) \right\|_{L_M(\mathbb{T},\omega)} &\le \left\| S_n(f) - f \right\|_{L_M(\mathbb{T},\omega)} + \sum_{k=0}^{\lambda_n-2} |\Delta_k A_{\lambda_n,k}| \left\| \sum_{i=1}^{k+1} i U_i(\cdot,f) \right\|_{L_M(\mathbb{T},\omega)} \\
&\quad + \frac{1}{\lambda_n} \left\| \sum_{k=1}^{\lambda_n} k U_k(\cdot,f) \right\|_{L_M(\mathbb{T},\omega)}. \qquad (2)
\end{aligned}$$

Then,

$$S_n(x,f) - \sigma_n(x,f) = \frac{1}{\lambda_n+1} \sum_{k=1}^{\lambda_n} k U_k(x,f).$$

Therefore, by Lemma 2

$$\left\|\sum_{i=1}^{\lambda_n} iU_i(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)} = (\lambda_n+1)\|S_n(f)-\sigma_n(f)\|_{L_M(\mathbb{T},\omega)} = O(1). \qquad (3)$$

If

$$\sum_{k=0}^{\lambda_n-1} |\Delta_k A_{\lambda_n,k}| = O\left(\lambda_n^{-1}\right),$$

then, from (2) and (3) we get

$$\begin{aligned}\left\|f(\cdot)-\tau_n^{\lambda}(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)} &= O\left(\lambda_n^{-1}\right)+O(1)\cdot\sum_{k=0}^{\lambda_n-1}|\Delta_k A_{\lambda_n,k}| \\ &= O\left(\lambda_n^{-1}\right).\end{aligned}$$

Therefore, the proof is completed. ◄

Remark 1. *When* $a_{\lambda_n,k} = \frac{p_{\lambda_n,k}}{P_k}$, *the matrix submethod* $\tau_n^{\lambda}(x,f)$ *turns into the Riesz submethod* $R_n^{\lambda}(x,f)$ *given as*

$$R_n^{\lambda}(x,f) := \frac{1}{P_n}\sum_{k=0}^{\lambda_n} p_{\lambda_n,k} S_k(x,f),$$

where $P_{\lambda_n} = p_0+p_1+p_2+...+p_{\lambda_n} \neq 0 \quad (n \geqslant 0)$ *and by convention* $p_{-1} = P_{-1} = 0$.

Remark 2. *Also, in the case* $p_n = 1$, $n \geq 0$, $\lambda_n = n$, $R_n^{\lambda}(x,f)$ *turns into the classical Cesàro method*

$$\sigma_n(x,f) = \frac{1}{n+1}\sum_{m=0}^{n} S_m(x,f).$$

Corollary 1. *Let* $f \in Lip(\gamma, L_M(\mathbb{T},\omega))$, $0 < \alpha_M \leq \beta_M < 1$, $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$, $0 < \gamma \leq 1$ *and let* (p_k) *be a sequence of positive real numbers. If one of the following conditions*

(i) $0 < \gamma < 1$ *and* $\{p_k\} \in AMIMS$,

(ii) $0 < \gamma < 1$, $\{p_k\} \in AMDMS$ *and* $(\lambda_n+1)\,p_{\lambda_n} = O(P_{\lambda_n})$,

(iii) $\gamma = 1$, $\sum_{k=0}^{\lambda_n-1}\left|\Delta_k \frac{p_k}{k+1}\right| = O\left(\frac{P_{\lambda_n}}{\lambda_n}\right)$

holds, then we have

$$\left\| f(\cdot) - R_n^{\lambda}(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} = O\left(\lambda_n^{-\gamma}\right).$$

Note: Our main theorems give the error(or degree) of estimation in terms of $\lambda_n^{-\alpha}$ $(0 < \alpha \leq 1)$, which is sharper than the results given using classical methods, because of $\lambda_n^{-\alpha} \leq n^{-\alpha}$ for $0 < \alpha \leq 1$.

3. Some Approximation Problems In Weighted Smirnov-Orlicz Spaces

Let Γ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$ and $\mathbf{G}^- := Ext$ Γ, $\mathbf{G} := Int$ Γ. Without loss of generality we may assume $0 \in \mathbf{G}$. Let $D := \{w \in \mathbb{C} : |w| < 1\}$, $\mathbb{T} : = \partial D$, $D^- : = ext\mathbb{T}$ and $w = \varphi(z)$ be the conformal mapping of $\mathbf{G}^-$ onto D^- normalized by the conditions

$$\varphi(\infty) = \infty, \ \lim_{z \to \infty} \frac{\varphi(z)}{z} > 0,$$

and let $\psi := \varphi^{-1}$ be the inverse mapping of φ.

We denote by $L_p(\Gamma)$ and $E_p(\mathbf{G})$, $1 < p < \infty$, the set of all measurable complex valued functions such that $|f|^p$ is Lebesgue integrable with respect to the arc length on Γ and Smirnov space of analytic functions in $\mathbf{G}$, respectively. Each function $f \in E_p(\mathbf{G})$ has a non-tangential limit almost everywhere on Γ, and if we use the same notation for the non tangential limit of f, then $f \in L_p(\Gamma)$ [17, p. 438].

Let h be a continuous function on $[0, 2\pi]$. Its modulus of continuity is defined by

$$\omega(t, h) := \sup\{|h(t_1) - h(t_2)| : t_1, t_2 \in [0, 2\pi], \ |t_1 - t_2| \leq t\}, \ t \geq 0.$$

The function h is called Dini-continuous if

$$\int_0^{\pi} \frac{\omega(t, h)}{t} dt < \infty.$$

Γ is called Dini-smooth curve if it has a parametrization

$$\Gamma : \varphi_0(\tau), \ 0 \leq \tau \leq 2\pi$$

such that $\varphi_0'(\tau)$ is Dini-continuous and $\varphi_0'(\tau) \neq 0$ [38, p. 48]. We denote the set of Dini smooth curves by $\mathbf{D}$.

If Γ is a Dini-smooth curve, then

$$\begin{aligned} 0 &< c_1 \leq \left|\psi'(\omega)\right| \leq c_2, \ |\omega| \geq 1, \\ 0 &< c_3 \leq \left|\varphi'(z)\right| \leq c_4, \ z \in G^-, \end{aligned} \tag{4}$$

hold. Here c_1, c_2 and c_3, c_4 are constants independent of ω and z, respectively.

We denote by $L_M(\Gamma)$ the linear space of Lebesgue measurable functions $f : \Gamma \to \mathbb{C}$ satisfying the condition

$$\int_\Gamma M[\alpha |f(z)|] |dz| < \infty$$

for some $\alpha > 0$.

The space $L_M(\Gamma)$ becomes a Banach space with the Luxemburg norm

$$\|f\|_{L_{(M)}(\Gamma)} := \inf\{\tau > 0 : \rho(f/\tau; M) \leq 1\},$$

and also with the Orlicz norm

$$\|f\|_{L_M(\Gamma)} := \sup\left\{\int_\Gamma |f(z)g(z)| |dz| : g \in L_N(\Gamma); \rho(g; N) \leq 1\right\},$$

where N is the complementary N-function to M and

$$\rho(g; N) := \int_\Gamma N[|g(z)|] |dz|.$$

The Banach space $L_M(\Gamma)$ is called Orlicz space.

The class of measurable functions f defined on Γ and satisfying the condition $\omega f \in L_M(\Gamma)$ is called weighted Orlicz space $L_M(\Gamma,\omega)$ with the norm

$$\|f\|_{L_M(\Gamma,\omega)} := \|\omega f\|_{L_M(\Gamma)}.$$

For $z \in \Gamma$ and $\epsilon > 0$, let $\Gamma(z, \epsilon)$ denote the portion on Γ contained in the open disc of radius ϵ and centered at z, i.e. $\Gamma(z, \epsilon) := \{t \in \Gamma : |t - z| < \epsilon\}$.

For fixed $p \in (1, \infty)$, we define $q \in (1, \infty)$ by $p^{-1} + q^{-1} = 1$. The set of all weights $\omega : \Gamma \to [0, \infty]$ satisfying the relation

$$\sup_{t\in\Gamma}\sup_{\epsilon>0}\left(\frac{1}{\epsilon}\int_{\Gamma(z,\epsilon)} \omega(\tau)^p |d\tau|\right)^{1/p}\left(\frac{1}{\epsilon}\int_{\Gamma(z,\epsilon)} \omega(\tau)^{-q} |d\tau|\right)^{1/q} < \infty$$

is denoted by $A_p(\Gamma)$.

Definition 1. *For a weight ω on Γ, the weighted Smirnov-Orlicz class $E_M(\mathbf{G},\omega)$ is defined as the sub-class of analytic functions of $E^1(\mathbf{G})$ whose boundary value functions belong to weighted Orlicz space $L_M(\Gamma,\omega)$.*

Let Γ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$. For $p>1$ a class of $L^p(\Gamma)$ is defined as Lebesgue measurable functions $f:\Gamma\to\mathbb{R}$ satisfying the condition

$$\left(\int\limits_{\Gamma}|f(z)|^p\,|dz|\right)^{\frac{1}{p}}<\infty.$$

The class $L^p(\Gamma)$ is a Banach space with respect to the norm

$$\|f\|_{L^p(\Gamma)}:=\left(\int\limits_{\Gamma}|f(z)|^p\,|dz|<\infty\right)^{\frac{1}{p}}<\infty.$$

We denote by $L^p(\Gamma,\omega)$ the set of all measurable functions $f:\Gamma\to\mathbb{C}$ such that $|f|\,\omega\in L^p(\Gamma)$, $1<p<\infty$.

Definition 2. *For a weight ω on Γ, the weighted Smirnov-Orlicz class $E_M(\mathbf{G},\omega)$ is defined as the sub-class of analytic functions of $E^1(\mathbf{G})$ whose boundary value functions belong to weighted Orlicz space $L_M(\Gamma,\omega)$.*

The Smirnov space $E^p(\mathbf{G})$ of analytic functions in $\mathbf{G}$ is defined as

$$E^p(\mathbf{G}):=\left\{f\in E^1(\mathbf{G}):f\in L^p(\Gamma)\right\}.$$

$E^p(\mathbf{G})$ is a Banach space with respect to the norm

$$\|f\|_{E^p(\mathbf{G})}:=\left(\int\limits_{\Gamma}|f(z)|^p\,|dz|<\infty\right)^{\frac{1}{p}}<\infty.$$

Definition 3. *For a weight ω on Γ, the weighted Smirnov-Orlicz class $E_M(\mathbf{G},\omega)$ is defined as the sub-class of analytic functions of $E^1(\mathbf{G})$ whose boundary value functions belong to weighted Orlicz space $L_M(\Gamma,\omega)$.*

The class of functions f analytic in $\mathbf{G}$ and satisfying

$$\sup_{0<r<1}\int_{\Gamma_r}M\left[|f(z)|\,|dz|\right]\leq c<\infty$$

with c independent of r, will be called Smirnov-Orlicz class and denoted by $E_M(\mathbf{G})$.

The weighted Smirnov-Orlicz space $E_M(\mathbf{G},\omega)$ is a generalization of the Smirnov space $E^p(\mathbf{G})$. In particular, if $M(x) := x^p$, $1 < p < \infty$, then the weighted Smirnov-Orlicz space $E_M(\mathbf{G},\omega)$ coincides with the weighted Smirnov space $E^p(\mathbf{G},\omega)$; if $\omega := 1$, then $E_M(\mathbf{G},\omega)$ coincides with the Smirnov-Orlicz space $E_M(\mathbf{G})$, defined in [26].

Let Γ be a rectifiable Jordan curve and $f \in L^1(\Gamma)$. Then the functions f^+ and f^- defined by

$$\begin{aligned} f^+(z) \; &: = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta,\ z \in \mathbf{G}, \\ f^-(z) \; &: = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta,\ z \in \mathbf{G}^- \end{aligned}$$

are analytic in $\mathbf{G}$ and $\mathbf{G}^-$ respectively, and $f^-(\infty) = 0$.

For $g \in L_M(\mathbb{T},\omega)$ we set

$$\sigma_h(g)(\omega) := \frac{1}{2h}\int_{-h}^{h} g\left(\omega e^{it}\right)dt,\ 0 < h < \pi,\ \omega \in \mathbb{T}.$$

If α_M and β_M are nontrivial and $\omega \in A_{\frac{1}{\alpha_M}} \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$, then by [19] we have

$$\|\sigma_h(g)\|_{L_M(\mathbb{T},\omega)} \preceq \|g\|_{L_M(\mathbb{T},\omega)}$$

and consequently $\sigma_h(g) \in L_M(\mathbb{T},\omega)$ for any $g \in L_M(\mathbb{T},\omega)$.

Let $\omega_0(w) := \omega(\psi(w))$ and $f_0(w) := f(\psi(w))$ for a weight ω on Γ, $f \in L_M(\Gamma,\omega)$ and $w \in \mathbb{T}$. By (4) we have

$$f_0 \in L_M(\mathbb{T},\omega_0) \text{ for } f \in L_M(\Gamma,\omega). \tag{5}$$

Using the nontangential boundary values of f_0^+ on $\mathbb{T}$ we define the rth modulus of smoothness of $f \in L_M(\Gamma,\omega)$ as

$$\Omega^r_{\Gamma,M,\omega}(f,\delta) := \Omega^r_{M,\omega_0}\left(f_0^+,\delta\right),\ \delta > 0,$$

for $r = 1, 2, 3, \ldots$.

Let

$$E_n(f,\mathbf{G})_{M,\omega} := \inf_{P \in \wp_n} \|f - P\|_{L_M(\Gamma,\omega)}$$

be the best approximation to $f \in E_M(\mathbf{G},\omega)$ in the class $\wp_n$ of algebraic polynomials of degree not greater than n.

Definition 4. *For $\alpha > 0$, let $r := \left[\frac{\alpha}{2}\right] + 1$. The set of functions $f \in E_M(\mathbf{G}, \omega)$ such that*

$$\Omega^r_{\Gamma,M,\omega}(f, \delta) = O(\delta^\alpha), \ \ \delta > 0$$

is called the generalized Lipschitz class $Lip_M(\mathbf{G}, \alpha)$.

Let

$$S_\Gamma(f)(z) := \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int\limits_{\Gamma\setminus\{\zeta\in\Gamma:|\zeta-z|<\varepsilon\}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

be a Cauchy singular integral of $f \in L_M(\Gamma, \omega)$. These Cauchy type integrals have the non-tangential inside and outside limits f^+ and f^-, respectively a.e. on Γ. In addition, the formulas

$$\begin{aligned} f^+(z) &= S_\Gamma(f)(z) + \frac{1}{2} f(z), \\ f^-(z) &= S_\Gamma(f)(z) - \frac{1}{2} f(z) \end{aligned}$$

hold, therefore, we get

$$f(z) = f^+(z) - f^-(z) \tag{6}$$

a.e. on Γ [17, p. 431].

Introduce the Faber polynomials F_k, $k = 0, 1, 2, ...$ defined as [41]

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{F_k(z)}{w^{k+1}}, \ \ w \in D^-, \ \ z \in \mathbf{G}. \tag{7}$$

Then the equalities

$$F_k(z) = \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{w^k \psi'(w)}{\psi(w) - z} dw, \ \ z \in \mathbf{G}, \tag{8}$$

$$F_k(z) = \varphi^k(z) + \int\limits_{\Gamma} \frac{\varphi^k(\zeta)}{\zeta - z} d\zeta, \ \ z \in \mathbf{G}^-,$$

hold [41, p. 33].

If $f \in E_M(\mathbf{G},\omega)$, then by definition $f \in E^1(\mathbf{G})$ and hence

$$f(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$= \frac{1}{2\pi i} \int_{\mathbb{T}} f(\psi(w)) \frac{\psi'(w)}{\psi(w) - z} dw, \ z \in \mathbf{G}.$$

Using (7) we have

$$f(z) \sim \sum_{k=0}^{\infty} a_k F_k(z), \ z \in \mathbf{G}, \tag{9}$$

where

$$a_k := a_k(f) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\psi(w))}{w^{k+1}} dw, \ k = 0, 1, 2, ...$$

The series (9) is called Faber series of $f \in E_M(\mathbf{G},\omega)$ and the coefficients a_k, $k = 0, 1, 2, ...$, are called Faber coefficients of $f \in E_M(\mathbf{G},\omega)$. The $\lambda_n th$ partial sum of the Faber series of f is defined as

$$S_{\lambda_n}^{\mathbf{G}}(f)(z) = \sum_{k=0}^{\lambda_n} a_k(f) F_k(z), \ n = 1, 2, 3, ...$$

Let $\wp := \{\text{all polynomials (with no restriction on the degree)}\}$, $\wp(D) := \{\text{traces of all members of } \wp \text{ on } D\}$ and let

$$T(P)(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{P(w)\psi'(w)}{\psi(w) - z} dw, \ z \in \mathbf{G}$$

be an operator T defined on $\wp(D)$.

Then by (8)

$$T\left(\sum_{k=0}^{\lambda_n} b_k w^k\right) = \sum_{k=0}^{\lambda_n} b_k F_k(z), \ z \in \mathbf{G}.$$

The Riesz submethod of the Faber series of $f \in E_M(\mathbf{G},\omega)$ with respect to the sequence (p_{λ_n}) is defined as

$$R_{\lambda_n}^{\mathbf{G}}(f)(z) := \frac{1}{P_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_{\lambda_n,k} S_k^{\mathbf{G}}(f)(z),$$

where $P_{\lambda_n} = \sum_{k=0}^{\lambda_n} p_k$. Clearly, if $p_{\lambda_n} = 1$ for all $n = 1, 2, 3, ...$, then $R_{\lambda_n}^{\mathbf{G}}(f)(z)$ coincides with the Cesàro mean $\sigma_n^{\mathbf{G}}(f)(z)$ defined as

$$\sigma_n^{\mathbf{G}}(f)(z) := \frac{1}{n+1} \sum_{k=0}^{n} S_k^{\mathbf{G}}(f)(z).$$

Let $A=(a_{\lambda_n,k})$ be an infinite lower triangular regular matrix with non-negative entries and let $s_{\lambda_n}^{(A)}$, $n=0,1,...$, denote the row sum of this matrix, that is $S_{\lambda_n}^{(A)}=\sum\limits_{k=0}^{\lambda_n} a_{\lambda_n,k}$. For a given $A=(a_{\lambda_n,k})$, we consider the matrix transform of Faber series of $f\in E_M(\mathbf{G},\omega)$ defined as

$$\tau_{\mathbf{G},\lambda_n}^{(A)}(f)(z):=\sum_{k=0}^{\lambda_n} a_{\lambda_n,k}S_k^{\mathbf{G}}(f)(z).$$

If we consider lower triangular matrix A with entries $a_{\lambda_n,k}=\frac{p_{\lambda_n,k}}{P_{\lambda_n}}$, then the matrix transform $T_{\mathbf{G},\lambda_n}^{(A)}(f)$ coincides with the Riesz submethod $R_{\lambda_n}^{\mathbf{G}}(f)$.

We say that the matrix $A=(a_{n,k})$ has almost monotone increasing (decreasing) rows if there is a constant $K_1,(K_2)$, depending only on A, such that $a_{n,k}\leq K_1 a_{n,m}\,(a_{n,m}\leq K_2 a_{n,k})$, where $0\leq k\leq m\leq n$.

Lemma 4. *[20] Let $0<\alpha_M,\beta_M<1$, $\omega\in A_{\frac{1}{\alpha_M}}(\Gamma)\cap A_{\frac{1}{\beta_M}}(\Gamma)$ and $f\in L_M(\Gamma,\omega)$. Then $f^+\in E_M(\mathbf{G},\omega)$ and $f^-\in E_M(\mathbf{G}^-,\omega)$.*

Lemma 5. *[20] Let Γ be a Dini-smooth curve, $0<\alpha_M,\beta_M<1$, $\omega\in A_{\frac{1}{\alpha_M}}(\Gamma)\cap A_{\frac{1}{\beta_M}}(\Gamma)$. Then the linear operator $T:\wp(D)\to E_M(\mathbf{G},\omega)$ is bounded.*

Lemma 6. *[20] Let Γ be a Dini-smooth curve, $0<\alpha_M,\beta_M<1$, $\omega\in A_{\frac{1}{\alpha_M}}(\Gamma)\cap A_{\frac{1}{\beta_M}}(\Gamma)$. Then the operator $T:E_M(D,\omega_0)\to E_M(\mathbf{G},\omega)$ is one-to-one and onto.*

The main theorems in this study are expressed and proved as follows:

Theorem 3. *Let Γ be a Dini-smooth curve. Let $f\in Lip_M(\mathbf{G},\gamma)$, $\gamma\in(0,1)$, $0<\alpha_M,\beta_M<1$, $\omega\in A_{\frac{1}{\alpha_M}}(\Gamma)\cap A_{\frac{1}{\beta_M}}(\Gamma)$ and let $A=(a_{\lambda_n,k})$ be a lower triangular matrix with $\left|S_{\lambda_n}^{(A)}-1\right|=O\left(\lambda_n^{-\gamma}\right)$. If one of the following conditions:*

(i) A has almost monotone decreasing rows and $(\lambda_n+1)\,a_{\lambda_n,0}=O(1)$,

(ii) A has almost monotone increasing rows and $(\lambda_n+1)\,a_{\lambda_n,r}=O(1)$, where r is the integer part of $\frac{\lambda_n}{2}$,

holds, then

$$\left\|f(\cdot)-T_{\mathbf{G},\lambda_n}^{(A)}(\cdot,f)\right\|_{L_M(\Gamma,\omega)}=O\left(\lambda_n^{-\gamma}\right).$$

Proof. Let Γ be a Dini-smooth curve. Let $f \in Lip_M(\mathbf{G},\gamma)$, $\gamma \in (0,1)$, $0<\alpha_M,\beta_M<1, \omega \in A_{\frac{1}{\alpha_M}}(\Gamma) \cap A_{\frac{1}{\beta_M}}(\Gamma)$ and let $A=(a_{\lambda_n,k})$ be a lower triangular matrix with $\left|S_{\lambda_n}^{(A)}-1\right| = O\left(\lambda_n^{-\gamma}\right)$. Then, we get $f_0^+ \in E_\Phi(D,\omega) \subset E^1(D)$ using (4) and Lemma 4 in the case of $f \in E_M(D)$. Therefore, the boundary function of f_0^+ belongs to $L_M(\mathbb{T},\omega)$. The analytic function on the unit disc has the Taylor series expansion. So, the function f_0^+ has the Taylor series expansion

$$\sum_{k=0}^{\infty} \beta_k\left(f_0^+\right) w^k,\ w \in D.$$

Let the Fourier series of the boundary function of $f_0^+ \in L_M(\mathbb{T},\omega) \subset L_1(\mathbb{T})$ be defined as

$$\sum_{k=-\infty}^{\infty} c_k\left(f_0^+\right) e^{ikt}.$$

Then, using [16, Theorem 3.4] we have

$$c_k\left(f_0^+\right) = \begin{cases} \beta_k\left(f_0^+\right), & if\ \ k \geq 1 \\ 0, & if\ \ k<0 \end{cases}.$$

Hence, we get

$$f_0^+(w) = \sum_{k=-\infty}^{\infty} c_k\left(f_0^+\right) w^k. \tag{10}$$

Also, using the equation (6), we can obtain the following relation showing the equality of the Faber coefficients of f and the Taylor coefficients of f_0^+:

$$a_k(f) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0(w)}{w^{k+1}}dw = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}}dw - \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0^-(w)}{w^{k+1}}dw$$

$$= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}}dw = \beta_k\left(f_0^+\right).$$

If $\sum_{k=0}^{\infty} a_k(f) F_k(z)$ is the Faber series expansion of $f \in E_M(D)$, then by (10) and (7) we get

$$T\left(\sum_{k=0}^{\lambda_n} c_k\left(f_0^+\right) w^k\right) = S_{\lambda_n}^{\mathbf{G}}(f)(z) \ \text{ and } T\left(T_{\lambda_n}^{(A)}\left(f_0^+\right)\right) = T_{\mathbf{G},\lambda_n}^{(A)}(f). \tag{11}$$

By considering (5), we get $\omega_0 \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$ and $f \in Lip_M(\mathbf{G},\omega,\gamma)$, $\gamma \in (0,1)$. Therefore, $f_0^+ \in Lip(\gamma, M, \omega_0)$. Hence, f_0^+ satisfies the conditions of Theorem 1. If one of the following conditions

(i) A has almost monotone decreasing rows and $(\lambda_n+1)\,a_{\lambda_n,0}=O(1)$,

(ii) A has almost monotone increasing rows and $(\lambda_n+1)\,a_{\lambda_n,r}=O(1)$, where r is the integer part of $\frac{\lambda_n}{2}$,

holds, then applying Theorem 1 for f_0^+, by Lemma 5 and Lemma 6 we have

$$\begin{aligned}\left\|f(\cdot)-T_{\mathbf{G},\lambda_n}^{(A)}(\cdot,f)\right\|_{L_M(\Gamma,\omega)} &= \left\|T\left(f_0^+\right)-T\left(T_{\lambda_n}^{(A)}\left(f_0^+\right)\right)\right\|_{L_M(\Gamma,\omega)}\\ &= \left\|T\left(f_0^+-T_{\lambda_n}^{(A)}\left(f_0^+\right)\right)\right\|_{L_M(\Gamma,\omega)}\\ &\preceq \left\|f_0^+-T_{\lambda_n}^{(A)}\left(f_0^+\right)\right\|_{L_M(\mathbb{T},\omega)}=O\left(\lambda_n^{-\gamma}\right).\end{aligned}$$

◀

Corollary 2. *Let Γ be a Dini-smooth curve. Let $f\in Lip_M(\mathbf{G},\gamma)$, $\gamma\in(0,1)$, $0<\alpha_M,\beta_M<1$, $\omega\in A_{\frac{1}{\alpha_M}}(\Gamma)\cap A_{\frac{1}{\beta_M}}(\Gamma)$ and let (p_{λ_n}) be a sequence of positive numbers. If one of the following conditions:*

(i) (p_{λ_n}) is almost monotone increasing and $(\lambda_n+1)\,p_{\lambda_n}=O(P_{\lambda_n})$,

(ii) (p_{λ_n}) is almost monotone decreasing,

holds, then

$$\left\|f(\cdot)-R_{\lambda_n}^{\mathbf{G}}(\cdot,f)\right\|_{L_M(\Gamma,\omega)}=O\left(\lambda_n^{-\gamma}\right).$$

Theorem 4. *Let Γ be a Dini-smooth curve. Let $f\in Lip_M(\mathbf{G},1)$, $0<\alpha_M,\beta_M<1$, $\omega\in A_{\frac{1}{\alpha_M}}(\Gamma)\cap A_{\frac{1}{\beta_M}}(\Gamma)$ and let $A=(a_{\lambda_n,k})$ be a lower triangular matrix with $\left|S_{\lambda_n}^{(A)}-1\right|=O\left(\lambda_n^{-1}\right)$. If*

$$\sum_{k=1}^{\lambda_n-1}(\lambda_n-k)\left|\Delta_k a_{\lambda_n,k-1}\right|=O(1),$$

then

$$\left\|f(\cdot)-T_{\mathbf{G},\lambda_n}^{(A)}(\cdot,f)\right\|_{L_M(\Gamma,\omega)}=O\left(\lambda_n^{-1}\right).$$

Proof. Let Γ be a Dini-smooth curve. Let $f\in Lip_M(\mathbf{G},1)$, $0<\alpha_M,\beta_M<1$, $\omega\in A_{\frac{1}{\alpha_M}}(\Gamma)\cap A_{\frac{1}{\beta_M}}(\Gamma)$ and let $A=(a_{\lambda_n,k})$ be a lower triangular matrix with $\left|S_{\lambda_n}^{(A)}-1\right|=O\left(\lambda_n^{-1}\right)$. By considering (5), we get $\omega_0\in A_{\frac{1}{\alpha_M}}(\mathbb{T})\cap A_{\frac{1}{\beta_M}}(\mathbb{T})$ and

$f \in Lip_M(\mathbf{G},1)$. Then $f_0^+ \in Lip(1, M, \omega_0)$. Hence f_0^+ satisfies the conditions of Theorem 2. If

$$\sum_{k=1}^{\lambda_n - 1} (\lambda_n - k) \left| a_{\lambda_n,k-1} - a_{\lambda_n,k} \right| = O(1),$$

then applying Theorem 2 for f_0^+, by (11), Lemma 5 and Lemma 6 we have

$$\begin{aligned} \left\| f(\cdot) - T^{(A)}_{\mathbf{G},\lambda_n}(\cdot, f) \right\|_{L_M(\Gamma,\omega)} &= \left\| T\left(f_0^+\right) - T\left(T^{(A)}_{\lambda_n}\left(f_0^+\right)\right) \right\|_{L_M(\Gamma,\omega)} \\ &= \left\| T\left(f_0^+ - T^{(A)}_{\lambda_n}\left(f_0^+\right)\right) \right\|_{L_M(\Gamma,\omega)} \\ &\preceq \left\| f_0^+ - T^{(A)}_{\lambda_n}\left(f_0^+\right) \right\|_{L_M(\mathbb{T},\omega)} = O\left(\lambda_n^{-1}\right). \end{aligned}$$

◄

Remark 3. *Since*

$$\sum_{k=1}^{\lambda_n - 1} (\lambda_n - k) \left| \Delta_k a_{\lambda_n,k-1} \right| \leq \lambda_n \sum_{k=1}^{\lambda_n - 1} \left| \Delta_k a_{\lambda_n,k-1} \right|,$$

from Theorem 4 we immediately get the following result.

Corollary 3. *Let* Γ *be a Dini-smooth curve. Let* $f \in Lip_M(\mathbf{G},1)$, $0 < \alpha_M, \beta_M < 1$, $\omega \in A_{\frac{1}{\alpha_M}}(\Gamma) \cap A_{\frac{1}{\beta_M}}(\Gamma)$ *and let* $A = (a_{\lambda_n,k})$ *be a lower triangular matrix with* $\left| S^{(A)}_{\lambda_n} - 1 \right| = O\left(\lambda_n^{-1}\right)$. *If*

$$\sum_{k=1}^{\lambda_n - 1} \left| \Delta_k a_{\lambda_n,k-1} \right| = O\left(\lambda_n^{-1}\right),$$

then

$$\left\| f(\cdot) - T^{(A)}_{\mathbf{G},\lambda_n}(\cdot, f) \right\|_{L_M(\Gamma,\omega)} = O\left(\lambda_n^{-1}\right).$$

Corollary 4. *Let* Γ *be a Dini-smooth curve. Let* $f \in Lip_M(\mathbf{G},1)$, $0 < \alpha_M, \beta_M < 1$, $\omega \in A_{\frac{1}{\alpha_M}}(\Gamma) \cap A_{\frac{1}{\beta_M}}(\Gamma)$ *and let* (p_{λ_n}) *be a sequence of positive numbers. If*

$$\sum_{k=1}^{\lambda_n - 1} \left| \Delta_k p_k \right| = O\left(\frac{P_{\lambda_n}}{\lambda_n}\right),$$

then

$$\left\| f(\cdot) - R^{\mathbf{G}}_{\lambda_n}(\cdot, f) \right\|_{L_M(\Gamma,\omega)} = O\left(\lambda_n^{-1}\right).$$

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