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The Darboux Problem with Nonlocal Boundary Conditions for a Hyperbolic System of First Order Equations

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Abstract. This paper considers the existence and uniqueness of the problem with generalized nonlocal (non-separated and integral) boundary conditions in curvilinear sector for linear and semilinear hyperbolic systems of the first order (Darboux type problem). Under some conditions on the coefficients of boundary conditions (solvability and compressibility conditions) the existence and uniqueness of generalized solution of the problem is proved. The conditions under which the generalized solution is piecewise smooth are specified. The question of the possibility of continuation of the solution for all $t > 0$ is considered.

Key Words and Phrases: hyperbolic system, characteristics, Darboux problem, nonlocal conditions

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1. Introduction

Solving modern applied problems requires qualitatively new formulations of problems, a particularity of which is, for example, that boundary conditions are given at interior points of the domain where the required function must satisfy the equation or nonlocal (non-separable or integral) boundary conditions.

This paper considers problems with nonlocal conditions for one-dimensional first-order hyperbolic systems in the curvilinear sector.

Note that problems with nonlocal (non-separated and integral) conditions for ordinary differential equations have been investigated before. For example, in [1] the following problem has been considered: find a solution of the equation

$$y' = k(x, \lambda)y + g(x),$$

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that satisfies the condition

$$\sum_i M^i(\lambda)y(a_i,\lambda) + \int_a^b M(x,\lambda)y(x,\lambda)dx = 0,$$

where a_i are the points of interval $[a,b]$.

Recently there appeared a work [2] with similar nonlocal conditions for nonlinear impulse ordinary differential equations.

Non-standard form of boundary conditions generates a number of peculiar phenomena: in some cases we obtain an infinite number of adjoint functions, there also arise nontrivial questions about convergence of eigenfunction expansions, about existence of global solution and continuous dependence on initial data.

Detailed review of works related to the problems with nonlocal (non-separated and integral) boundary conditions for hyperbolic systems and equations on the plane is available in [3].

In this paper we consider nonlocal boundary value problems for hyperbolic systems of first order. We will consider the initial conditions degenerate in the sense that the segment on which the initial conditions are given is degenerate into a point, i.e. the boundary condition setting lines exit from one point and do not intersect anywhere else. Problems in such domains are called Darboux type problems [4]. In this case some characteristics of the system exiting from the intersection point of the boundary curves can enter the domain of solution and the boundary conditions are given in nonlocal form.

The study of mixed problems for hyperbolic systems with discontinuous coefficients in the case where the discontinuity lines of the initial data have common points leads to the consideration of such domains, for example, in aerodynamics and gas dynamics [5].

The problems with nonlocal (integral) conditions for hyperbolic systems and equations appear in biology, ecology, mechanics, demography, etc. [3].

For example, in biological and demographic studies [6, 7], "continuous" models of population dynamics of the form

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial r} = \mu(r,t)\rho(r,t) + \omega(r,t), \quad r > 0, \quad t > 0,$$

$$\rho(r,0) = \rho_0(r), \quad \rho(0,t) = \beta(t)\int_{r_1}^{r_2} h(r,t)K(r,t)\rho(r,t)dr$$

are used. Here $\rho(r,t)$ is the density of population grouping by growth r at the time t; $\omega(r,t)$ is the migrant density; $\mu(r,t)$ is the mortality rate; $p_0(r)$, $\beta(t)$, $h(r,t)$, $K(r,t)$ are the standard demographic indices.

Many other applied problems that lead to nonlocal boundary conditions for hyperbolic type equations and systems have been treated in [3, 8, 9]. Some variants of such problems have been considered for nonlinear hyperbolic systems with unknown boundaries [10] and optimal control [11].

Correct solvability of the problems stated in this paper was proved by the method of characteristics, using the methodology of [12, 13].

2. Statement of the Problem

Let G be a curvilinear sector in upper half-plane $t > 0$ of plane xOt, bounded by curves γ_0 and γ_{m+1}, which are given by equations $x = a_0(t)$, $x = a_{m+1}(t)$, $m \geqslant 0$, $a_0(0) = a_{m+1}(0) = 0$, $a_{m+1}(t) > a_0(t)$, respectively, for all $t > 0$. The curves $\gamma_s : x = a_s(t)$, $s = \overline{0, m+1}$, $a_s \in C^1(R_+)$ $(R_+ = [0, \infty))$, $a_{s+1}(t) > a_s(t)$ for all $t > 0$, $a_s(0) = 0$ divide G into $m+1$ connectivity components G^s $(s = \overline{0, m})$, which are numbered from left to right.

In the domain $G \setminus \cup\gamma_s$ (Fig.1) we consider a system (with the assumption that some characteristics of the system that pass through $(0, 0)$ may fall into $G \setminus \cup\gamma_s$) for which conditions are formulated that change the boundary conditions defined at γ_0 and γ_{m+1} and the conjugation conditions at $\gamma_1, \ldots, \gamma_m$, if $m > 0$:

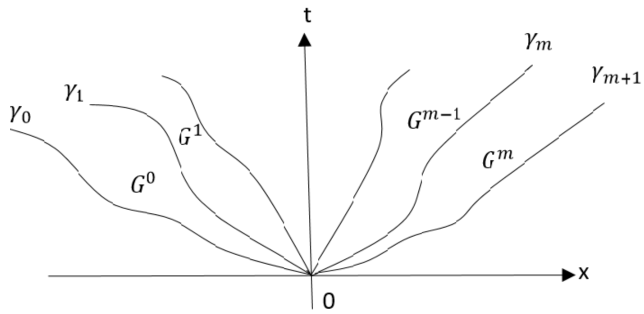

Figure 1:

$$\frac{\partial u_i}{\partial t} + \lambda_i(x, t)\frac{\partial u_i}{\partial x} = \sum_{j=1}^{n} a_{ij}(x, t)u_j + f_i(x, t), \quad i = \overline{1, n}, \tag{1}$$

where λ_i, a_{ij}, f_i are the given functions that are uniformly continuous in each domain G^s.

All functions F defined on G or on $G_\varepsilon(\varepsilon > 0)$ will be assumed to be uniformly continuous in each domain G^s (respectively G^s_ε) and by F^s we mean the continuous extension of F from G^s to $\bar{G}^s$ (respectively, from G^s_ε to $\bar{G}^s_\varepsilon$). Let us denote

$$\Phi_i^s(x,t,u^s)=\sum_{j=1}^{n} a_{ij}^s(x,t)u_j^s(x,t)+f_i^s(x,t),\quad i=\overline{1,n},\quad s=\overline{0,m}$$

for convenience.

Assume that for all $t\geqslant 0$ and for each $s=\overline{0,m}$ the conditions

$$\begin{gathered}\lambda_i^s(a_s(t),t)-a_s'(t)>0,\ i=\overline{1,p_s},\quad \lambda_i^s(a_s(t),t)-a_s'(t)<0,\ i=\overline{p_s+1,n},\\ \lambda_i^s(a_{s+1}(t),t)-a_{s+1}'(t)>0,\quad i=\overline{1,q_s},\\ \lambda_i^s(a_{s+1}(t),t)-a_{s+1}'(t)<0,\quad i=\overline{q_s+1,n},\\ 0\leqslant p_s,\quad q_s\leqslant n,\quad s=\overline{0,m}\end{gathered}\tag{2}$$

are valid. Since $p_s(q_s)$ is the set of indices i for which $\lambda_i^s(0,0)>a_s'(0)$ (respectively, $a_{s+1}'(0)$), and $a_s'(0)\leqslant a_{s+1}'(0)$, we have $p_s\geqslant q_s$ for all $s=\overline{0,m}$. Let

$$N=\sum_{s=0}^{m}(p_s-q_s)+(m+1)n.$$

Conditions (2) regulate the relations between angular coefficients of λ_i^s characteristics of system (1) and angular coefficients of a_k' ($k=\overline{s,s+1}$) boundaries of domains G^s ($s=\overline{0,m}$). The number of boundary conditions for the corresponding problem for system (1) depends on conditions (2). Particularly, it follows from (2) that p_s-q_s characteristics of system (1) exit from the origin and fall into the middle of the domain G^s. Therefore, for a correct formulation of the boundary value problem for system (1) the number of boundary conditions increases by the number p_s-q_s with respect to the number of equations of system (1).

For system (1), impose the conditions that replace the boundary conditions on γ_0 and γ_{m+1} and the conjugate conditions on $\gamma_1,\dots,\gamma_m$, if $m>0$:

$$\sum_{i=1}^{n}\sum_{s=0}^{m}\Big(\sum_{k=s}^{s+1}\alpha_{is}^{kp}(t)u_i^s(a_k(t),t)+\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^p(y,t)u_i^s(y,t)dy\Big)=h^p(t),\ p=\overline{1,q},\tag{3}$$

$$\sum_{i=1}^{n}\sum_{s=0}^{m}\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^p(y,t)u_i^s(y,t)dy=h^p(t),\ p=\overline{q+1,N},\ q=\overline{0,N}.\tag{4}$$

Here $\alpha_{is}^{kp}(t)$, $h^p(t)$, $\beta_{is}^p(y,t)$ are the given continuous functions on $[0,\infty)$ and $\bar{G}^s$, respectively.

By substituting $t=0$ into (4), we immediately obtain the necessary condition for the solvability of the problem: $h^p(0)=0$, $p=\overline{q+1,N}$. Let us assume that this condition is always satisfied.

Additionally assume that $h^p \in C^1(\mathbb{R}_+)$, $\beta_{is}^p \in C^1(\bar{G}^s)$, $s = \overline{0,m}$, $p = \overline{q+1,N}$, $i = \overline{1,n}$.

Let us denote by $\varphi_i^s(\tau, x, t)$ the solution of the Cauchy problem

$$\frac{d\xi}{d\tau} = \lambda_i^s(\xi, \tau), \;\; \xi(t) = x, \;\; (x,t) \in \bar{G}^s, \;\; i = \overline{1,n}, \;\; s = \overline{0,m}. \tag{5}$$

Denote the corresponding integral curves by $Q_i^s(x,t)$. Let us assume that all $\lambda_i^s \in C^2(\bar{G}^s)$, which ensures that each solution is uniquely extended. Let $t_i^s(x,t)$ be the smallest value of τ for such solution. Obviously, $0 \leqslant t_i^s(x,t) \leqslant t$. If $t_i^s(x,t) > 0$, then $\varphi_i^s(t_i^s(x,t), x, t)$ equals either $a_s(t_i^s(x,t))$ or $a_{s+1}(t_i^s(x,t))$.

Accordingly, the characteristic $Q_i^s(0,0)$ is defined if $a_s'(0) < \lambda_i^s(0,0) < a_{s+1}'(0)$, that is, $q_s < i \leqslant p_s$; in this case, it divides G^s into two components G^{si-} and G^{si+} (Fig.2).

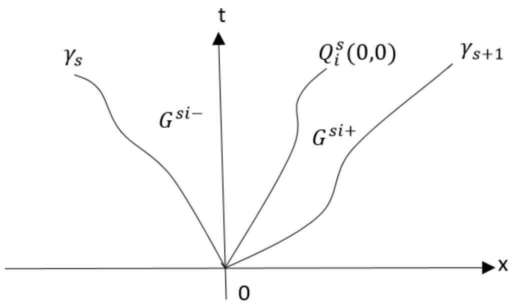

Figure 2:

Similarly, for $\lambda_i^s(0,0) < a_s'(0)$ ($\lambda_i^s(0,0) > a_{s+1}'(0)$) we will assume $G^{si-} = \varnothing$, $G^{si+} = G^s$ (respectively, $G^{si-} = G^s$, $G^{si+} = \varnothing$). Then for $t > 0$, the condition $\varphi_i^s(t_i^s(x,t), x, t) = a_s(t_i^s(x,t))$ ($a_{s+1}(t_i^s(x,t))$) is equivalent to $(x,t) \in \bar{G}^{si-}$ (respectively, $\bar{G}^{si+}$).

Let us introduce the following matrices:

$$\alpha_s^1(t) = \left\| \alpha_{is}^{sp}(t) \right\|, \; p = \overline{1,q}, \; i = \overline{1,p_s}; \; \alpha_s^2(t) = \left\| \alpha_{is}^{s+1,p}(t) \right\|, \; p = \overline{1,q}, \; i = \overline{q_s+1,n};$$

$$\alpha_s^3(0) = -\left\| \alpha_{is}^{s+1,p}(0) \right\|, \; p = \overline{1,q}, \; i = \overline{1,q_s};$$

$$\alpha_s^4(0) = -\left\| \alpha_{is}^{sp}(0) \right\|, \; p = \overline{1,q}, \; i = \overline{p_s+1,n};$$

$$\beta_s^1(t) = \left\| \beta_{is}^p(a_s(t),t)\Big(\lambda_i^s(a_s(t),t) - a_s'(t)\Big) \right\|, \; p = \overline{q+1,N}, \; i = \overline{1,p_s};$$

$$\beta_s^2(t) = -\left\| \beta_{is}^p(a_{s+1}(t),t)\Big(\lambda_i^s(a_{s+1}(t),t) - a_{s+1}'(t)\Big) \right\|, \; p = \overline{q+1,N}, \; i = \overline{q_s+1,n};$$

$$\beta_s^3(0)=\left\|\beta_{is}^p(0,0)\Big(\lambda_i^s(0,0)-a'_{s+1}(0)\Big)\right\|,\ p=\overline{q+1,N},\ i=\overline{1,q_s};$$
$$\beta_s^4(0)=-\left\|\beta_{is}^p(0,0)\Big(\lambda_i^s(0,0)-a'_s(0)\Big)\right\|,\ p=\overline{q+1,N},\ i=\overline{p_s+1,n};$$
$$s=\overline{0,m},\ t\geqslant 0$$

and in addition, let us introduce square matrices of order N

$$A(t)=\left\|\begin{array}{cccccc}\alpha_0^1(t) & \dots & \alpha_m^1(t) & \alpha_0^2(t) & \dots & \alpha_m^2(t)\\ \beta_0^1(t) & \dots & \beta_m^1(t) & \beta_0^2(t) & \dots & \beta_m^2(t)\end{array}\right\|,$$

$$B(0)=\left\|\begin{array}{cccccc}\alpha_0^3(0)0_0^1 & \dots & \alpha_m^3(0)0_m^1 & 0_0^1\alpha_0^4(0) & \dots & 0_m^1\alpha_m^4(0)\\ \beta_0^3(0)0_0^2 & \dots & \beta_m^3(0)0_m^2 & 0_0^2\beta_0^4(0) & \dots & 0_m^2\beta_m^4(0)\end{array}\right\|.$$

Here 0_s^k are null matrices of dimension $q\times(p_s-q_s)$ if $k=1$ and of dimension $(N-q)(p_s-q_s)$ if $k=2$ ($s=\overline{0,m}$).

Let us assume that

$$\det A(t)\neq 0,\ \forall\, t\geqslant 0, \tag{6}$$

$$|A(0)^{-1}B(0)|<1 \tag{7}$$

and conditions of agreement

$$\sum_{p=1}^{N}(\delta_{l_i^s,p}-\delta_{k_i^s,p})H^p(0)=0,\ i=\overline{q_s+1,p_s},\ s=\overline{0,m} \tag{8}$$

are satisfied at point $(0,0)$ $\sum\limits_{s=0}^{m}(p_s-q_s)$, where δ_{jp} are elements of the matrix $[I-A(0)^{-1}B(0)]^{-1}$,

$$l_i^s=i,\ k_i^s=ns+\sum_{r=0}^{m}p_r-\sum_{r=0}^{s}q_r+i,$$
$$H^p(0)=h^p(0)\ (p=\overline{1,q}),\ H^p(0)=h^{p\prime}(0)\ (p=\overline{q+1,N}).$$

3. Auxiliary Lemmas

Lemma 1. *The functions φ_i^s ($i=\overline{1,n},\ s=\overline{0,m}$) for $0\leqslant t<\infty$, $a_s(t)\leqslant x\leqslant a_{s+1}(t)$, $0\leqslant\tau\leqslant t$ are continuously differentiable, and the formulas*

$$\frac{\partial\varphi_i^s}{\partial x}(\tau,x,t)=\exp\left(-\int\limits_{\tau}^{t}{\lambda_{ix}^s}'(\varphi_i^s(\sigma,x,t),\sigma)d\sigma\right), \tag{9}$$

$$\frac{\partial\varphi_i^s}{\partial t}(\tau,x,t)=-\lambda_i^s(x,t)\exp\left(-\int\limits_{\tau}^{t}\lambda_{ix}^s{}'(\varphi_i^s(\sigma,x,t),\sigma)d\sigma\right) \tag{10}$$

are valid.

Proof. The continuous differentiability of the function φ_i^s is well known from the theorems of the theory of ordinary differential equations, and formulas (9), (10) are obtained by integrating the variation equation for (5) (see, for example, [14, p. 92]). ◄

Lemma 2. *The functions* t_i^s $(i=\overline{1,n},\ s=\overline{0,m})$ *are continuously differentiable for* $(x,t)\in\bar{G}^{si-}\cup\bar{G}^{si+}$ *and*

$$\begin{gathered}\frac{\partial t_i^s}{\partial x}(x,t)=\frac{-1}{\lambda_i^s(a_s(t_i^s(x,t)),t_i^s(x,t))-a_s'(t_i^s(x,t))}\times\\ \times\exp\left(-\int\limits_{t_i^s(x,t)}^{t}\lambda_{ix}^s{}'(\varphi_i^s(\sigma,x,t),\sigma)d\sigma\right)\left((x,t)\in\bar{G}^{si-}\right),\end{gathered} \tag{11}$$

$$\begin{gathered}\frac{\partial t_i^s}{\partial x}(x,t)=\frac{-1}{\lambda_i^s(a_{s+1}(t_i^s(x,t)),t_i^s(x,t))-a_{s+1}'(t_i^s(x,t))}\times\\ \times\exp\left(-\int\limits_{t_i^s(x,t)}^{t}\lambda_{ix}^s{}'(\varphi_i^s(\sigma,x,t),\sigma)d\sigma\right)\left((x,t)\in\bar{G}^{si+}\right),\end{gathered} \tag{12}$$

$$\begin{gathered}\frac{\partial t_i^s}{\partial t}(x,t)=\frac{\lambda_i^s(x,t)}{\lambda_i^s(a_s(t_i^s(x,t)),t_i^s(x,t))-a_s'(t_i^s(x,t))}\times\\ \times\exp\left(-\int\limits_{t_i^s(x,t)}^{t}\lambda_{ix}^s{}'(\varphi_i^s(\sigma,x,t),\sigma)d\sigma\right)\left((x,t)\in\bar{G}^{si-}\right),\end{gathered} \tag{13}$$

$$\begin{gathered}\frac{\partial t_i^s}{\partial t}(x,t)=\frac{\lambda_i^s(x,t)}{\lambda_i^s(a_{s+1}(t_i^s(x,t)),t_i^s(x,t))-a_{s+1}'(t_i^s(x,t))}\times\\ \times\exp\left(-\int\limits_{t_i^s(x,t)}^{t}\lambda_{ix}^s{}'(\varphi_i^s(\sigma,x,t),\sigma)d\sigma\right)\left((x,t)\in\bar{G}^{si+}\right).\end{gathered} \tag{14}$$

Proof. Since $t_i^s(x,t)$ is the ordinate of the i-th characteristic's intersection point, which exits from the point $(x,t) \in \bar{G}^s$, with the curve γ_s or γ_{s+1}, we have the identities

$$\begin{aligned} \varphi_i^s(t_i^s(x,t),x,t) \equiv a_s(t_i^s(x,t)),\ (x,t) \in \bar{G}^{si-}, \\ \varphi_i^s(t_i^s(x,t),x,t) \equiv a_{s+1}(t_i^s(x,t)),\ (x,t) \in \bar{G}^{si+}. \end{aligned} \tag{15}$$

Let us rewrite the first identity as $\varphi_i^s(v,x,t) - a_s(v) = 0$, where $v = t_i^s(x,t)$, and note that according to (2)

$$\frac{\partial[\varphi_i^s(v,x,t) - a_s(v)]}{\partial v} = \lambda_i^s(\varphi_i^s(v,x,t),v) - a_s'(v) = \lambda_i^s(a_s(v),v) - a_s'(v) \neq 0,$$

by the implicit function theorem, the function t_i^s is continuously differentiable in G^{si-}. Therefore we can differentiate the first identity of (15) with respect to x to obtain

$$\frac{\partial \varphi_i^s}{\partial \tau}(\tau,x,t)\Big|_{\tau=t_i^s(x,t)} \frac{\partial t_i^s}{\partial x}(x,t) + \frac{\partial \varphi_i^s}{\partial x}(\tau,x,t)\Big|_{\tau=t_i^s(x,t)} = a_s'(t_i^s(x,t)) \frac{\partial t_i^s}{\partial x}(x,t).$$

Given (5) and formula (9) of Lemma 1, we have

$$\Big[\lambda_i^s(a_s(t_i^s(x,t))) - a_s'(t_i^s(x,t))\Big] \frac{\partial t_i^s}{\partial x}(x,t) = -\exp\left(\int\limits_{t_i^s(x,t)}^{t} \lambda_{ix}^{s\prime}(\varphi_i^s(\sigma,x,t),\sigma)d\sigma \right).$$

Thus we obtain (11).

Similarly, by differentiating identities (15) with respect to x and t and applying formula (10), we obtain formulas (12), (13) and (14).

Lemma 2 is proved. ◄

4. Local Theorem on Existence and Uniqueness of a Continuous Generalized Solution

Introduce additional auxiliary unknown functions for $s = \overline{0,m}$:

$$\mu_i^s(t) = u_i^s(a_s(t),t),\ i = \overline{1,p_s};\ \ \nu_i^s(t) = u_i^s(a_{s+1}(t),t),\ i = \overline{q_s+1,n}. \tag{16}$$

We will call a function piecewise continuous on G if it is uniformly continuous on some G^s and piecewise smooth on G if its first-order derivatives are piecewise continuous.

The following lemma is true.

Lemma 3. *For piecewise smooth functions u on G (or on G_ε for $\varepsilon > 0$) , the system* (1) *is equivalent to a system of integro-functional equations*

$$u_i^s(x,t) = \omega_i^s(x,t) + \int\limits_{t_i^s(x,t)}^{t} \Phi_i^s(\varphi_i^s(\tau,x,t),\tau,u^s)d\tau,\ (x,t)\in \bar{G}^s,\ i=\overline{1,n}, \tag{17}$$

$$\omega_i^s(x,t) = \begin{cases} \mu_i^s(t_i^s(x,t)),\ \text{if } \varphi_i^s(t_i^s(x,t),x,t) = a_s(t_i^s(x,t)), \\ \qquad\qquad \text{i.e. } (x,t)\in \bar{G}^{si-}\ (\text{if } t>0) \text{ or } x=t=0, \\ \nu_i^s(t_i^s(x,t)),\ \text{if } \varphi_i^s(t_i^s(x,t),x,t) = a_{s+1}(t_i^s(x,t)), \\ \qquad\qquad \text{i.e. } (x,t)\in \bar{G}^{si+}\ (\text{if } t>0) \text{ or } x=t=0, \end{cases}$$

for any predefined functions μ_i^s and ν_i^s, $s=\overline{0,m}$.

Proof. We obtain the transition from (1) to (17) by substituting $\xi = \varphi_i^s(\tau,x,t)$, τ instead of x, t in (1) and further integrating over τ from $t_i^s(x,t)$ to t, i.e., integrating along the characteristics of the i-th family. The inverse transition occurs by the same substitution in (17) and further differentiation by τ at $\tau = t$, i.e., differentiation along the characteristics of the i-th family. Note that relation (16) follows from (17). ◄

Definition 1. *The piecewise continuous generalized solution of problem* (1) – (4) *is defined as a piecewise continuous solution of system* (17) *that satisfies conditions* (3), (4) *for each t.*

Theorem 1. *Suppose all assumptions formulated in Section 2 are satisfied. Then for some $\varepsilon > 0$ the problem* (1) – (4) *has a unique generalized piecewise continuous solution in $\bar{G}_\varepsilon$. This solution is continuous (in the sense of a uniform metric) and depends on the given functions $h^p(p=\overline{1,q})$, $h^{p\prime}(p=\overline{q+1,N})$ and all f_i^s. Moreover, there exists a constant C, which does not depend on the functions h^p and f_i^s, such that*

$$|u_i(x,t)| \leqslant C\Big[\sum_{i=1}^{q}\max_{0\leqslant t\leqslant\varepsilon}|h^p(t)| + \sum_{i=q+1}^{N}\max_{0\leqslant t\leqslant\varepsilon}|h^{p\prime}| + \\ + \sum_{r=0}^{m}\sum_{i=1}^{n}\max_{(x,t)\in\bar{G}_\varepsilon^r}|f_i^r(x,t)|\Big],\ (x,t)\in\bar{G}_\varepsilon^s,\ s=\overline{0,m},\ i=\overline{1,n}.$$

Proof. Let us rewrite conditions (3) and (4) as follows:

$$\sum_{s=0}^{m}\Big[\sum_{i=1}^{p_s}\alpha_{is}^{sp}(t)u_i^s(a_s(t),t) + \sum_{i=p_s+1}^{n}\alpha_{is}^{sp}(t)u_i^s(a_s(t),t) +$$

$$+\sum_{i=1}^{q_s}\alpha_{is}^{s+1,p}(t)u_i^s(a_{s+1}(t),t)+\sum_{i=q_s+1}^{n}\alpha_{is}^{s+1,p}(t)u_i^s(a_{s+1}(t),t)+$$

$$+\sum_{i=1}^{q_s}\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^p(y,t)u_i^s(y,t)dy+\sum_{i=q_s+1}^{p_s}\int\limits_{a_s(t)}^{\varphi_i^s(t,0,0)}\beta_{is}^p(y,t)u_i^s(y,t)dy+$$

$$+\sum_{i=q_s+1}^{p_s}\int\limits_{\varphi_i^s(t,0,0)}^{a_{s+1}(t)}\beta_{is}^p(y,t)u_i^s(y,t)dy+$$

$$+\sum_{i=p_s+1}^{n}\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^p(y,t)u_i^s(y,t)dy\Big]=h^p(t),\ p=\overline{1,q};$$

$$\sum_{s=0}^{m}\Big[\sum_{i=1}^{q_s}\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^p(y,t)u_i^s(y,t)dy+\sum_{i=q_s+1}^{p_s}\int\limits_{a_s(t)}^{\varphi_i^s(t,0,0)}\beta_{is}^p(y,t)u_i^s(y,t)dy+$$

$$+\sum_{i=q_s+1}^{p_s}\int\limits_{\varphi_i^s(t,0,0)}^{a_{s+1}(t)}\beta_{is}^p(y,t)u_i^s(y,t)dy+$$

$$+\sum_{i=p_s+1}^{n}\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^p(y,t)u_i^s(y,t)dy\Big]=h^p(t),\ p=\overline{q+1,N}.$$

(If $q_s=p_s$, then the sum from q_s+1 to p_s is zero).

Substituting (17) here, we obtain

$$\sum_{s=0}^{m}\Big[\sum_{i=1}^{p_s}\alpha_{is}^{sp}(t)\mu_i^s(t_i^s(a_s(t),t))+\sum_{i=1}^{p_s}\alpha_{is}^{sp}(t)\int\limits_{t_i^s(a_s(t),t)}^{t}\Phi_i^s(\varphi_i^s(\tau,a_s(t),t),\tau,u^s)d\tau+$$

$$+\sum_{i=p_s+1}^{n}\alpha_{is}^{sp}(t)\nu_i^s(t_i^s(a_s(t),t))+\sum_{i=p_s+1}^{n}\alpha_{is}^{sp}(t)\int\limits_{t_i^s(a_s(t),t)}^{t}\Phi_i^s(\varphi_i^s(\tau,a_s(t),t),\tau,u^s)d\tau+$$

$$+\sum_{i=1}^{q_s}\alpha_{is}^{s+1,p}(t)\mu_i^s(t_i^s(a_{s+1}(t),t))+$$

$$+\sum_{i=1}^{q_s}\alpha_{is}^{s+1,p}(t)\int\limits_{t_i^s(a_{s+1}(t),t)}^{t}\Phi_i^s(\varphi_i^s(\tau,a_{s+1}(t),t),\tau,u^s)d\tau+$$

$$+\sum_{i=q_s+1}^{n}\alpha_{is}^{s+1,p}(t)\nu_i^s(t_i^s(a_{s+1}(t),t))+$$

$$+\sum_{i=q_s+1}^{n}\alpha_{is}^{s+1,p}(t)\int\limits_{t_i^s(a_{s+1}(t),t)}^{t}\Phi_i^s(\varphi_i^s(\tau,a_{s+1}(t),t),\tau,u^s)d\tau+$$

$$+\sum_{i=1}^{q_s}\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^p(y,t)\mu_i^s(t_i^s(y,t))dy+\sum_{i=q_s+1}^{p_s}\int\limits_{a_s(t)}^{\varphi_i^s(t,0,0)}\beta_{is}^p(y,t)\mu_i^s(t_i^s(y,t))dy+$$

$$+\sum_{i=q_s+1}^{p_s}\int\limits_{\varphi_i^s(t,0,0)}^{a_{s+1}(t)}\beta_{is}^p(y,t)\nu_i^s(t_i^s(y,t))dy+\sum_{i=p_s+1}^{n}\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^p(y,t)\nu_i^s(t_i^s(y,t))dy+$$

$$+\sum_{i=1}^{n}\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^p(y,t)\left(\int\limits_{t_i^s(y,t)}^{t}\Phi_i^s(\varphi_i^s(\tau,y,t),\tau,u^s)d\tau\right)dy\Big]=h^p(t),\ p=\overline{1,q}; \quad (18)$$

$$\sum_{s=0}^{m}\Big[\sum_{i=1}^{q_s}\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^p(y,t)\mu_i^s(t_i^s(y,t))dy+\sum_{i=q_s+1}^{p_s}\int\limits_{a_s(t)}^{\varphi_i^s(t,0,0)}\beta_{is}^p(y,t)\mu_i^s(t_i^s(y,t))dy+$$

$$+\sum_{i=q_s+1}^{p_s}\int\limits_{\varphi_i^s(t,0,0)}^{a_{s+1}(t)}\beta_{is}^p(y,t)\nu_i^s(t_i^s(y,t))dy+\sum_{i=p_s+1}^{n}\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^p(y,t)\nu_i^s(t_i^s(y,t))dy+$$

$$+\sum_{i=1}^{n}\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^p(y,t)\left(\int\limits_{t_i^s(y,t)}^{t}\Phi_i^s(\varphi_i^s(\tau,y,t),\tau,u^s)d\tau\right)dy\Big]=h^p(t),\ p=\overline{q+1,N}. \quad (19)$$

Given that $t_i^s(a_s(t),t)\equiv t$ for $i=\overline{1,p_s}$, and $t_i^s(a_{s+1}(t),t)\equiv t$ for $i=\overline{q_s+1,n}$, $s=\overline{0,m}$, we rewrite equality (18) as

$$\sum_{s=0}^{m}\Big[\sum_{i=1}^{p_s}\alpha_{is}^{sp}(t)\mu_i^s(t)+\sum_{i=q_s+1}^{n}\alpha_{is}^{s+1,p}(t)\nu_i^s(t)\Big]=$$

$$=\sum_{s=0}^{m}\Big[-\sum_{i=1}^{q_s}\alpha_{is}^{s+1,p}(t)\mu_i^s(t_i^s(a_{s+1}(t),t))-\sum_{i=p_s+1}^{n}\alpha_{is}^{sp}(t)\nu_i^s(t_i^s(a_s(t),t))-$$

$$-\sum_{i=1}^{q_s}\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^p(y,t)\mu_i^s(t_i^s(y,t))dy-\sum_{i=q_s+1}^{p_s}\int\limits_{a_s(t)}^{\varphi_i^s(t,0,0)}\beta_{is}^p(y,t)\mu_i^s(t_i^s(y,t))dy-$$

$$-\sum_{i=q_s+1}^{p_s}\int\limits_{\varphi_i^s(t,0,0)}^{a_{s+1}(t)}\beta_{is}^p(y,t)\nu_i^s(t_i^s(y,t))dy-\sum_{i=p_s+1}^{n}\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^p(y,t)\nu_i^s(t_i^s(y,t))dy-$$

$$-\sum_{i=1}^{n}\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^p(y,t)\left(\int\limits_{t_i^s(y,t)}^{t}\Phi_i^s(\varphi_i^s(\tau,y,t),\tau,u^s)d\tau\right)dy-$$

$$-\sum_{i=1}^{q_s}\alpha_{is}^{s+1,p}(t)\int\limits_{t_i^s(a_{s+1}(t),t)}^{t}\Phi_i^s(\varphi_i^s(\tau,a_{s+1}(t),t),\tau,u^s)d\tau-$$

$$-\sum_{i=p_s+1}^{n}\alpha_{is}^{sp}(t)\int\limits_{t_i^s(a_s(t),t)}^{t}\Phi_i^s(\varphi_i^s(\tau,a_s(t),t),\tau,u^s)d\tau\Big]+h^p(t),\quad p=\overline{1,q}. \quad (20)$$

Thus, the problem is reduced to finding a system of continuous functions $\{u_i^s(x,t)\}$, $\{\mu_i^s(t)\}$ and $\{\nu_i^s(t)\}$ for which all relations (17), (19) and (20) are fulfilled.

Next, we will need the derivatives

$$\frac{\partial\varphi_i^s(\tau,x,t)}{\partial x},\ \frac{\partial\varphi_i^s(\tau,x,t)}{\partial t},\ \frac{\partial t_i^s(x,t)}{\partial x},\ \frac{\partial t_i^s(x,t)}{\partial t},$$

the image of which was found in Section 3.

From formula (11) we obtain $\frac{\partial t_i^s}{\partial x}\neq 0$ in $\bar{G}^{si-}$, and therefore, according to the implicit function theorem, in $\bar{G}^{si-}$ the equation $\tau=t_i^s(x,t)$ can be solved with respect to x. Let us denote the obtained function by $x=\rho_i^{s-}(\tau,t)$; according to the same theorem, it is continuously differentiable. This function is defined for $0\leqslant\tau<\infty$, $\tau\leqslant t<\tau_i^{s-}(\tau)\leqslant\infty$, with $(\rho_i^{s-}(\tau,t),t)\in\bar{G}^{si-}$ and if $\tau_i^{s-}(\tau)<\infty$, we can assume that $t=t_i^{s-}(\tau)$; then $\varphi_i^s(\tau_i^{s-}(\tau),a_i(\tau),\tau)=a_{i+1}(\tau_i^{s-}(\tau))$. Similarly, in $\bar{G}^{si+}$ we can solve the equation $\tau=t_i^s(x,t)$ with respect to x, which will give a continuously differentiable function $x=\rho_i^{s+}(\tau,t)$, defined for $0\leqslant\tau<\infty$, $\tau\leqslant t<\tau_i^{s+}(\tau)\leqslant\infty$, with $(\rho_i^{s+}(\tau,t),t)\in\bar{G}^{si+}$ and if $\tau_i^{s+}(\tau)<\infty$, then $\varphi_i^s(\tau_i^{s+}(\tau),a_{i+1}(\tau),\tau)=a_i(\tau_i^{s+}(\tau))$. For $1\leqslant i\leqslant q_s$ the function ρ_i^{s+} is undefined, for $p_s+1\leqslant i\leqslant n$ the function ρ_i^{s-} is undefined, for $q_s+1\leqslant i\leqslant p_s$ (if $q_s<p_s$) both functions and $\tau_i^{s-}(\tau)\equiv\tau_i^{s+}(\tau)\equiv\infty$ are defined.

Using the introduced functions ρ_i^{s+}, given that $t_i^s(\varphi_i^s(t,0,0),t)\equiv 0$, $i=\overline{q_s+1,p_s}$ and making simple transformations of integrals (19) and (20), we obtain the equalities

$$\begin{aligned}
&\sum_{s=0}^{m}\Big[\sum_{i=1}^{p_s}\alpha_{is}^{sp}(t)\mu_i^s(t)+\sum_{i=q_s+1}^{n}\alpha_{is}^{s+1,p}(t)\nu_i^s(t)\Big]=\\
&\quad=\sum_{s=0}^{m}\Big[-\sum_{i=1}^{q_s}\alpha_{is}^{s+1,p}(t)\mu_i^s(t_i^s(a_{s+1}(t),t))-\sum_{i=p_s+1}^{n}\alpha_{is}^{sp}(t)\nu_i^s(t_i^s(a_s(t),t))+\\
&\qquad+\sum_{i=1}^{q_s}\int\limits_{t_i^s(a_{s+1}(t),t)}^{t}\beta_{is}^{p}(\rho_i^{s-}(\tau,t),t)\frac{\partial\rho_i^{s-}(\tau,t)}{d\tau}\mu_i^s(\tau)d\tau+\\
&\qquad+\sum_{i=q_s+1}^{p_s}\int\limits_{0}^{t}\beta_{is}^{p}(\rho_i^{s-}(\tau,t),t)\frac{\partial\rho_i^{s-}(\tau,t)}{\partial\tau}\mu_i^s(\tau)d\tau-\\
&\qquad-\sum_{i=q_s+1}^{p_s}\int\limits_{0}^{t}\beta_{is}^{p}(\rho_i^{s+}(\tau,t),t)\frac{\partial\rho_i^{s+}(\tau,t)}{\partial\tau}\nu_i^s(\tau)d\tau-\\
&\qquad-\sum_{i=p_s+1}^{n}\int\limits_{t_i^s(a_s(t),t)}^{t}\beta_{is}^{p}(\rho_i^{s+}(\tau,t),t)\frac{\partial\rho_i^{s+}(\tau,t)}{\partial\tau}\nu_i^s(\tau)d\tau-\\
&\qquad-\sum_{i=1}^{q_s}\alpha_{is}^{s+1,p}(t)\int\limits_{t_i^s(a_{s+1}(t),t)}^{t}\Phi_i^s(\varphi_i^s(\tau,a_{s+1}(t),t),\tau,u^s)d\tau-\\
&\qquad-\sum_{i=p_s+1}^{n}\alpha_{is}^{sp}(t)\int\limits_{t_i^s(a_s(t),t)}^{t}\Phi_i^s(\varphi_i^s(\tau,a_s(t),t),\tau,u^s)d\tau-
\end{aligned}$$

$$-\sum_{i=1}^{q_s}\int\limits_{t_i^s(a_{s+1}(t),t)}^{t}d\tau\int\limits_{\varphi_i^s(t,a_s(\tau),\tau)}^{a_{s+1}(\tau)}\beta_{is}^p(\varphi_i^s(t,y,\tau),t)\frac{\partial\varphi_i^s}{\partial y}\Phi_i^s(y,\tau,u^s)dy-$$

$$-\sum_{i=q_s+1}^{p_s}\int\limits_0^t d\tau\int\limits_{\varphi_i^s(t,a_s(\tau),\tau)}^{\varphi_i^s(t,0,0)}\beta_{is}^p(\varphi_i^s(t,y,\tau),t)\frac{\partial\varphi_i^s}{\partial y}\Phi_i^s(y,\tau,u^s)dy-$$

$$-\sum_{i=q_s+1}^{p_s}\int\limits_0^t d\tau\int\limits_{\varphi_i^s(t,0,0)}^{\varphi_i^s(t,a_{s+1}(\tau),\tau)}\beta_{is}^p(\varphi_i^s(t,y,\tau),t)\frac{\partial\varphi_i^s}{\partial y}\Phi_i^s(y,\tau,u^s)dy-$$

$$-\sum_{i=p_s+1}^{n}\int\limits_{t_i^s(a_s(t),t)}^{t}d\tau\int\limits_{a_s(\tau)}^{\varphi_i^s(t,a_{s+1}(\tau),\tau)}\beta_{is}^p(\varphi_i^s(t,y,\tau),t)\frac{\partial\varphi_i^s}{\partial y}\Phi_i^s(y,\tau,u^s)dy\Big]+$$

$$+h^p(t),\quad p=\overline{1,q};\quad (21)$$

$$\sum_{s=0}^{m}\Big[-\sum_{i=1}^{q_s}\int\limits_{t_i^s(a_{s+1}(t),t)}^{t}\beta_{is}^p(\rho_i^{s-}(\tau,t),t)\frac{\partial\rho_i^{s-}(\tau,t)}{\partial\tau}\mu_i^s(\tau)d\tau-$$

$$-\sum_{i=q_s+1}^{p_s}\int\limits_0^t\beta_{is}^p(\rho_i^{s-}(\tau,t),t)\frac{\partial\rho_i^{s-}(\tau,t)}{\partial\tau}\mu_i^s(\tau)d\tau+$$

$$+\sum_{i=q_s+1}^{p_s}\int\limits_0^t\beta_{is}^p(\rho_i^{s+}(\tau,t),t)\frac{\partial\rho_i^{s+}(\tau,t)}{\partial\tau}\nu_i^s(\tau)d\tau+$$

$$+\sum_{i=p_s+1}^{n}\int\limits_{t_i^s(a_s(t),t)}^{t}\beta_{is}^p(\rho_i^{s+}(\tau,t),t)\frac{\partial\rho_i^{s+}(\tau,t)}{\partial\tau}\nu_i^s(\tau)d\tau+$$

$$+\sum_{i=1}^{q_s}\int\limits_{t_i^s(a_{s+1}(t),t)}^{t}d\tau\int\limits_{\varphi_i^s(t,a_s(\tau),\tau)}^{a_{s+1}(\tau)}\beta_{is}^p(\varphi_i^s(t,y,\tau),t)\frac{\partial\varphi_i^s}{\partial y}\Phi_i^s(y,\tau,u^s)dy+$$

$$+\sum_{i=q_s+1}^{p_s}\int\limits_0^t d\tau\int\limits_{\varphi_i^s(t,a_s(\tau),\tau)}^{\varphi_i^s(t,0,0)}\beta_{is}^p(\varphi_i^s(t,y,\tau),t)\frac{\partial\varphi_i^s}{\partial y}\Phi_i^s(y,\tau,u^s)dy+$$

$$+\sum_{i=q_s+1}^{p_s}\int\limits_0^t d\tau\int\limits_{\varphi_i^s(t,0,0)}^{\varphi_i^s(t,a_{s+1}(\tau),\tau)}\beta_{is}^p(\varphi_i^s(t,y,\tau),t)\frac{\partial\varphi_i^s}{\partial y}\Phi_i^s(y,\tau,u^s)dy+$$

$$+\sum_{i=p_s+1}^{n}\int\limits_{t_i^s(a_s(t),t)}^{t}d\tau\int\limits_{a_s(\tau)}^{\varphi_i^s(t,a_{s+1}(\tau),\tau)}\beta_{is}^p(\varphi_i^s(t,y,\tau),t)\frac{\partial\varphi_i^s}{\partial y}\Phi_i^s(y,\tau,u^s)dy\Big]=$$

$$=h^p(t),\quad p=\overline{q+1,N}.\quad (22)$$

Let us calculate the derivatives that appear in formulas (21), (22)

$$\frac{\partial\rho_i^{s-}(\tau,t)}{\partial\tau},\quad\frac{\partial\rho_i^{s+}(\tau,t)}{\partial\tau}.$$

From formula (9) we obtain

$$\frac{\partial\varphi_i^s(t,y,\tau)}{\partial y}=\exp\left(-\int\limits_{\tau}^{t}\lambda_{ix}^{s}{}'(\varphi_i^s(\sigma,y,\tau),\sigma)d\sigma\right),\ \ i=\overline{1,n},\ s=\overline{0,m}. \tag{23}$$

To find $\frac{\partial\rho_i^{s+}(\tau,t)}{\partial\tau}$, write down the first identity of (15) at $x=\rho_i^{s-}(\tau,t)$:

$$\varphi_i^s(\tau,\rho_i^{s-}(\tau,t),t)\equiv a_s(\tau).$$

Differentiating the resulting equality by τ, we have

$$\frac{\partial\varphi_i^s(\tau,\rho_i^{s-}(\tau,t),t)}{\partial\tau}+\frac{\partial\varphi_i^s(\tau,\rho_i^{s-}(\tau,t),t)}{\partial x}\frac{\partial\rho_i^{s-}(\tau,t)}{\partial\tau}=a_s'(\tau).$$

Given (2), we get

$$\frac{\partial\rho_i^{s-}(\tau,t)}{\partial\tau}=-(\lambda_i^s(a_s(\tau),\tau)-a_s'(\tau))\exp\left(\int\limits_{\tau}^{t}\lambda_{ix}^{s}{}'(\varphi_i^s(\sigma,\rho_i^{s-}(\tau,t),t),\sigma)d\sigma\right). \tag{24}$$

Substituting $x=\rho_i^{s+}(\tau,t)$ into the second identity of (15) and differentiating by τ, we similarly find

$$\frac{\partial\rho_i^{s+}(\tau,t)}{\partial\tau}=-(\lambda_i^s(a_{s+1}(\tau),\tau)-a_{s+1}'(\tau))\exp\left(\int\limits_{\tau}^{t}\lambda_{ix}^{s}{}'(\varphi_i^s(\sigma,\rho_i^{s+}(\tau,t),t),\sigma)d\sigma\right). \tag{25}$$

Let

$$\begin{aligned}R_{is}^{p-}(\tau,t)\equiv&\beta_{is}^{p}(\rho_i^{s-}(\tau,t),t)(\lambda_i^s(a_s(\tau),\tau)-a_s'(\tau))\times\\&\times\exp\left(\int\limits_{\tau}^{t}\lambda_{ix}^{s}{}'(\varphi_i^s(\sigma,\rho_i^{s-}(\tau,t),t),\sigma)d\sigma\right),\end{aligned} \tag{26}$$

$$s=\overline{0,m},\ i=\overline{1,p_s},\ 0\leqslant\tau<\infty,\ \tau\leqslant t<\tau_i^{s-}(\tau);$$

$$\begin{aligned}R_{is}^{p+}(\tau,t)\equiv&\beta_{is}^{p}(\rho_i^{s+}(\tau,t),t)(\lambda_i^s(a_{s+1}(\tau),\tau)-a_{s+1}'(\tau))\times\\&\times\exp\left(\int\limits_{\tau}^{t}\lambda_{ix}^{s}{}'(\varphi_i^s(\sigma,\rho_i^{s+}(\tau,t),t),\sigma)d\sigma\right),\end{aligned} \tag{27}$$

$$s=\overline{0,m},\ i=\overline{q_s+1,n},\ 0\leqslant\tau<\infty,\ \tau\leqslant t<\tau_i^{s+}(\tau);$$

$$Q_{is}^{p}(y,\tau,t)\equiv\beta_{is}^{p}(\varphi_i^s(t,y,\tau),t)\exp\left(\int\limits_{\tau}^{t}\lambda_{ix}^{s}{}'(\varphi_i^s(\sigma,y,t),\sigma)d\sigma\right),\tag{28}$$

$$i=\overline{1,n},\ s=\overline{0,m}.$$

Let us rewrite equations (21), (22), taking into account (23)–(25) and the introduced notations (26)–(28), in the following form (double integrals for $i=\overline{q_s+1,p_s}$ are combined):

$$\begin{aligned}
&\sum_{s=0}^{m}\Big[\sum_{i=1}^{p_s}\alpha_{is}^{sp}(t)\mu_i^s(t)+\sum_{i=q_s+1}^{n}\alpha_{is}^{s+1,p}(t)\nu_i^s(t)\Big]=\\
&\quad=\sum_{s=0}^{m}\Big[-\sum_{i=p_s+1}^{n}\alpha_{is}^{sp}(t)\nu_i^s(t_i^s(a_s(t),t))-\sum_{i=1}^{q_s}\alpha_{is}^{s+1,p}(t)\mu_i^s(t_i^s(a_{s+1}(t),t))-\\
&\qquad-\sum_{i=1}^{q_s}\int\limits_{t_i^s(a_{s+1}(t),t)}^{t}R_{is}^{p-}(\tau,t)\mu_i^s(\tau)d\tau-\sum_{i=q_s+1}^{p_s}\int\limits_{0}^{t}R_{is}^{p-}(\tau,t)\mu_i^s(\tau)d\tau+\\
&\qquad+\sum_{i=q_s+1}^{p_s}\int\limits_{0}^{t}R_{is}^{p+}(\tau,t)\nu_i^s(\tau)d\tau+\sum_{i=p_s+1}^{n}\int\limits_{t_i^s(a_s(t),t)}^{t}R_{is}^{p+}(\tau,t)\nu_i^s(\tau)d\tau-\\
&\qquad-\sum_{i=1}^{q_s}\alpha_{is}^{s+1,p}(t)\int\limits_{t_i^s(a_{s+1}(t),t)}^{t}\Phi_i^s(\varphi_i^s(\tau,a_{s+1}(t),t),\tau,u^s)d\tau-\\
&\qquad-\sum_{i=1}^{q_s}\int\limits_{t_i^s(a_{s+1}(t),t)}^{t}d\tau\int\limits_{\varphi_i^s(t,a_s(\tau),\tau)}^{a_{s+1}(\tau)}Q_{is}^{p}(y,\tau,t)\Phi_i^s(y,\tau,u^s)dy-\\
&\qquad-\sum_{i=q_s+1}^{p_s}\int\limits_{0}^{t}d\tau\int\limits_{\varphi_i^s(t,a_s(\tau),\tau)}^{\varphi_i^s(t,a_{s+1}(\tau),\tau)}Q_{is}^{p}(y,\tau,t)\Phi_i^s(y,\tau,u^s)dy-\\
&\quad-\sum_{i=p_s+1}^{n}\int\limits_{t_i^s(a_s(t),t)}^{t}d\tau\int\limits_{a_s(\tau)}^{\varphi_i^s(t,a_{s+1}(\tau),\tau)}Q_{is}^{p}(y,\tau,t)\Phi_i^s(y,\tau,u^s)dy\Big]+h^p(t),\ p=\overline{1,q};
\end{aligned}\tag{29}$$

$$\begin{aligned}
&\sum_{s=0}^{m}\Big[\sum_{i=1}^{q_s}\int\limits_{t_i^s(a_{s+1}(t),t)}^{t}R_{is}^{p-}(\tau,t)\mu_i^s(\tau)d\tau+\sum_{i=q_s+1}^{p_s}\int\limits_{0}^{t}R_{is}^{p-}(\tau,t)\mu_i^s(\tau)d\tau-\\
&\qquad-\sum_{i=q_s+1}^{p_s}\int\limits_{0}^{t}R_{is}^{p+}(\tau,t)\nu_i^s(\tau)d\tau-\sum_{i=p_s+1}^{n}\int\limits_{t_i^s(a_s(t),t)}^{t}R_{is}^{p+}(\tau,t)\nu_i^s(\tau)d\tau+\\
&\qquad+\sum_{i=1}^{q_s}\int\limits_{t_i^s(a_{s+1}(t),t)}^{t}d\tau\int\limits_{\varphi_i^s(t,a_s(\tau),\tau)}^{a_{s+1}(\tau)}Q_{is}^{p}(y,\tau,t)\Phi_i^s(y,\tau,u^s)dy+
\end{aligned}$$

$$+\sum_{i=q_s+1}^{p_s}\int_0^t d\tau \int_{\varphi_i^s(t,a_s(\tau),\tau)}^{\varphi_i^s(t,a_{s+1}(\tau),\tau)} Q_{is}^p(y,\tau,t)\Phi_i^s(y,\tau,u^s)dy+$$

$$+\sum_{i=p_s+1}^{n}\int_{t_i^s(a_s(t),t)}^{t} d\tau \int_{a_s(\tau)}^{\varphi_i^s(t,a_{s+1}(\tau),\tau)} Q_{is}^p(y,\tau,u^s)\Phi_i^s(y,\tau,u^s)dy\Big] =$$

$$= h^p(t),\ p=\overline{q+1,N}. \quad (30)$$

Equalities (29) have the form of Volterra equations of the second kind, and equalities (30) have the form of Volterra equations of the first kind with respect to the functions $\mu_i^s(t)$ and $\nu_i^s(t)$.

In order to present equations (30) as a Volterra equation of the second kind, we will need formulas that easily follow from (26)–(28) and (11)–(15):

$$R_{is}^{p-}(t,t) \equiv \beta_{is}^p(a_s(t),t)(\lambda_i^s(a_s(t),t)-a_s'(t)),$$

$$R_{is}^{p+}(t,t) \equiv \beta_{is}^p(a_{s+1}(t),t)(\lambda_i^s(a_{s+1}(t),t)-a_{s+1}'(t)),$$

$$R_{is}^{p-}(t_i^s(a_{s+1}(t),t))\frac{d}{dt}t_i^s(a_{s+1}(t),t) \equiv \beta_{is}^p(a_{s+1}(t),t)(\lambda_i^s(a_{s+1}(t),t)-a_{s+1}'(t))\times$$

$$\times\exp\left(-\int_{t_i^s(a_{s+1}(t),t)}^{t} {\lambda_{ix}^s}'(\varphi_i^s(\sigma,a_{s+1}(t),t),\sigma)d\sigma\right),$$

$$R_{is}^{p+}(t_i^s(a_s(t),t))\frac{d}{dt}t_i^s(a_s(t),t) \equiv \beta_{is}^p(a_s(t),t)(\lambda_i^s(a_s(t),t)-a_s'(t))\times$$

$$\times\exp\left(-\int_{t_i^s(a_s(t),t)}^{t} {\lambda_{ix}^s}'(\varphi_i^s(\sigma,a_s(t),t),\sigma)d\sigma\right).$$

Since the left-hand and right-hand sides of (30) coincide at $t=0$ and have continuous derivatives in t, then fulfilling these equalities is equivalent to fulfilling the corresponding differentiated equalities. Differentiating with respect to t, taking into account the derived formulas and grouping similar terms, we obtain the relations

$$\sum_{s=0}^{m}\Big[\sum_{i=1}^{p_s}\beta_{is}^p(a_s(t),t)(\lambda_i^s(a_s(t),t)-a_s'(t))\mu_i^s(t)-$$

$$-\sum_{i=q_s+1}^{n}\beta_{is}^p(a_{s+1}(t),t)(\lambda_i^s(a_{s+1}(t),t)-a_{s+1}'(t))\nu_i^s(t)\Big] =$$

$$=\sum_{s=0}^{m}\Big[\sum_{i=1}^{q_s}\beta_{is}^p(a_{s+1}(t),t)(\lambda_i^s(a_{s+1}(t),t)-a_{s+1}'(t))\times$$

$$\times\exp\left(-\int_{t_i^s(a_{s+1}(t),t)}^{t} {\lambda_{ix}^s}'(\varphi_i^s(\sigma,a_{s+1}(t),t),\sigma)d\sigma\right)\mu_i^s(t_i^s(a_{s+1}(t),t))-$$

$$-\sum_{i=p_s+1}^{n}\beta_{is}^{p}(a_s(t),t)(\lambda_i^s(a_s(t),t)-a_s'(t))\times$$
$$\times\exp\Bigg(-\int\limits_{t_i^s(a_s(t),t)}^{t}\lambda_{ix}^{s}{}'(\varphi_i^s(\sigma,a_s(t),t),\sigma)d\sigma\Bigg)\nu_i^s(t_i^s(a_s(t),t))-$$
$$-\sum_{i=1}^{q_s}\int\limits_{t_i^s(a_{s+1}(t),t)}^{t}R_{is_t}^{p-}{}'(\tau,t)\mu_i^s(\tau)d\tau-\sum_{i=q_s+1}^{p_s}\int\limits_{0}^{t}R_{is_t}^{p-}{}'(\tau,t)\mu_i^s(\tau)d\tau+$$
$$+\sum_{i=q_s+1}^{p_s}\int\limits_{0}^{t}R_{is_t}^{p+}{}'(\tau,t)\nu_i^s(\tau)d\tau+\sum_{i=p_s+1}^{n}\int\limits_{t_i^s(a_s(t),t)}^{t}R_{is_t}^{p+}{}'(\tau,t)\nu_i^s(\tau)d\tau+$$
$$+G_{is}^{p}(\Phi_i^s(y,\tau,u^s(y,\tau)),\tau,t)\Big],\ p=\overline{q+1,N},\quad (31)$$

where

$$G_{is}^{p}(\Phi_i^s(y,\tau,u^s(y,\tau)),\tau,t)\equiv$$
$$\equiv-\sum_{i=1}^{n}\frac{d}{dt}\int\limits_{\gamma_i^s(t)}^{t}\int\limits_{\psi_i^s(\tau,t)}^{\chi_i^s(\tau,t)}Q_{is}^{p}(y,\tau,t)\Phi_i^s(y,\tau,u^s)dy+h^{p}{}'(t),\ p=\overline{q+1,N},$$

$$\gamma_i^s(t)=\begin{cases}t_i^s(a_{s+1}(t),t), & \text{if } i=\overline{1,q_s},\\ 0, & \text{if } i=\overline{q_s+1,p_s},\\ t_i^s(a_s(t),t), & \text{if } i=\overline{p_s,n},\end{cases}$$

$$\psi_i^s(\tau,t)=\begin{cases}\varphi_i^s(t,a_s(\tau),\tau), & \text{if } i=\overline{1,p_s},\\ a_s(t), & \text{if } i=\overline{p_s+1,n},\end{cases}$$

$$\chi_i^s(\tau,t)=\begin{cases}a_{s+1}(t), & \text{if } i=\overline{1,q_s},\\ \varphi_i^s(t,a_{s+1}(\tau),\tau), & \text{if } i=\overline{q_s+1,n}.\end{cases}$$

Equalities (28), (31) form a system of linear integro-functional equations with respect to the functions $\mu_i^s(t)$ and $\nu_i^s(t)$, and the matrix formed by the coefficients of these functions in the left-hand sides of the equations is $A(t)$. According to (6), the system (28), (31) can be written in the form

$$\nu(t)=(M\nu)(t)+(K\nu)(t)+(Lu)(t)+(\tilde{L}f)(t)+H(t),\quad (32)$$

where $\nu(t)=col(\mu_1^0(t),\dots,\mu_{p_0}^0(t),\dots,\mu_1^m(t),\dots,\mu_{p_m}^m(t),\ \nu_{q_0+1}^0(t),\dots,\nu_n^0(t),\dots,$ $\nu_{q_m+1}^m(t),\dots,\nu_n^m(t))$; K is a Volterra type matrix linear integral operator, elements of which are linear combinations of integrals of the form $\int\limits_{\gamma_i^s(t)}^{t}K_i^{s-}(t,\tau)\mu_i^s(\tau)d\tau$

and $\int\limits_{\gamma_i^s(t)}^{t} K_i^{s-}(t,\tau)\nu_i^s(\tau)d\tau$ with continuous kernels, with $\gamma_i^s(t) = t_i^s(a_{s+1}(t), t)$ if $i = \overline{1, q_s}$, $\gamma_i^s(t) = 0$ for $i = \overline{q_s+1, p_s}$, and $\gamma_i^s(t) = t_i^s(a_s(t), t)$ for $i = \overline{p_s+1, n}$. L and $\tilde{L}$ are Volterra type matrix linear integral operators elements of which have continuous kernels, that act on the vector-function u with components $u_i^s(x,t)$ and the vector-function f with components $f_i^s(x,t)$, respectively. $H(t)$ is a known continuous column vector of height N with elements $h^p(t)(p = \overline{1,q})$, $h^{p'}(t)(p = \overline{q+1, N})$; M is an operator that has the form

$$(M\nu)(t) = A(t)^{-1}B(t)(P\nu)(t),$$

where P is a shift operator defined by the formula

$$(P\mu_i^s)(t) = \mu_i^s(t_i^s(a_{s+1}(t), t)), \quad (P\nu_i^s)(t) = \nu_i^s(t_i^s(a_s(t), t)),$$

and $B(t)$ is the coefficient matrix of the functions $(P\nu)(t)$ in the right-hand sides of equations (29) and (31). Obviously, $B(0)$ coincides with the matrix introduced in Section 2.

Since $0 \leqslant t_i^s(a_r(t), t) \leqslant t$, $r = \overline{s, s+1}$, $s = \overline{0, m}$, $i = \overline{1, n}$, for arbitrary $\varepsilon > 0$ the operator P maps the elements of the space $\Big[C[0,\varepsilon]\Big]^N$ to the elements of the same space. Let the norm in this space be defined as follows: if $\nu = \{\nu_l\} \in \Big[C[0,\varepsilon]\Big]^N$. Then $\|\nu\| = \Big|\max\limits_{0\leqslant t\leqslant\varepsilon} |\nu_l(t)|\Big|$, where the outer vertical lines denote an arbitrary of the norm in $\mathbb{R}^N$. Then from the obvious inequality

$$\max_{0\leqslant t\leqslant\varepsilon} |(P\nu)(t)| \leqslant \max_{0\leqslant t\leqslant\varepsilon} |\nu_l(t)|,$$

which turns into an equality for $\nu_l(t) \equiv \mathit{const}$, it follows that the norm of the operator P is 1.

Returning to condition (7) (in which the matrix norm is considered to be consistent with the norm of the vectors in $\mathbb{R}^N$, so that $|Aa| \leqslant |A|\cdot|a|$), we obtain, from the continuity of the elements of matrices $A(t)$ and $B(t)$, the existence of such $\varepsilon > 0$ such that the operator M in the space $\Big[C[0,\varepsilon]\Big]^N$ has a norm less than 1. But then equation (32) can be written as

$$\nu(t) = \Big[(I-M)^{-1}(K\nu + Lu + \tilde{L}f + H)\Big](t), \tag{33}$$

that is

$$\Big(\Big[I - (I-M)^{-1}K\Big]\nu\Big)(t) = \Big[(I-M)^{-1}(Lu + \tilde{L}f + H)\Big](t).$$

Since K is a Volterra type integral operator which, with sufficiently small $\varepsilon > 0$, has a norm as small as desired, the operator $I - (I - M)^{-1}K$ can be continuously inverted (this may require reducing the value of $\varepsilon > 0$), that is, we can arrive at the equation

$$\nu(t) = \Big[(I - (I - M)^{-1}K)^{-1}(I - M)^{-1}(Lu + \tilde{L}f + H)\Big](t). \tag{34}$$

On the other hand, equation (17) has the form

$$u(x,t) = (Q\nu)(x,t) + (L_1 u)(x,t) + (\tilde{L}_1 f)(x,t), \tag{35}$$

where operator Q is a shift operator with norm 1, and L_1 and $\tilde{L}_1$ are Volterra type matrix linear operators with continuous kernels. Moreover, for the operator Q to be continuously defined and for a continuous vector-function $\nu(t)$ to determine the continuity of the vector-function $(Q\nu)(x,t)$, it is necessary and sufficient that the vector-function $\nu(t)$ satisfies the condition

$$\mu_i^s(0) = \nu_i^s(0) \quad (i = \overline{q_s + 1, p_s},\ s = \overline{0, m}). \tag{36}$$

If the vector-function $\nu(t)$ satisfies condition (33), then

$$\nu(0) = \Big[(I - M)^{-1}H\Big](0) = \Big[I - A^{-1}(0)B(0)\Big]^{-1} H(0). \tag{37}$$

Hence we get

$$\mu_i^s(0) = \sum_{p=1}^{N} \delta_{l_i^s, p} H^p(0), \quad \nu_i^s(0) = \sum_{p=1}^{N} \delta_{k_i^s, p} H^p(0),$$

where the notations l_i^s and k_i^s are given in Section 2.

Thus, condition (36) coincides with (8) and therefore, if the latter is satisfied, then the vector-function ν, defined from (34), always generates a continuous vector-function $Q\nu$. Substituting ν from (34) into (35), we rewrite the last equation as

$$\begin{aligned}\Big(\Big[I - Q(I - (I - M)^{-1}K)^{-1}(I - M)^{-1}L - L_1\Big]u\Big)(x,t) = \\ = \Big[Q(I - (I - M)^{-1}K)^{-1}(I - M)^{-1}H\Big](x,t) + \\ + \Big(\Big[Q(I - (I - M)^{-1}K)^{-1}(I - M)^{-1}\tilde{L} + \tilde{L}_1\Big]f\Big)(x,t).\end{aligned} \tag{38}$$

Thus, the system of equations (17), (29), (31), which gives the solution to the boundary value problem under consideration is equivalent to the system of equations (38), (34). Moreover, equation (38) does not contain ν. Since L and L_1 are Volterra type integral operators, the operator $I - Q(I - (I - M)^{-1}K)^{-1}(I - M)^{-1}L - L_1$ can be continuously inverted again, from which we obtain the whole statement of Theorem 1. Note that all operator transformations mentioned here can be done using the standard iteration method.

This concludes the proof of Theorem 1. Note that this theorem, due to its local character, holds even if conditions (6) are replaced by $\det A(0) \neq 0$. ◄

5. Corresponding Nonlocal Theorem

Let us first consider a mixed problem similar to the one considered in Sections 2 – 4, in the case where the interval in which the initial conditions are given is nondegenerate. For $0 < \varepsilon < T < \infty$ we denote $G_{\varepsilon,T} = \{(x,t) \in G : \varepsilon < t < T\}$ and consider the system of equations (1) with boundary conditions (3), (4) and initial conditions

$$u_i^s(x,\varepsilon) = g_i^s(x), \ \ (a_s(\varepsilon) \leqslant x \leqslant a_{s+1}(\varepsilon), \ i = \overline{1,n}, \ s = \overline{0,m}) \tag{39}$$

in the above domain. Let us assume that all functions g_i^s are continuous, and the other given functions satisfy the conditions formulated in Section 2. Moreover, condition (7) is dropped, and the agreement condition, instead of (8), takes the form ($s = \overline{0,m}$)

$$\sum_{i=1}^{n}\sum_{s=0}^{m}\Big[\sum_{k=s}^{s+1}\alpha_{is}^{kp}(\varepsilon)g_i^s(a_k(\varepsilon)) + \int\limits_{a_s(\varepsilon)}^{a_{a+1}(\varepsilon)}\beta_{is}^p(y,\varepsilon)g_i^s(y)dy\Big] = h^p(\varepsilon), \ p = \overline{1,q};$$

$$\begin{aligned}\sum_{i=1}^{n}\sum_{s=0}^{m}\Big\{&a'_{s+1}(\varepsilon)\beta_{is}^p(a_{s+1}(\varepsilon),\varepsilon)g_i^s(a_{s+1}(\varepsilon)) - a'_s(\varepsilon)\beta_{is}^p(a_s(\varepsilon),\varepsilon)g_i^s(a_s(\varepsilon)) +\\ &+ \int\limits_{a_s(\varepsilon)}^{a_{s+1}(\varepsilon)}\tfrac{\partial\beta_{is}^p(y,\varepsilon)}{\partial t}g_i^s(y)dy + \int\limits_{a_s(\varepsilon)}^{a_{s+1}(\varepsilon)}\beta_{is}^p(y,\varepsilon)\Big[-\lambda_i^s(y,\varepsilon)\tfrac{dg_i^s(y)}{dy} +\\ &+ \sum_{k=1}^{n}a_{ik}^s(y,\varepsilon)g_k^s(y) + f_i^s(y,\varepsilon)\Big]dy\Big\} = \tfrac{dh^p(\varepsilon)}{dt}, \ p = \overline{q+1,N},\end{aligned} \tag{40}$$

$$\sum_{i=1}^{n}\sum_{s=0}^{m}\int\limits_{a_s(\varepsilon)}^{a_{s+1}(\varepsilon)}\beta_{is}^p(y,\varepsilon)g_i^s(y)dy = h^p(\varepsilon), \ p = \overline{q+1,N}.$$

Definition 2. *The piecewise continuous generalized solution of the system of equations* (1) *in* $G_{\varepsilon,T}$ *is defined as a function* $u(x,t)$ *that is uniformly continuous* *in every domain* $G^s_{\varepsilon,T} = \{(x,t) \in G^s : \varepsilon < t < T\}$ *and satisfies a system of integro-functional equations of the same form* (17) *as before, where*

$$\omega^s_i(x,t) = \begin{cases} u^s_i(\varphi^s_i(\varepsilon,x,t),\varepsilon), & \text{if } t^s_{i,\varepsilon}(x,t) = \varepsilon, \\ u^s_i(a_s(t_{i,\varepsilon}(x,t)), t^s_{i,\varepsilon}(x,t)), & \text{if } t^s_{i,\varepsilon}(x,t) > \varepsilon, \\ & \varphi^s_i(t^s_{i,\varepsilon}(x,t),x,t) = a_s(t^s_{i,\varepsilon}(x,t)), \\ u^s_i(a_{s+1}(t^s_{i,\varepsilon}(x,t)), t^s_{i,\varepsilon}(x,t)), & \text{if } t^s_{i,\varepsilon}(x,t) > \varepsilon, \\ & \varphi^s_i(t^s_{i,\varepsilon}(x,t),x,t) = a_{s+1}(t_{i,\varepsilon}(x,t)), \end{cases}$$

and by $t^s_{i,\varepsilon}(x,t)$ *we mean the smallest value* $\tau \geqslant \varepsilon$ *at which the function* $\varphi^s_i(\tau,x,t)$ *is defined. Such solution will be called a piecewise continuous generalized solution of problems* (1), (3), (4), (39) *in* $\bar{G}_{\varepsilon,T}$ *if it satisfies conditions* (3), (4), (39) *in the ordinary sense.*

From Definition 2 a simple but very important corollary follows immediately.

Corollary 1. *Let* $\tilde{u}(\tilde{\tilde{u}})$ *be a piecewise continuous generalized solution of the system of equations* (1) *in* $\bar{G}_{\varepsilon,T_1}$ *(respectively* $\bar{G}_{T_1,T}$*), where* $0 < \varepsilon < T_1 < T$, *and* $\tilde{u}(x,T_1) \equiv \tilde{\tilde{u}}(x,T_1)$. *Then the vector-function* u*, that is equal to* $\tilde{u}$ *in* $\bar{G}_{\varepsilon,T_1}$ *and* $\tilde{\tilde{u}}$ *in* $\bar{G}_{T_1,T}$*, is a piecewise continuous generalized solution of the system of equations* (1) *in* $\bar{G}_{\varepsilon,T}$. *This corollary extends directly to "gluing" solutions in* $\bar{G}_\varepsilon$ *and* $\bar{G}_{\varepsilon,T}$*, as well as to the case of "gluing" solutions in several domains* $\bar{G}_{\varepsilon,T_1}, \bar{G}_{T_1,T_2}, \dots, \bar{G}_{T_p,T}$.

In the statement of the theorem on generalized solvability of the problem (1), (3), (4), (39) let us list all assumptions about the given functions.

Theorem 2. *Let for all* $i = \overline{1,n}$, $s = \overline{0,m}$, $k = \overline{s,s+1}$*:*

1) *the functions* $\lambda^s_i \in C^2(\bar{G}^s_{\varepsilon,T})$, a^s_{ij}, $f^s_i \in C(\bar{G}^s_{\varepsilon,T})$;

2) *the coefficients* α^{kp}_{is}, $h^p \in C[\varepsilon,T]$, $\beta^p_{is} \in C(\bar{G}^s_{\varepsilon,T})$, $(p = \overline{1,q})$;

3) *the coefficients* $\beta^p_{is} \in C^1(\bar{G}^s_{\varepsilon,T})$, $h^p \in C^1[\varepsilon,T]$, $(p = \overline{1+q,N})$;

4) $g^s_i \in C[a_s(\varepsilon), a_{s+1}(\varepsilon)]$;

5) *conditions* (2), (6) *for all* $t \in [\varepsilon,T]$ *be fulfilled;*

6) *the agreement conditions* (40) *be fulfilled.*

Then the problem (1), (3), (4), (39) *has a unique piecewise continuous generalized solution in* $\bar{G}_{\varepsilon,T}$. *This solution is continuous (in the sense of a uniform metrics) and depends on the given functions* $h^p (p = \overline{1,q})$, $h^{p\prime} (p = \overline{1+q,N})$ *and all* f^s_i *and* g^s_i *(a similar explanation of this is as in Theorem 1).*

Proof. The proof is not essentially different from the one of Theorem 1. Here it is convenient to divide the whole interval $[\varepsilon, T]$ into equal smaller intervals $[\varepsilon, T_1], [T_1, T_2], \ldots, [T_p, T]$, so that in no domain $\bar{G}^s_{T_k,T_{k+1}}(k = 0, \ldots, p,\ T_0 = \varepsilon,\ T_{p+1} = T)$ can any characteristic join one from the "sides" of the other. Then, according to Corollary 1, it suffices to prove it for any of the domains $\bar{G}^s_{T_k,T_{k+1}}$; in other words, without loss of generality, we can assume that the domain $\bar{G}_{\varepsilon,T}$ itself already possesses the above property. In that case, the transformation of the boundary conditions described in the proof of Theorem 1 will make the terms with multipliers $\mu^s_i(t^s_{i,\varepsilon}(a_{s+1}(t), t))(i = \overline{1, q_s})$ and $\nu^s_i(t^s_{i,\varepsilon}(a_s(t), t))(i = \overline{p_s + 1, n})$ disappear, that is, instead of (32) we get the equation

$$\nu(t) = (K\nu + Lu + \tilde{L}f + Rg + H)(t), \tag{41}$$

where $K, L, \tilde{L}, H$ remain the same as before, and R is a linear bounded functional operator, which maps $\prod\limits_{s=0}^{n}\Big[C[a_s(\varepsilon), a_{s+1}(\varepsilon)]\Big]^n$ into $\Big[C[\varepsilon, T]\Big]^N$.

In the above transformation, during the differentiation of the second group of boundary conditions, we apply the last group of agreement conditions (40) instead of equality $h^p(0) = 0$.

From equation (41) it follows, without any assumption on the norm of the operator K, that the operator $I - K$ is inverse, that is, (41) is equivalent to

$$\nu(t) = \Big[(I - K)^{-1}(Lu + \tilde{L}f + Rg + H)\Big](t).$$

Substituting this expression into (35) (the piecewise continuity of the function $(Q\nu)(x, t)$ is ensured by the first two groups of agreement conditions (40)) leads us to an equation with respect to u

$$\begin{aligned}\Big[(I - Q(I - K)^{-1}L - L_1)u\Big](x, t) =& \\ = \Big[Q - (I - K)^{-1}(\tilde{L}f + Rg + H)\Big]&(x, t) + (\tilde{L}_1 f)(x, t). \quad (42)\end{aligned}$$

Note that for sufficiently small ε the norm of the operators $Q(I - K)^{-1}L$ and L_1 is as small as desired. By choosing ε such that $\Big|Q(I - K)^{-1}L - L_1\Big| < 1$, we can find the inverse operator to the operator $I - Q(I - K)^{-1}L - L_1$ in equation (42), which in turn completes the proof of Theorem 2. ◄

From Theorems 1 and 2 we have

Theorem 3. *Under the conditions of Theorem 1, the problem* (1)–(4) *has a unique generalized piecewise continuous solution in* $\bar{G}$*, which at each finite change interval of* t*, depends continuously (in the sense of uniform metrics) on the given functions* $h^p(p = \overline{1, q})$, $h^{p\prime}(p = \overline{q + 1, N})$ *and all* f^s_i.

To prove this, we first use Theorem 1 to construct a solution in $\bar{G}_\varepsilon$, and then apply Theorem 2 in $\bar{G}_{\varepsilon,T}$ for an arbitrary $T > \varepsilon$ to the initial condition

$$u_i^s(x,\varepsilon)\Big|_{\bar{G}^s_{\varepsilon,T}} = g_i^s(x) = u_i^s(x,\varepsilon)\Big|_{\bar{G}^s_\varepsilon}.$$

Then, from the boundary conditions (3), (4) and the conditions (4) differentiated with respect to t (which are fulfilled in the ordinary sense) for $t = \varepsilon$ it follows that the agreement conditions (40) are fulfilled in $\bar{G}_{\varepsilon,T}$. According to Corollary 1, we obtain a piecewise continuous generalized solution of problem (1)–(4) in $\bar{G}_T$ which, in addition, is unique. From the previous considerations we obtain the desired solution in $\bar{G}$ as $t \to \infty$.

6. The Case of a Semilinear System

Consider now in G^s, $s = \overline{0,m}$, a semilinear system

$$\frac{\partial u^s}{\partial t} + \Lambda^s \frac{\partial u^s}{\partial x} = \Phi^s(x,t,u^s), \quad s = \overline{0,m}, \tag{43}$$

where $u^s = (u_1^s, \dots, u_n^s)$, $\Lambda^s(x,t)$ is a given real continuously differentiable diagonal matrix with elements $\lambda_i^s(x,t)(i = \overline{1,n})$ that satisfy condition (2), Φ^s is a given function of x, t, u^s, nonlinear in general with respect to u^s.

Consider the problem: find in G a piecewise continuous generalized solution of system (43) that satisfies conditions (3), (4), with the notion of generalized solution introduced in Section 4 using equations (16), (17).

The following theorem is true.

Theorem 4. *Assume that for all $i = \overline{1,n}$, $s = \overline{0,m}$ the following holds:*

1) *the coefficients $\lambda_i^s \in C^2(\bar{G}^s)$ and satisfy conditions* (2)*;*

2) *the functions $\Phi^s(x,t,u^s) \in C(G^s \times \mathbb{R}^n)$ and satisfy the Lipschitz condition locally with respect to u^s:*

$$\forall \varepsilon > 0,\ \forall U > 0,\ \exists L > 0 : \Big|\Phi^s(x,t,\bar{u}) - \Phi^s(x,t,\bar{\bar{u}})\Big| \leqslant L|\bar{u} - \bar{\bar{u}}|$$

for $a_s(t) \leqslant x \leqslant a_{s+1}(t)$, $t \in [0,\varepsilon]$, $|\bar{u}|$, $|\bar{\bar{u}}| \leqslant U$;

3) *the functions α_{is}^p, $h^p \in C(\mathbb{R}_+)$, $\beta_{is}^p \in C(\bar{G})$, $p = \overline{1,q}$;*

4) *the functions $h^p \in C^1(\mathbb{R}_+)$, $\beta_{is}^p \in C^1(\bar{G})$, $p = \overline{1+q,N}$;*

5) *conditions* (6)–(8) *are fulfilled.*

Then for some $\varepsilon > 0$ *the problem* (43), (3), (4) *has a unique piecewise continuous generalized solution in* $\bar{G}_\varepsilon$*.*

The proof of Theorem 4 is similar to the one of Theorem 1, so let us limit ourselves to some remarks only.

The transformation of boundary conditions is done similarly as in the proof of Theorem 1 and leads to an equation of the form (32) in which a nonlinear matrix operator of Volterra type $(Vu)(t)$ appears instead of $(Lu)(t) + (\tilde{L}f)(t)$, which satisfies the local Lipschitz condition. By putting $t = 0$, we find the value of (37) for $\nu(0)$ the same as in Theorem 1. Instead of (38), we obtain the equation in $\bar{G}_\varepsilon$

$$\begin{aligned} u(x,t) = &\Big[Q(I-(I-M)^{-1}K)^{-1}(I-M)^{-1}V + V_1\Big]u+ \\ &+\Big[Q(I-(I-M)^{-1}K)^{-1}(I-M)^{-1}H\Big](x,t), \end{aligned} \tag{44}$$

where the operator V has the same properties as L. Consider an arbitrary $U > |\nu(0)|$. Then we see from formula (44) that if ε_0 is sufficiently small, then for $0 < \varepsilon < \varepsilon_0$ the operator is defined by the right-hand side of this formula and maps the ball $S_\varepsilon = \{u : \|u\| \leqslant U\}$ into itself. According to condition 2) of Theorem 4, the function Φ in $\bar{G}_{\varepsilon_0} \times S_{\varepsilon_0}$ satisfies the Lipschitz condition. It follows from the forms of the operators V and V_1 that for any sufficiently small ε the operator defined by the right-hand side of (44) satisfies the Lipschitz condition with respect to u with some sufficiently small constant, that is, it is compressible. The application of Banach theorem on compressible mappings completes the proof of Theorem 4; note that according to the same Banach theorem, the desired solution can be obtained by iteration method.

In contrast to Theorem 3, the generalized solution of (43), (3), (4), in general cannot be extended to the whole domain $\bar{G}$. It is not difficult to show that there exists a maximal $T \in (0,\infty]$ for which the problem considered is solved in the domain $\bar{G} \cap \{(x,t) : 0 \leqslant t < T\}$. Moreover, if $T < \infty$, then

$$\sum_{s=0}^{m}\sum_{i=1}^{n} \max_{a_s(t)\leqslant x\leqslant a_{s+1}(t)} |u_i^s(x,t)| \xrightarrow[t\to T]{} \infty.$$

Therefore, if an a priori estimate can be derived for the desired solution on an arbitrary finite change interval of t, then $T = \infty$, that is, the solution can be continued to the entire domain $\bar{G}$. This will be the case, in particular, if the functions $\Phi_i^s(x,t,u)$ have no more than linear growth in u, that is, there exist continuous (possibly unbounded) functions $F_1(t)$, $F_2(t)$ $(0 \leqslant t < \infty)$ such that for all $s = \overline{0,m}$, $i = \overline{1,n}$, $(x,t) \in \bar{G}^s$, $u \in \mathbb{R}^n$ the equality

$$\Big|\Phi_i^s(x,t,u)\Big| \leqslant F_1(t)u + F_2(t)$$

is fulfilled.

Remark 1. *The issue of constructing global solutions of nonlinear hyperbolic problems is considered, for example, in [10, 11], but the sign-constancy and monotonicity of the initial data are significantly used therein.*

7. Comments

I. Considering the problems in Sections 2 – 6, we assumed the fulfillment of some conditions (conditions (6), (7)). All other conditions of smoothness and data agreement are natural, while the appearance of the mentioned conditions may seem artificial at first glance.

Let us show by simple examples that the conditions introduced by us for solving the considered problems are significant.

Example 1. *Let G be a sector in the plane xOt, bounded by rays l_1 and l_2, defined by equations $x = -kt$ and $x = kt$, $0 < k < 1$. In G consider the system*

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x} = 0, \quad \frac{\partial u_2}{\partial t} - \frac{\partial u_2}{\partial x} = 0, \quad t > 0, \ -kt < x < kt, \tag{45}$$

with boundary conditions

$$u_1(-kt,t) - 2u_2(-kt,t) = 0, \quad u_2(kt,t) - \frac{1}{2}u_1(kt,t) = 0, \quad 0 \leqslant t < \infty. \tag{46}$$

We obtain this problem from problem (1)–(4) *if $n = 2$, $m = 0$, $p_0 = q_0 = p_1 = q_1 = 1$, $N = q = 2$, and the coefficients and free terms are equal to the corresponding constants. All assumptions of Sections 2 – 4 are satisfied here except condition* (7). *Since*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -2 \\ -\frac{1}{2} & 0 \end{pmatrix},$$

the eigenvalues of the matrix $A^{-1}B$ are equal to ± 1, so none of its norms can be less than 1.

Let

$$u_1(-kt,t) = \mu_1(t), \quad u_2(kt,t) = \nu_2(t).$$

Applying the method of characteristics, that is, equation (17), *we obtain*

$$u_1(x,t) = \mu_1\Big(\frac{t-x}{1+k}\Big), \quad u_2(x,t) = \nu_2\Big(\frac{t+x}{1+k}\Big).$$

Substituting the found $u_1(x,t)$ *and* $u_2(x,t)$ *into the boundary conditions* (46), *we have*

$$\mu_1(t) = 2\nu_2\Big(\frac{1-k}{1+k}t\Big), \ \ \nu_2(t) = \frac{1}{2}\mu_1\Big(\frac{1-k}{1+k}t\Big).$$

Hence $\mu_1(t) = \mu_1(\lambda^2 t)$, $\lambda = \frac{1-k}{1+k}$, *and therefore* $\mu_1(t) = c$, $c = const$.

From the second equation we have $\nu_2(t) = \frac{1}{2}c$. *Thus, the problem has an infinite set of solutions:* $u_1(x,t) \equiv c$, $u_2(x,t) \equiv \frac{1}{2}c$.

Example 2. *Suppose we need to find a solution to the system* (45) *in the domain* G *of Example 1 that satisfies the conditions*

$$u_1(-kt,t) - u_2(kt,t) = h_1(t), \ \ \int\limits_{-kt}^{kt} \big(u_2(x,t) - u_1(x,t)\big)dx = h_2(t), \tag{47}$$

where $h_1(t)$ *and* $h_2(t)$ *are the given functions continuous in* $\mathbb{R}_+$, $h_2(t)$ *being continuously differentiable and* $h_2(0) = 0$. *Condition* (6) *is not fulfilled here.*

From equation (45) *we have*

$$u_1(x,t) = g_1(t-x), \ \ u_2(x,t) = g_2(t+x),$$

where g_1 *and* g_2 *are arbitrary continuous functions in* $\mathbb{R}_+$. *Requiring condition* (47) *to be fulfilled, we obtain the relations*

$$g_1\big((1+k)t\big) - g_2\big((1+k)t\big) = h_1(t), \ \ \int\limits_{-kt}^{kt} \big[-g_1(t-x) + g_2(t+x)\big]dx = h_2(t).$$

After changing the variable, the second condition becomes

$$\int\limits_{(1-k)t}^{(1+k)t} \big[g_2(y) - g_1(y)\big]dy = h_2(t).$$

By differentiating it, we have

$$(1+k)\big[g_2((1+k)t) - g_1((1+k)t)\big] - (1-k)\big[g_2((1-k)t) - g_1((1-k)t)\big] = h_2'(t).$$

Comparing it to the first condition, we see that this problem is solvable if and only if

$$(1+k)h_1(t) + (1-k)h_1\Big(\frac{1-k}{1+k}t\Big) + h_2'(t) \equiv 0.$$

If this condition is satisfied, then

$$g_1(t) = g_2(t) + h_1\Big(\frac{t}{1+k}\Big)$$

and the function $g_2(t)$ remains arbitrary.

This means that the problem either has no solution or has many solutions, since it contains an arbitrary continuous function of one variable.

II. Let the characteristics of the system (1) that exit from the intersection point of the boundary curves not fall into the domain of solution, and the additional conditions have the form

$$\int\limits_{a(t)}^{b(t)} \sum_{i=1}^{n} \beta_i^p(y,t)u_i(y,t)dy = h^p(t), \quad p = \overline{1,n}. \tag{48}$$

Problem (1), (48) can be obtained from problem (1)–(4) if $m = 0$, $p_0 = q_0$, $q = 0$, $a_0(t) = a(t)$, $a_1(t) = b(t)$.

Suppose that $a'(0) < b'(0)$ and

$$\det A_a^b(t) \neq 0, \quad \forall t \geqslant 0, \tag{49}$$

where

$$A_a^b(t) = \begin{Vmatrix} \beta_1^1(a(t),t) & \dots & \beta_k^1(a(t),t) & \beta_{k+1}^1(b(t),t) & \dots & \beta_n^1(b(t),t) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \beta_1^n(a(t),t) & \dots & \beta_k^n(a(t),t) & \beta_{k+1}^n(b(t),t) & \dots & \beta_n^n(b(t),t) \end{Vmatrix},$$

with the index value $i = \overline{1,k}$ ($\overline{k+1,n}$) corresponding to $\lambda_i(0,0) > b'(0)$ (respectively, $\lambda_i(0,0) < a'(0)$).

Let us show that in this case conditions (6) and (7) are fulfilled. Indeed, from the definitions of $A(t)$ and $B(0)$ it is not difficult to see, with the assumptions made, that

$$\begin{aligned} A(t) = A_b^a(t) diag\{&\lambda_1(a(t),t) - a'(t), \dots, \lambda_k(a(t),t) - a'(t), \\ &b'(t) - \lambda_{k+1}(b(t),t), \dots, b'(t) - \lambda_n(b(t),t)\}, \\ B(0) = A_a^b(0) diag\{&\lambda_1(0,0) - b'(0), \dots, \lambda_k(0,0) - b'(0), \\ &a'(0) - \lambda_{k+1}(0,0), \dots, a'(0) - \lambda_n(0,0)\}. \end{aligned}$$

Since by condition all diagonal elements are nonzero, the conditions (49) and (6) are equivalent. In addition,

$$A(0)^{-1}B(0) = diag\bigg\{\frac{\lambda_1(0,0) - b'(0)}{\lambda_1(0,0) - a'(0)}, \dots, \frac{\lambda_k(0,0) - b'(0)}{\lambda_k(0,0) - a'(0)},$$

$$\left.\frac{a'(0)-\lambda_{k+1}(0,0)}{b'(0)-\lambda_{k+1}(0,0)},\ldots,\frac{a'(0)-\lambda_n(0,0)}{b'(0)-\lambda_n(0,0)}\right\}.$$

Since all the diagonal elements here are defined on the interval $(0,1)$, the norm of the matrix $A(0)^{-1}B(0)$ is less than one. Thus, problem (1), (48) reduces to a system of Volterra integro-functional equations of the second kind solvable by the iteration method. In the considered case, conditions (7) follow from conditions (6).

III. Condition (6) is an analogue of the well-known Lopatinsky condition for classical boundary value problems for the elliptic type equations. For the class of considered boundary value problems, it is, in general, vital and so cannot be dropped.

IV. The scheme proposed in Sections 2 – 4 for investigating problem (1)–(4) without modifications can also be used for the case where the curves $\gamma_s(s=\overline{0,m+1})$ (or some of them) are characteristics of the system (1) or (43). Thus, the method described covers, in particular, some variant of the characteristic problem for the system (1) or (43) (the Goursat-Darboux problem).

V. If the conditions of Theorem 1 are fulfill and, in addition, the smoothness of all given functions is increased by 1, then it can be shown that the constructed solution u is continuously differentiable in each domain G^s everywhere except the characteristic $x=\varphi_i^s(t,0,0)$ $(i=\overline{q_s+1,p_s})$, along which, it has in general a discontinuity of the first kind for the derivatives. To avoid this discontinuity, that is, for the solution to be piecewise smooth, it is necessary and sufficient that $\sum\limits_{s=0}^{m}(p_s-q_s)$ additional first-order agreement conditions fulfil. These conditions are rather cumbersome and can be obtained as follows. We differentiate by t the relations (29) and (31) and, taking $t=0$ after that, we obtain a system of equations from which, since the values of $\mu_i^s(0)$ and $\nu_i^s(0)$ are already known, we can find all the values of $\mu_i^{s\prime}(0)$ and $\nu_i^{s\prime}(0)$. (It is not difficult to check that the corresponding determinant is nonzero; we have to use the formulas

$$\left.\frac{dt_i^s(a_{s+1}(t),t)}{dt}\right|_{t=0}=\begin{cases}\frac{\lambda_i^s(0,0)-a'_{s+1}(0)}{\lambda_i^s(0,0)-a'_s(0)}, & i=\overline{1,q_s},\\ \frac{a'_s(0)-\lambda_i^s(0,0)}{a'_{s+1}(0)-\lambda_i^s(0,0)}, & i=\overline{p_s+1,n}\end{cases}$$

to calculate it). On the other hand, for arbitrary $s=\overline{0,m}$, $i=\overline{q_s+1,p_s}$ from equation (1) we have

$$\left.\frac{du_i^s(\varphi_i^s(t,0,0),t)}{dt}\right|_{t=0}=\sum_{j=1}^{n}a_{ij}^s(0,0)\omega_i^s(0,0)+f_i^s(0,0)$$

(the value of ω_i^s is defined by formula (17), i.e., we can also find it). From the condition of no discontinuity on the characteristic $x = \varphi_i^s(t,0,0)$ we obtain the relation

$$\left.\frac{du_i^s(\varphi_i^s(t,0,0),t)}{dt}\right|_{t=0} = \frac{1}{a'_{i+1}(0) - a'_i(0)}\times$$
$$\times\left[(\lambda_i^s(0,0) - a'_i(0))\mu_i^{s\prime}(0) + (a'_{i+1}(0) - \lambda_i^s(0,0))\nu_i^s(0)\right].$$

If we replace here the expression on the left-hand side and also $\mu_i^{s\prime}(0)$ and $\nu_i^{s\prime}(0)$ by the values at zero of the given functions and their derivatives, we obtain the necessary agreement conditions.

VI. If $q_s = p_s$ ($s = \overline{0,m}$), i.e., from the origin the characteristics of the system (1) or (43) do not fall into G^s, then the number of boundary conditions (3), (4) equals the number of equations of the system (1) or (43) for each G^s, i.e., equal to $(m+1)n$. In this case, the agreement conditions (8) are not needed.

VII. For the generalized solution to be not only piecewise continuous but also continuous in $\bar{G}$, conditions (3) must contain the equality

$$u_i^{s-1}(a_s(t),t) = u_i^s(a_s(t),t),\ s = \overline{1,m},\ i = \overline{1,n}.$$

Other conditions like (3) and (4) should be $n + \sum\limits_{s=0}^{m}(p_s - q_s)$.

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