Azerbaijan Journal of Mathematics
V. 13 , No 1, 2023, January
ISSN 2218-6816

Existence and Uniqueness of Solution for Caputo-Fabrizio Fractional Bratu-Type Initial Value Problem

A. Khalouta

Abstract. In this paper, we investigate and prove the existence and uniqueness of solution for Caputo-Fabrizio fractional Bratu-type initial value problem by using the Banach fixed point theorem. In addition, we present an algorithm of the homotopy perturbation transform method (HPTM) to obtain an approximate analytical solution of this problem. The HPTM is combined form of the ZZ transform and homotopy perturbation method. We establish the convergence results of the proposed method. The algorithm is applied to numerical example. To show the accuracy of the obtained results and the effectiveness of the proposed method, the approximate solutions are compared with the exact solutions.

Key Words and Phrases: fractional Bratu's type initial value problem, Caputo-Fabrizio fractional derivative, existence and uniqueness, Banach fixed point theorem, ZZ transform, homotopy perturbation method.

2010 Mathematics Subject Classifications: 34A08, 26A33, 34A12, 47H10

1. Introduction

The Bratu-type equation is one of the important equations in mathematical physics which has attracted the attention of many researchers due to its appearance in a wide variety of fields of application such as physics, chemistry and engineering. Among these applications, we can mention the fuel ignition model of the thermal combustion theory, the model of thermal reaction process, the Chandrasekhar model of the expansion of the universe, questions in geometry and relativity about the Chandrasekhar model, chemical reaction theory, radiative heat transfer and nanotechnology, electrospinning process for the manufacturing of the nanofibers [3, 5, 7, 8, 17].

Fractional calculus theory is a mathematical analysis tool for the study of integrals and derivatives of arbitrary order, which can be real or complex. In recent years, it has turned out that many phenomena in fluid mechanics, viscoelasticity, biology, economy, bioengineering and other areas of science can be successfully modeled by the use of fractional derivatives and integrals [1, 6, 11, 13, 15]. Therefore, the researchers put forward numerous different approaches to define the fractional derivatives to study the solutions of fractional order differential equations, which are generalizations of classical integer order differential equations. The most used ones are: Riemann-Liouville [14], Caputo [16], Hilfer [4], Hadamard [9], etc.

Recently, Michele Caputo and Mauro Fabrizio [2] modified the Caputo fractional derivative to come up with a new derivative, the Caputo-Fabrizio fractional derivative. This new fractional derivative was obtained by substituting the kernel in the Caputo fractional derivative with an exponential function to get the fractional derivative without singular kernel.

The main goal of this paper is to discuss the existence and uniqueness of solutions for Caputo-Fabrizio fractional Bratu-type initial value problem of the form

$$\begin{cases} \mathcal{D}^{\mu}u(t) + \lambda \exp\left(u(t)\right) = 0, \\ u(0) = \gamma_0, u'(0) = \gamma_1, \end{cases} \tag{1}$$

where $\mathcal{D}^{\mu}$ is the Caputo fractional derivative of the function $u(t)$ of order $\mu = \alpha+1$ with $0 < \alpha \leq 1$, $0 < t < 1$ and $\lambda \in \mathbb{R}$.

This paper is arranged as follows. In Section 2, we give the basic definitions and important properties of the Caputo-Fabrizio fractional derivative and ZZ transform. In Section 3, we present the main results of the existence and uniqueness of the solution for the Caputo-Fabrizio fractional Bratu-type initial value problem (1) using the Banach fixed point theorem. In Section 4, we describe the HPTM and establish its convergence to solve the problem (1) and demonstrate it by solving a numerical example. Finally the conclusion is given in Section 5.

2. Definitions and Preliminary Results

In this section, we will present the basic definitions and several properties for the fractional calculus and our results regarding the ZZ transform of the Caputo-Fabrizio fractional derivative that we shall use in this paper.

Definition 1. *[2] Let $u \in H^1(0,1), 0 < \alpha \leq 1$. Then the Caputo-Fabrizio fractional derivative of order α is defined as*

$$\mathcal{D}^{\alpha}u(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t u'(\tau)\exp\left(-\frac{\alpha\,(t-\tau)}{1-\alpha}\right)d\tau, \tag{2}$$

where $M(\alpha)$ is a normalization function that satisfies $M(0) = M(1) = 1$.

If the function does not belong to $H^1(0,1)$, then its fractional derivative is redefined as [2]

$$\mathcal{D}^{\alpha}u(t) = \frac{\alpha M(\alpha)}{1-\alpha}\int_0^t (u(t)-u(\tau))\exp\left(-\frac{\alpha\,(t-\tau)}{1-\alpha}\right)d\tau, 0 \leq t \leq 1. \tag{3}$$

Obviously [2], if we change the notation as

$$\sigma = \frac{1-\alpha}{\alpha} \in [0,\infty) \text{ and } \alpha = \frac{1}{1+\sigma} \in (0,1]\,, \tag{4}$$

then the Caputo-Fabrizio fractional derivative becomes

$$\mathcal{D}^{\alpha}u(t) = \frac{N(\sigma)}{\sigma}\int_0^t u'(\tau)\exp\left(-\frac{t-\tau}{\sigma}\right)d\tau, 0 \leq t \leq 1, \tag{5}$$

where $N(\sigma)$ is the normalization term corresponding to $M(\alpha)$ such that $N(0) = N(\infty) = 1$.

In addition,

$$\lim_{\sigma \longrightarrow 0}\frac{1}{\sigma}\exp\left(-\frac{t-\tau}{\sigma}\right) = \delta(t-\tau), \tag{6}$$

where $\delta(t-\tau)$ is the Dirac delta function.

For $n \geq 1$ and $0 < \alpha \leq 1$, the fractional derivative of order $\mu = \alpha + n$ is defined by

$$\mathcal{D}^{\mu}u(t) = \mathcal{D}^{\alpha+n}u(t) = \mathcal{D}^{\alpha}(\mathcal{D}^{n}u(t)). \tag{7}$$

The above Caputo-Fabrizio fractional derivative was later modified by Jorge Losada and Juan José Nieto [12] as

$$\mathcal{D}^{\alpha}u(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_0^t u'(\tau)\exp\left(-\frac{\alpha\,(t-\tau)}{1-\alpha}\right)d\tau, 0 \leq t \leq 1. \tag{8}$$

The fractional integral corresponding to the derivative in equation (8) was defined by Jorge Losada and Juan José Nieto in 2015 as follows.

Definition 2. *[12] Let* $0 < \alpha \leq 1$. *The fractional integral of order* α *of a function* u *is defined by*

$$\mathcal{I}^{\alpha}u(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t u(\tau)d\tau, 0 \leq t \leq 1. \tag{9}$$

From (9) it follows that the fractional integral of Caputo-Fabrizio type of a function u of order $0 < \alpha \leq 1$ is an average between function u and its first order integral, i.e.,

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1. \tag{10}$$

Therefore,

$$M(\alpha) = \frac{2}{2-\alpha}, 0 < \alpha \leq 1. \tag{11}$$

Due to this, Losada and Nieto remarked that Caputo-Fabrizio fractional derivative can redefined as

Definition 3. *[12] Let* $0 < \alpha \leq 1$. *The fractional Caputo-Fabrizio derivative of order* α *of a function* u *is given by*

$$\mathcal{D}^{\alpha}u(t) = \frac{1}{1-\alpha}\int_0^t u'(\tau)\exp\left(-\frac{\alpha\,(t-\tau)}{1-\alpha}\right)d\tau, 0 \leq t \leq 1, \tag{12}$$

and its fractional integral is defined as

$$\mathcal{I}^{\alpha}u(t) = (1-\alpha)u(t) + \alpha\int_0^t u(\tau)d\tau, 0 \leq t \leq 1. \tag{13}$$

Definition 4. *[18] Let* $u(t)$ *be a function defined for all* $t \geq 0$. *The ZZ transform of* $u(t)$ *is the function* $T(s,v)$ *defined by the following integral*

$$\mathbb{Z}\left[u(t)\right] = T(s,v) = \frac{s}{v}\int_0^{\infty}\exp\left(-\frac{st}{v}\right)u(t)dt, s, v > 0,$$

provided the integral exists for some variables s *and* v.

Some basic properties of the ZZ transform are given as follows.

Property 1: The ZZ transform is a linear operator

$$\mathbb{Z}\left[\lambda u(t) \pm \mu w(t)\right] = \lambda\mathbb{Z}\left[u(t)\right] \pm \mu\mathbb{Z}\left[w(t)\right], \lambda, \mu \in \mathbb{R}.$$

Property 2: If $u^{(n)}(t)$ is the n-th derivative of the function $u(t)$ with respect to the time variable t, then its ZZ transform is given by

$$\mathbb{Z}\left[u^{(n)}(t)\right] = T_n(s,v) = \frac{s^n}{v^n}T(s,v) - \sum_{k=0}^{n-1}\frac{s^{n-k}}{v^{n-k}}u^{(k)}(0).$$

Property 3: (Convolution property) Suppose $T(s,v)$ and $W(s,v)$ are the natural transforms of $u(t)$ and $w(t)$, respectively, both defined in the set A. Then the natural transform of their convolution is given by

$$\mathbb{Z}\left[(u*w)(t)\right] = \frac{v}{s}T(s,v)W(s,v),$$

where the convolution of two functions is defined by

$$(u*v)(t) = \int_0^t u(\xi)v(t-\xi)d\xi = \int_0^t u(t-\xi)v(\xi)d\xi.$$

Property 4: Some special ZZ transforms

$$\begin{aligned}
\mathbb{Z}[1] &= 1,\\
\mathbb{Z}[t] &= \frac{v}{s},\\
\mathbb{Z}[\exp[at]] &= \frac{s}{s-va},\\
\mathbb{Z}\left[\frac{t^n}{n!}\right] &= \frac{v^n}{s^n},\ n=0,1,2,....\\
\mathbb{Z}\left[\frac{t^\alpha}{\Gamma(\alpha+1)}\right] &= \frac{v^\alpha}{s^\alpha}, \alpha \geq 0.
\end{aligned}$$

Theorem 1. *The ZZ transform of the Caputo-Fabrizio fractional derivative of the function $u(t)$ of order $\alpha+n$, where $0<\alpha\leq 1$ and $n\in\mathbb{N}\cup\{0\}$, is given by*

$$\mathbb{Z}\left[\mathcal{D}^{\alpha+n}u(t)\right] = \frac{1}{s-\alpha(s-v)}\left[\frac{s^{n+1}}{v^n}\mathbb{Z}(u(t)) - \sum_{k=0}^{n}\frac{s^{n-k+1}}{v^{n-k}}u^{(k)}(0)\right]. \tag{14}$$

Proof. By Definition 3 of the Caputo-Fabrizio fractional derivative and using the relation (7), we have

$$\begin{aligned}
\mathbb{Z}\left[\mathcal{D}^{\alpha+n}u(t)\right] &= \mathbb{Z}\left[\mathcal{D}^{\alpha}(\mathcal{D}^{n}u(t))\right]\\
&= \frac{1}{1-\alpha}\frac{s}{v}\int_0^{+\infty}\exp\left(-\frac{st}{v}\right)\left(\int_0^t u^{(n+1)}(\tau)\exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau\right)dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1-\alpha}\frac{s}{v}\int_0^{+\infty} \exp\left(-\frac{st}{v}\right)\left(u^{(n+1)}(t) * \exp\left[-\frac{\alpha t}{1-\alpha}\right]\right) dt \\
&= \frac{1}{1-\alpha}\mathbb{Z}\left(u^{(n+1)}(t) * \exp\left[-\frac{\alpha t}{1-\alpha}\right]\right).
\end{aligned}
$$

Hence, from the Properties (2) and (3) of the ZZ transform, we have

$$
\begin{aligned}
\mathbb{Z}\left[\mathcal{D}^{\alpha+n}u(t)\right] &= \frac{1}{1-\alpha}\frac{v}{s}\mathbb{Z}\left(u^{(n+1)}(t)\right)\mathbb{Z}\left(\exp\left[-\frac{\alpha t}{1-\alpha}\right]\right) \\
&= \frac{v}{s-\alpha(s-v)}\left[\frac{s^{n+1}}{v^{n+1}}\mathbb{Z}\left(u(t)\right) - \sum_{k=0}^{n}\frac{s^{n-k+1}}{v^{n-k+1}}u^{(k)}(0)\right] \\
&= \frac{1}{s-\alpha(s-v)}\left[\frac{s^{n+1}}{v^{n}}\mathbb{Z}\left(u(t)\right) - \sum_{k=0}^{n}\frac{s^{n-k+1}}{v^{n-k}}u^{(k)}(0)\right].
\end{aligned}
$$

The proof is complete. ◀

Theorem 2. *[19] (Banach contraction principle) Let (X, d) be a complete metric space. Then each contraction mapping $T : X \longrightarrow X$ has a unique fixed point x in X, i.e.*

$$Tx = x.$$

3. Main results

In this section, we show the existence and uniqueness of the solution for the Caputo-Fabrizio fractional Bratu-type initial value problem using the fixed point theorem. Our main results are the following.

Theorem 3. *The Caputo-Fabrizio fractional Bratu-type initial value problem (1) has a solution in terms of the integral given by*

$$
\begin{aligned}
u(t) &= c_1 - \int_0^t \tau f(\tau)\exp\left(-\frac{\alpha\tau}{1-\alpha}\right) d\tau \\
&\quad + t\left(c_2 + \int_0^t f(\tau)\exp\left(-\frac{\alpha\tau}{1-\alpha}\right) d\tau\right),
\end{aligned} \tag{15}
$$

where

$$f(t) = -\lambda\left[(1-\alpha)\frac{d}{dt}\left(\exp u(t)\right) + \alpha \exp u(t)\right]\exp\left(\frac{\alpha t}{1-\alpha}\right), \tag{16}$$

$$c_1 = \gamma_0 + \int_0^t \tau f(\tau) \exp\left(-\frac{\alpha\tau}{1-\alpha}\right) d\tau \Big|_{t=0}, \tag{17}$$

$$c_2 = \gamma_1 - \int_0^t f(\tau) \exp\left(-\frac{\alpha\tau}{1-\alpha}\right) d\tau \Big|_{t=0}. \tag{18}$$

Proof. By using Definition 3 of the fractional Caputo-Fabrizio derivative and the relation (7), we have

$$\int_0^t u''(\tau) \exp\left(\frac{\alpha\tau}{1-\alpha}\right) d\tau = -\lambda(1-\alpha) \exp\left(\frac{\alpha t}{1-\alpha}\right) \exp\left(u(t)\right). \tag{19}$$

Integration by parts twice gives

$$\begin{aligned} & \frac{d}{dt}\left(u(t)\exp\left(\frac{\alpha t}{1-\alpha}\right)\right) - \frac{2\alpha}{1-\alpha} u(t) \exp\left(\frac{\alpha t}{1-\alpha}\right) \\ & + \left(\frac{\alpha}{1-\alpha}\right)^2 \int_0^t u(\tau) \exp\left(\frac{\alpha\tau}{1-\alpha}\right) d\tau \\ = \; & -\lambda(1-\alpha)\exp\left(\frac{\alpha t}{1-\alpha}\right)\exp\left(u(t)\right) + u'(0) - \frac{\alpha}{1-\alpha} u(0). \end{aligned} \tag{20}$$

We consider the new function

$$v(t) = u(t) \exp\left(\frac{\alpha t}{1-\alpha}\right), \tag{21}$$

and we differentiate (20) once. Then we get

$$v''(t) - k_1 v'(t) + k_2 v(t) = f(t), \tag{22}$$

where

$$\begin{aligned} k_1 &= \frac{2\alpha}{1-\alpha}, \qquad k_2 = \left(\frac{\alpha}{1-\alpha}\right)^2, \\ f(t) &= -\lambda\left[(1-\alpha)\frac{d}{dt}(\exp u(t)) + \alpha \exp u(t)\right] \exp\left(\frac{\alpha t}{1-\alpha}\right). \end{aligned}$$

The general solution of the equation (22) can be written as follows

$$\begin{aligned} v(t) &= \exp\left(\frac{\alpha t}{1-\alpha}\right)\left(c_1 - \int_0^t \tau f(\tau) \exp\left(-\frac{\alpha\tau}{1-\alpha}\right) d\tau \right. \\ & \left. + t\left(c_2 + \int_0^t f(\tau)\exp\left(-\frac{\alpha\tau}{1-\alpha}\right) d\tau\right)\right). \end{aligned} \tag{23}$$

Considering (21), we obtain

$$\begin{aligned} u(t) &= c_1 - \int_0^t \tau f(\tau) \exp\left(-\frac{\alpha\tau}{1-\alpha}\right) d\tau \\ &\quad + t\left(c_2 + \int_0^t f(\tau) \exp\left(-\frac{\alpha\tau}{1-\alpha}\right) d\tau\right), \end{aligned} \tag{24}$$

where

$$c_1 = \gamma_0 + \int_0^t \tau f(\tau) \exp\left(-\frac{\alpha\tau}{1-\alpha}\right) d\tau\Big|_{t=0},$$

and

$$c_1 = \gamma_1 - \int_0^t f(\tau) \exp\left(-\frac{\alpha\tau}{1-\alpha}\right) d\tau\Big|_{t=0},$$

and the theorem is proved. ◀

Now, let $C([0,1])$ be the Banach space of all continuous functions on $[0,1]$ with the norm

$$\|u\|_{C([0,1])} = \max_{t\in[0,1]} |u(t)|. \tag{25}$$

Theorem 4. *Let* $\exp(.) : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ *be continuous function and let the following conditions hold*

(H_1) *There exists a non-negative function* $k_1(t), k_2(t) \in L([0,1])$ *with*

$$\begin{aligned} |\exp(x) - \exp(y)| &\leq k_1 |x-y|, \\ \left|\frac{d}{dt}(\exp(x) - \frac{d}{dt}(\exp(y))\right| &\leq k_2 |x-y|, \end{aligned}$$

for all $x, y \in \mathbb{R}$.

(H_2) *Suppose that*

$$L = \max_{t\in[0,1]} \left|2\lambda\left((1-\alpha)\int_0^t k_1(\tau)d\tau + \alpha\int_0^t k_2(\tau)d\tau\right)\right| < 1.$$

Then, the Caputo-Fabrizio fractional Bratu-type initial value problem (1) has a unique solution $u(t) \in C([0,1])$.

Proof. By Theorem 3, we know that a function u is a solution to the Caputo-Fabrizio fractional Bratu-type initial value problem (1) if and only if u satisfies integral equation (15).

Let the operator $T : C([0,1]) \to C([0,1])$ be defined by

$$\begin{aligned} Tu(t) &= c_1 - \int_0^t \tau f(\tau) \exp\left(-\frac{\alpha\tau}{1-\alpha}\right) d\tau \\ &\quad + t\left(c_2 + \int_0^t f(\tau) \exp\left(-\frac{\alpha\tau}{1-\alpha}\right) d\tau\right), \end{aligned}$$

where

$$f(t) = -\lambda\left[(1-\alpha)\frac{d}{dt}(\exp u(t)) + \alpha \exp u(t)\right] \exp\left(\frac{\alpha t}{1-\alpha}\right),$$

$$c_1 = \gamma_0 + \int_0^t \tau f(\tau) \exp\left(-\frac{\alpha\tau}{1-\alpha}\right) d\tau\Big|_{t=0},$$

$$c_2 = \gamma_1 - \int_0^t f(\tau) \exp\left(-\frac{\alpha\tau}{1-\alpha}\right) d\tau\Big|_{t=0}.$$

Then the fixed point of operator T is equivalent to the solution of the Caputo-Fabrizio fractional Bratu-type initial value problem (1).

We put

$$\begin{aligned} g(t) &= t f(t) \exp\left(-\frac{\alpha t}{1-\alpha}\right) \\ &= -\lambda t\left[(1-\alpha)\frac{d}{dt}(\exp u(t)) + \alpha \exp u(t)\right]. \end{aligned}$$

Then, the operator T becomes

$$Tu(t) = c_1 - \int_0^t g(\tau) d\tau + t\left(c_2 + \int_0^t \frac{g(\tau)}{t} d\tau\right).$$

For $u_1(t), u_2(t) \in C([0,1])$, we get

$$\begin{aligned} |Tu_1(t) - Tu_2(t)| &\leq 2\lambda \int_0^t \left[(1-\alpha)\left|\frac{d}{dt}(\exp u_1(\tau)) - \frac{d}{dt}(\exp u_2(\tau))\right|\right. \\ &\quad \left. + \alpha\left|\exp u_1(\tau) - \exp u_2(\tau)\right|\right] d\tau \\ &\leq 2\lambda\left((1-\alpha)\int_0^t k_1(\tau)\left|u_1(\tau) - u_2(\tau)\right| d\tau\right. \\ &\quad \left. + \alpha \int_0^t k_2(\tau)\left|u_1(\tau) - u_2(\tau)\right| d\tau\right) \\ &\leq 2\lambda\left((1-\alpha)\int_0^t k_1(\tau) d\tau + \alpha \int_0^t k_2(\tau) d\tau\right)\left|u_1(t) - u_2(t)\right|. \end{aligned}$$

Therefore,

$$\begin{aligned}\|Tu_1(t) - Tu_2(t)\|_{C([0,1])} &\leq \max_{t\in[0,1]} \left| 2\lambda \left((1-\alpha) \int_0^t k_1(\tau) d\tau \right.\right. \\ &\quad \left.\left. +\alpha \int_0^t k_2(\tau) d\tau \right) \right| \times \|u_1(t) - u_2(t)\|_{C([0,1])} \\ &\leq L \|u_1(t) - u_2(t)\|_{C([0,1])} \\ &\leq \|u_1(t) - u_2(t)\|_{C([0,1])} .\end{aligned}$$

Since $L < 1$, T is a contraction operator.

By the Banach contraction principle in Theorem 2, we can conclude that the operator T has a unique fixed point $u(t) \in C([0,1])$, which means that there exists a unique solution $u(t) \in C([0,1])$ for the Caputo-Fabrizio fractional Bratu-type initial value problem (1). This completes the proof. ◀

4. Application of HPTM to Solve the Caputo-Fabrizio Fractional Bratu-Type Initial Value Problem

In this section, we present an algorithm of HPTM to solve the Caputo-Fabrizio fractional Bratu-type initial value problem. Then we examine the convergence analysis and error estimate of the proposed algorithm. Finally, we demonstrate it by solving a numerical example.

Theorem 5. *Consider the Caputo-Fabrizio fractional Bratu-type initial value problem (1). The HPTM gives the solution of (1) in the form of infinite series that rapidly converges to the exact solution as follows*

$$u(t) = \sum_{n=0}^{\infty} u_n(t). \tag{26}$$

Proof. To prove the above theorem, we first define the nonlinear operator

$$\mathcal{N}u(t) = \exp\left(u(t)\right). \tag{27}$$

Then, equation (1) is written in the form

$$\mathcal{D}^{\mu}u(t) + \lambda\mathcal{N}u(t) = 0, 0 < t < 1, \lambda \in \mathbb{R}. \tag{28}$$

Applying the ZZ transform on both sides of equation (28) and using Theorem 1, we get

$$\mathbb{Z}\left[u(t)\right] = u(0) + \frac{v}{s}u'(0) - \frac{v\left(s - \alpha(s-v)\right)}{s^2}\mathbb{Z}\left[\lambda\mathcal{N}u(t)\right]. \tag{29}$$

Taking the inverse ZZ transform on both sides of equation (29), we have

$$u(t) = \gamma_0 + \gamma_1 t - \lambda \mathbb{Z}^{-1}\left(\frac{v(s-\alpha(s-v))}{s^2}\mathbb{Z}\left[\mathcal{N}u(t)\right]\right). \tag{30}$$

Now, using the homotopy perturbation method [10], we can assume that the solution can be written as a power series of p as follows

$$u(t) = \sum_{n=0}^{\infty} p^n u_n(t), \tag{31}$$

where $p \in [0,1]$ is the homotopy parameter.

The nonlinear terms can be decomposed as

$$\mathcal{N}u = \sum_{n=0}^{\infty} p^n H_n(u), \tag{32}$$

where $H_n(u)$ are He's polynomials, of $u_0, u_1, ..., u_n$ and they can be calculated by the formulas given below

$$H_n(u_0, u_1, ..., u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[\mathcal{N}\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, n = 0, 1, 2, ... \tag{33}$$

Substituting equations (31) and (32) into equation (30), we get

$$\sum_{n=0}^{\infty} p^n u_n(t) = \gamma_0 + \gamma_1 t - \lambda \mathbb{Z}^{-1}\left(\frac{v(s-\alpha(s-v))}{s^2}\mathbb{Z}\left[\sum_{n=0}^{\infty} p^n H_n(u)\right]\right). \tag{34}$$

Using the coefficients of the like powers of p in equation (34), the following approximations are obtained

$$\begin{aligned}
p^0 &: \quad u_0(t) = \gamma_0 + \gamma_1 t, \\
p^1 &: \quad u_1(t) = -\lambda \mathbb{Z}^{-1}\left(\frac{v(s-\alpha(s-v))}{s^2}\mathbb{Z}\left[H_0(u)\right]\right), \\
p^2 &: \quad u_2(t) = -\lambda \mathbb{Z}^{-1}\left(\frac{v(s-\alpha(s-v))}{s^2}\mathbb{Z}\left[H_1(u)\right]\right), \\
p^3 &: \quad u_3(t) = -\lambda \mathbb{Z}^{-1}\left(\frac{v(s-\alpha(s-v))}{s^2}\mathbb{Z}\left[H_2(u)\right]\right), \\
&\quad \vdots
\end{aligned}$$

Therefore, the solution of (1) is obtained as $p \to 1$

$$u(t) = \sum_{n=0}^{\infty} u_n(t). \tag{35}$$

This completes the proof. ◀

Theorem 6. *Suppose that $u(t)$ and $u_n(t)$ are defined in the Banach space $(C[0,1], \|.\|)$. Then the series solution given by (26) converges to the solution of (1), if there exists δ, $0 < \delta < 1$, such that*

$$\|u_n\| \leq \delta \|u_{n-1}\|, \forall n \in \mathbb{N}.$$

Proof. Recall that $(C[0,1], \|.\|)$ is the Banach space of all continuous functions on $[0,1]$ with the norm defined by (25).

Define $\{S_n\}$ as a sequence of partial sums

$$\begin{aligned} S_0 &= u_0(t), \\ S_1 &= u_0(t) + u_1(t), \\ S_2 &= u_0(t) + u_1(t) + u_2(t), \\ &\vdots \\ S_n &= u_0(t) + u_1(t) + u_2(t) + ... + u_n(t). \end{aligned}$$

We need to show that $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $(C[0,1], \|.\|)$. For this purpose, we consider

$$\|S_{n+1} - S_n\| \leq \|u_{n+1}\| \leq \delta \|u_n\| \leq \delta^2 \|u_{n-1}\| \leq ... \leq \delta^{n+1} \|u_0\|. \tag{36}$$

For every $n, m \in \mathbb{N}$, $n \geq m$, by using (36) and the triangle inequality successively, we have

$$\begin{aligned} \|S_n - S_m\| &= \|S_{m+1} - S_m + S_{m+2} - S_{m+1} + ... + S_n - S_{n-1}\| \\ &\leq \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| + ... + \|S_n - S_{n-1}\| \\ &\leq \delta^{m+1} \|u_0\| + \delta^{m+2} \|u_0\| + ... + \delta^n \|u_0\| \\ &= \delta^{m+1} \left(1 + \delta + ... + \delta^{n-m-1}\right) \|u_0\| \\ &\leq \delta^{m+1} \left(\frac{1 - \delta^{n-m}}{1 - \delta}\right) \|u_0\|. \end{aligned}$$

Since $0 < \delta < 1$, we have $1 - \delta^{n-m} \leq 1$. Then

$$\|S_n - S_m\| \leq \frac{\delta^{m+1}}{1 - \delta} \|u_0\|.$$

Since $\|u_0\|$ is bounded, we have

$$\lim_{n,m\longrightarrow\infty} \|S_n - S_m\| = 0.$$

Therefore, $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $(C[0,1], \|.\|)$, so the series solution defined in (26) converges. This completes the proof. ◀

Theorem 7. *The maximum absolute truncation error of the series solution (26) of (1) is estimated to be*

$$\max_{t\in[0,1]} \left| u_n(t) - \sum_{k=0}^{m} u_k(t) \right| \leq \frac{\delta^{m+1}}{1-\delta} \max_{t\in[0,1]} |u_0(t)| . \tag{37}$$

Proof. From Theorem 6, we have

$$\|S_n - S_m\| \leq \frac{\delta^{m+1}}{1-\delta} \max_{t\in[0,1]} |u_0(t)| . \tag{38}$$

But we assume that $S_n = \sum\limits_{k=0}^{n} u_k(t)$ and since $n \to \infty$, we obtain $S_n \to u_n(t)$, so (38) can be rewritten as

$$\|u_n(t) - S_m\| = \left\| u_n(t) - \sum_{k=0}^{m} u_k(t) \right\| \leq \frac{\delta^{m+1}}{1-\delta} \max_{t\in[0,1]} |u_0(t)| .$$

So, the maximum absolute truncation error in $[0,1]$ is

$$\max_{t\in[0,1]} \left| u_n(t) - \sum_{k=0}^{m} u_k(t) \right| \leq \frac{\delta^{m+1}}{1-\delta} \max_{t\in[0,1]} |u_0(t)| .$$

This completes the proof. ◀

Example 1. *Consider the following Caputo-Fabrizio fractional Bratu-type initial value problem*

$$\begin{cases} \mathcal{D}^{\mu} u(t) - 2\exp(u(t)) = 0, \\ u(0) = u'(0) = 0, \end{cases} \tag{39}$$

where $\mathcal{D}^{\mu}$ is the Caputo-Fabrizio fractional derivative of the function $u(t)$ of order $\mu = \alpha + 1$ with $0 < \alpha \leq 1$ and $0 < t < 1$.

For $\mu = 2$ (or $\alpha = 1$), the exact solution of (39) is (see [8])

$$u(t) = -2\ln(\cos(t)).$$

Following the description of HPTM presented in Section 4, gives

$$\sum_{n=0}^{\infty} u_n(t) = 2\mathbb{Z}^{-1}\left(\frac{v\left(s-\alpha(s-v)\right)}{s^2}\mathbb{Z}\left[\sum_{n=0}^{\infty} H_n(u)\right]\right),$$

where $H_n(u)$ are He's polynomials.

For the convenience of the reader, the first few components of He's polynomials are given by

$$\begin{aligned}
H_0(u) &= \exp(u_0(t)),\\
H_1(u) &= u_1(t)\exp(u_0(t)),\\
H_2(u) &= \left(u_2(t)+\frac{1}{2}u_1^2(t)\right)\exp(u_0(t)),\\
H_3(u) &= \left(u_3(t)+u_1(t)u_3(t)+\frac{1}{6}u_1^3(t)\right)\exp(u_0(t)),\\
&\vdots
\end{aligned}$$

Comparing the coefficients of various powers of p, we get

$$\begin{aligned}
p^0 &: u_0(t) = 0,\\
p^1 &: u_1(t) = 2(1-\alpha)t+\alpha t^2,\\
p^2 &: u_2(t) = 2(1-\alpha)^2t^2+\frac{4\alpha(1-\alpha)}{3}t^3+\frac{\alpha^2}{6}t^4,\\
p^3 &: u_3(t) = \frac{8}{3}(1-\alpha)^3t^3+\frac{7}{3}\alpha(1-\alpha)^2t^4+\frac{3}{5}\alpha^2(1-\alpha)t^5+\frac{2}{45}\alpha^3t^6,\\
&\vdots
\end{aligned}$$

Therefore, the solution of (39) is given by

$$u(t) = \sum_{n=0}^{\infty} u_n(t) = u_0(t)+u_1(t)+u_2(t)+u_3(t)+...$$

Remark 1. *In this work, only 4−term HPTM-approximate solution is used to calculate the numerical solution and HPTM can provide a more precise solution with less absolute error by calculating a higher order approximation.*

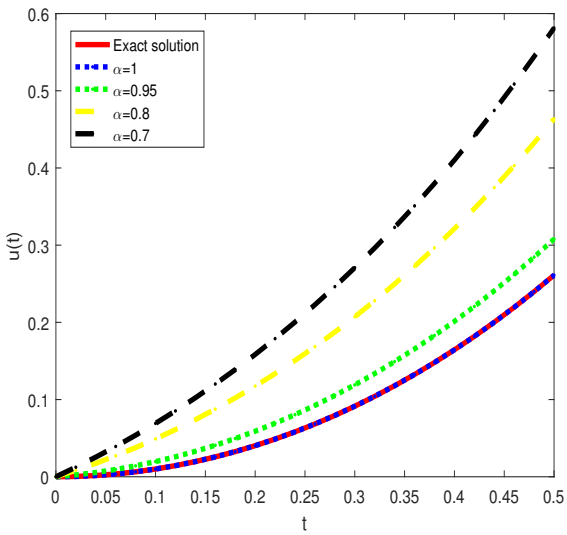

Figure 1: The behavior of the exact solution and $4-$term approximate solutions by HPTM for different values of α

Table 1: The numerical values of the exact solution and $4-$term approximate solutions by HPTM for different values of α

t	$\alpha = 0.8$	$\alpha = 0.95$	$\alpha = 1$	Exact solution	$\lvert u_{exact} - u_{HPTM} \rvert$
0.1	0.049054	0.019630	0.010017	0.010017	1.3536×10^{-10}
0.2	0.11739	0.058970	0.040270	0.040270	3.4994×10^{-8}
0.3	0.20721	0.11903	0.091382	0.091383	9.1185×10^{-7}
0.4	0.32134	0.20130	0.16445	0.16446	9.3270×10^{-6}
0.5	0.46342	0.30790	0.26111	0.26117	5.7370×10^{-5}

5. Conclusion

In this paper, the existence and uniqueness of solution for Caputo-Fabrizio fractional Bratu-type initial value problem has been studied and proved by using the Banach fixed point theorem. In addition, an algorithm of HPTM was applied to obtain an approximate analytical solution for this problem. The approximate solutions are compared with exact solutions and also with other existing solutions obtained by other methods. The results show that the HPTM can be used as a very accurate algorithm for solving this type of problem. We can conclude that the proposed method is a powerful and effective method for finding approximate analytical solutions of nonlinear fractional differential equations. In a future work we plan to use this method to approximate analytical solutions of some other nonlinear equations with higher order fractional derivatives.

Acknowledgments

The author thanks Professor Bilal T. Bilalov (AJM Editor in Chief) as well as the anonymous referee for his/her careful reading of the paper and his/her valuable remarks that improved the final version of the paper.

References

[1] H.M. Baskonus, T. Mekkaoui, Z. Hammouch, H. Bulut, *Active Control of a Chaotic Fractional Order Economic System*, Entropy, **17(8)**, 2015, 5771-5783.

[2] M. Caputo, M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progress in Fractional Differentiation and Applications, **1(2)**, 2015, 73-85.

[3] S. Chandrasekhar, *Introduction to the Study of Stellar Structure*, Dover, New York, 1967.

[4] K.M. Furati, M.D. Kassim, N.T.Tatar, *Existence and uniqueness for a problem involving Hilfer fractional derivative*, Computers & Mathematics with Applications, **64(6)**, 2012, 1616-1626.

[5] J.H. He, H.Y. Kong, R.X. Chen, M.S. Hu, Q.L. Chen, *Variational iteration method for Bratu-like equation arising in electrospinning*, Carbohydrate Polymers, **105**, 2014, 229-230.

[6] H.H. Hilton, *Generalized Fractional Derivative Anisotropic Viscoelastic Characterization*, Materials, **5(1)**, 2012, 169-191.

[7] J. Jacobsen, K. Schmitt, *The Liouville-Bratu-Gelf and Problem for Radial Operators*, Journal of Differential Equations, **184(1)**, 2002, 283-298.

[8] H. Jafari, H. Tajadodi, *Electro-Spunorganic Nanofibers Elaboration Process Investigations using BPs Operational Matrices*, Iranian Journal of Mathematical Chemistry, **7(1)**, 2016, 19-27.

[9] F. Jarad, A. Abdeljawad, D. Baleanu, *Caputo-type modification of the Hadamard fractional derivatives*, Advances in Difference Equations, **2012(421)**, 2012, 1-12.

[10] S. Javeed, D. Baleanu, A. Waheed, M.S. Khanv, H. Affan, *Analysis of Homotopy Perturbation Method for Solving Fractional Order Differential Equations*, Mathematics, **7(1)**, 2019, 1-14.

[11] A. Khalouta, *The Existence and Uniqueness of Solution for Fractional Newel-Whitehead-Segel Equation within Caputo-Fabrizio Fractional Operator*, Applications and Applied Mathematics, **16(2)**, 2021, 894-909.

[12] J. Losada, J.J. Nieto, *Properties of a new fractional derivative without singular kernel*, Progress in Fractional Differentiation and Applications, **1(2)**, 2015, 87-92.

[13] R.L. Magin, *Fractional Calculus in Bioengineering*, Begell House Inc. Publishers, 2006.

[14] K.M. Owolabi, *Riemann-Liouville Fractional Derivative and Application to Model Chaotic Differential Equations*, Progress in Fractional Differentiation and Applications, **4(2)**, 2018, 99-110.

[15] T. Sandev, R. Metzler, Z. Tomovski, *Fractional diffusion equation with a generalized Riemann-Liouville time fractional derivative*, Journal of Physics. A. Mathematical and Theoretical, **44**, 2011, 1-21.

[16] P. Veeresha, D.G. Prakasha, H.M. Baskonus, *New numerical surfaces to the mathematical model of cancer chemotherapy effect in Caputo fractional derivatives*, Chaos, **29(1)**, 2019, 013119.

[17] Y.Q. Wan, Q. Guo, N. Pan, *Thermo-Electro-Hydrodynamic Model for Electrospinning Process*, International Journal of Nonlinear Sciences and Numerical Simulation, **5(1)**, 2004, 5-8.

[18] Z.U.A. Zafar, *ZZ Transform Method*, Journal of Advanced Engineering and Global Technology, **4(1)**, 2016, 1605-1611.

[19] Y. Zhou, *Basic Theory of Fractional Differential Equations*, Singapore: World Scientific, 2014.

Ali Khalouta
Laboratory of Fundamental and Numerical Mathematics, Department of Mathematics, Faculty of Sciences, Ferhat Abbas Sétif University 1, 19000 Sétif, Algeria
E-mail: nadjibkh@yahoo.fr ; ali.khalouta@univ-setif.dz

Received 27 October 2021
Accepted 27 March 2022