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Some Knopp's Core Type Theorems Via Ideals

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Abstract. In this paper, we characterize the matrix class $(\mathcal{I}_c \cap l_\infty, \mathcal{I}_c \cap l_\infty)_{reg}$, where $\mathcal{I}_c$ is the space of all ideal convergent sequences and l_∞ denotes the space of all bounded sequences. We use this class to establish some core theorems analogous to Knopp's core theorem.

Key Words and Phrases: $I-$convergence, I_c-convergence, matrix transformations, Knopp's core, $I-$core, Knopp's core theorem.

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1. Introduction

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$ denote the set of all complex, real and natural numbers, respectively, and $\mathcal{T} = (t_{nk})_{n,k=1}^{\infty}$ be an infinite matrix of complex entries t_{nk}. By $\mathcal{T}\eta = (\mathcal{T}_n(\eta))$ we denote the $\mathcal{T}$-transform of the sequence $\eta = (\eta_k)_{k=1}^{\infty}$, where $\mathcal{T}_n(\eta) = \sum_k t_{nk}\eta_k$, provided that the series on the right-hand side converges for each $n \in \mathbb{N}$. For any two sequence spaces $\mathcal{X}$ and $\mathcal{Y}$, we write $(\mathcal{X}, \mathcal{Y})$ for a class of matrices $\mathcal{T}$ such that $\mathcal{T}\eta \in \mathcal{Y}$ for $\eta \in \mathcal{X}$. If in addition $\lim \mathcal{T}\eta = \lim \eta$, then we denote such a class by $(\mathcal{X}, \mathcal{Y})_{reg}$. Let l_∞ and c denote the spaces of all bounded and convergent sequences, respectively. The matrix $\mathcal{T}$ is said to be regular, i.e. $\mathcal{T} \in (c, c)_{reg}$ if $\mathcal{T}\eta \in c$ for $\eta \in c$ with $\lim \mathcal{T}\eta = \lim \eta$. The necessary and sufficient conditions (cf. Cook [5]) for $\mathcal{T}$ to be regular are:

Lemma 1. $\mathcal{T} \in (c, c)_{reg}$ *if and only if the following conditions hold:*

(i) $||\mathcal{T}|| = \sup_n \sum_k |t_{nk}| < \infty$;

(ii) $\lim_n t_{nk} = 0$*, for each* k*;*

(iii) $\lim_n \sum_k t_{nk} = 1$.

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The Knopp core or $\mathcal{K}-$core of a real bounded sequence $\eta = (\eta_k)_{k=1}^{\infty}$ is defined to be the closed interval $[\ell(\eta), \mathcal{L}(\eta)]$, where $\ell(\eta) = \liminf \eta$; $\mathcal{L}(\eta) = \limsup \eta$. The well-known Knopp's core theorem states that (cf. Knopp [21], Maddox [28]): In order that $\mathcal{L}(\mathcal{T}\eta) \leq \mathcal{L}(\eta)$ for every real bounded sequence η, it is necessary and sufficient that $\mathcal{T}$ should be regular and $\lim_n \sum_k \mid t_{nk} \mid = 1$. Note that $\mathcal{L}(\mathcal{T}\eta) \leq \mathcal{L}(\eta)$ means $\mathcal{K} - core\{\mathcal{T}\eta\} \subseteq \mathcal{K} - core\{\eta\}$. Shcherbakov [36] has shown that for every bounded complex sequence η,

$$\mathcal{K} - core\{\eta\} = \bigcap_{z \in \mathbb{C}} K_\eta(z),$$

where

$$K_\eta(z) := \{w \in \mathbb{C} : \mid w - z \mid \leq \limsup_k \mid \eta_k - z \mid\} .$$

The concept of $\mathcal{K}-$core has been extended to the statistical core [18] and $\mathcal{I}-$core [8] for a complex number sequence η.

Let $S \neq \varnothing$. Recall that a non-empty class $\mathcal{I} \subseteq 2^{\mathcal{S}}$ of subsets of $\mathcal{S}$ is called ideal if (i) $\emptyset \in \mathcal{I}$, (ii) $\mathcal{D}_1 \cup \mathcal{D}_2 \in \mathcal{I}$ for $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{I}$, (iii) $\mathcal{D}_1 \in \mathfrak{I}$, $\mathcal{D}_2 \subseteq \mathcal{D}_1 \Longrightarrow \mathcal{D}_2 \in \mathcal{I}$. An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \varnothing$, $\mathcal{S} \notin \mathcal{I}$, and is called admissible if $\{s\} \in \mathcal{I}$, for each $s \in \mathcal{S}$. A non-empty class $\mathcal{F} \subseteq 2^{\mathcal{S}}$ of subsets of $\mathcal{S}$ is called Filter if (i) $\emptyset \notin \mathcal{F}$, (ii) $\mathcal{D}_1 \cap \mathcal{D}_2 \in \mathcal{F}$ for $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{F}$, (iii) $\mathcal{D}_1 \in \mathcal{F}$, $\mathcal{D}_2 \supseteq \mathcal{D}_1 \Longrightarrow \mathcal{D}_2 \in \mathcal{F}$. Let $\mathcal{I}$ be a non-trivial ideal in $\mathcal{S}$. Then the filter $\mathcal{F}(\mathcal{I}) = \{\mathcal{M} = \mathcal{S} \setminus \mathcal{U} : \mathcal{U} \in \mathcal{I}\}$ is called the filter associated with the ideal $\mathcal{I}$. The concepts of $\mathcal{I}$ and $\mathcal{I}^*-$convergence have been introduced and studied by Kostyrko et al. [23]. Throughout the paper, $\mathcal{I}$ will be a non-trivial admissible ideal in $\mathbb{N}$.

2. Preliminaries

Definition 1. *A real sequence $\eta = (\eta_k)$ is said to be $\mathcal{I}-$convergent to $\xi \in \mathbb{R}$ if $\{k : |\eta_k - \xi| \geq \epsilon,$ for every $\epsilon > 0\} \in \mathcal{I}$, and we write $\mathcal{I} - \lim_k \eta_k = \xi$. We denote the set of all $\mathcal{I}-$convergent sequences by $\mathcal{I}_c$.*

Definition 2. *A real sequence $\eta = (\eta_k)$ is said to be $\mathcal{I}^*-$convergent to $\xi \in \mathbb{R}$ if there is a set $\mathcal{M} = \mathbb{N} \setminus \mathcal{U} = \{m_i\}_{i=1}^{\infty} \in \mathcal{F}(\mathcal{I})$ such that $\lim_i \eta_{m_i} = \xi$. In this case, we write $\mathcal{I}^* - \lim \eta_k = \xi$ and we denote the set of all $\mathcal{I}^*-$convergent sequences by $\mathcal{I}_c^*$.*

Remark 1. *(a) $c \subseteq \mathcal{I}_c$.*

(b) $\mathcal{I}_c^* \subseteq \mathcal{I}_c$, and equality hold if and only if $\mathcal{I}$ satisfy $(\mathcal{AP})$ condition [23], i.e. if for every sequence $(\mathcal{A}_n)$ of pairwise disjoint sets from $\mathcal{I}$ there are sets $\mathcal{B}_n \subset \mathbb{N}$, $n \in \mathbb{N}$ such that the symmetric difference $\mathcal{A}_n \Delta \mathcal{B}_n$ is finite for every n and $\cup_n \mathcal{B}_n \in \mathcal{I}$.

(c) *If* $\mathcal{I} = \mathcal{I}_\delta = \left\{ \mathcal{U} \subseteq \mathbb{N} : \delta(\mathcal{U}) = \lim_n \frac{|\{k \leq n : k \in U\}|}{n} = 0 \right\}$, *where* $|.|$ *denotes the cardinality of the enclosed set, then* $\mathcal{I}-$*convergence coincide with the statistical convergence due to [13], and we denote the set of all statistically convergent sequences by st.*

(d) *If* $\eta \in \mathcal{I}_c$, *then* η *need not be bounded. For example, let* $\mathcal{U}$ *be any infinite set such that* $\mathcal{U} \in \mathcal{I}$ *and let* $\eta = (\eta_k)$ *be defined as*

$$\eta_k = \begin{cases} k & , \text{ if } \ k \in U, \\ 0 & , \text{otherwise} . \end{cases}$$

Then η *is* $\mathcal{I}$ *convergent to zero but not bounded.*

In [23], [8] and [24], the concepts of statistical bounded, statistical cluster point and statistical limit superior and inferior [17] have been extended to $\mathcal{I}-$bounded, $\mathcal{I}-$cluster point and $\mathcal{I}-$limit superior and inferior of a real sequence $\eta = (\eta_k)$ and some related properties have been proved.

Definition 3. *A sequence* $\eta = (\eta_k)$ *is said to be* $\mathcal{I}-$*bounded if there is a number* $t > 0$ *such that* $\{k : |\eta_k| > t\} \in \mathcal{I}$.

Definition 4. *A number* ξ *is said to be* $\mathcal{I}-$*cluster point of a sequence* $\eta = (\eta_k)$ *if the set* $\{k : |\eta_k - \xi| < \epsilon\} \notin \mathcal{I}$ *for each* $\epsilon > 0$ *and we denote the set of all* $\mathcal{I}-$*cluster points by* $\mathcal{I}(\Gamma_\eta)$.

Definition 5. *The concept of* $\mathcal{I}-$*limit superior and inferior of a real sequence* $\eta = (\eta_k)$ *is defined as*

$$\mathcal{I} - \limsup \eta = \begin{cases} \sup \mathcal{B}_\eta & , \textit{ if } \mathcal{B}_\eta \neq \varnothing, \\ -\infty & , \textit{ if } \mathcal{B}_\eta = \varnothing, \end{cases}$$

$$\mathcal{I} - \liminf \eta = \begin{cases} \inf \mathcal{C}_\eta & , \textit{ if } \mathcal{C}_\eta \neq \varnothing, \\ \infty & , \textit{ if } \mathcal{C}_\eta = \varnothing, \end{cases}$$

where

$$\mathcal{B}_\eta = \{g \in \mathbb{R} : \{k : \eta_k > g\} \notin \mathcal{I}\} \quad \textit{and } \mathcal{C}_\eta = \{g \in \mathbb{R} : \{k : \eta_k < g\} \notin \mathcal{I}\} .$$

Remark 2. (a) *If* $\mathcal{I} = \mathcal{I}_\delta$, *then we have statistical bounded, statistical cluster point and statistical limit superior and inferior.*

(b) *If* $\eta \in l_\infty$ *or* $\eta \in \mathcal{I}_c$, *then* η *is* $\mathcal{I}-$*bounded.*

(c) *If* η *is* $\mathcal{I}-$ *bounded, then* $\mathcal{I}-$*limit superior and inferior are finite.*

Lemma 2. *(i) (see [8]) For every real sequence $\eta = (\eta_k)$*

$$\liminf \eta \leq \mathcal{I} - \liminf \eta \leq \mathcal{I} - \limsup \eta \leq \limsup \eta.$$

(ii) (see [8]) The $\mathcal{I}-$bounded sequence η is $\mathcal{I}-$convergent if and only if $\mathcal{I} - \liminf \eta = \mathcal{I} - \limsup \eta$.

(iii) (see [24]) Let $\eta \in l_\infty$. Then

$$\mathcal{I} - \limsup \eta = \max \mathcal{I}(\Gamma_\eta), \quad \mathcal{I} - \liminf \eta = \min \mathcal{I}(\Gamma_\eta).$$

Remark 3. *From (ii) and (iii) of Lemma 2, we can say that: If $\eta \in l_\infty$, then η is $\mathcal{I}-$convergent if and only if $\mathcal{I}(\Gamma_\eta) = \{\xi\}$.*

Demirci [8] defined $\mathcal{I}-$core of a complex sequence η as follows.

Definition 6. *Let η be an $\mathcal{I}-$ bounded sequence and let for each $z \in \mathbb{C}$*

$$B_\eta(z) = \left\{ w \in \mathbb{C} : |w - z| \leq \mathcal{I} - \limsup_k |\eta_k - z| \right\}.$$

Then

$$\mathcal{I}\text{-}core\{\eta\} = \bigcap_{z \in \mathbb{C}} B_\eta(z).$$

Remark 4. *For any $\mathcal{I}-$bounded real sequence η, we have*

(a) $\mathcal{I}-core\{\eta\} = [\mathcal{I} - \liminf \eta, \mathcal{I} - \limsup \eta]$.

(b) From Lemma 2 (i) and the definition of $\mathcal{K}-$core, we have $\mathcal{I} - core\{\eta\} \subseteq \mathcal{K} - core\{\eta\}$.

(c) From Lemma 2 (iii), we have $\mathcal{I}(\Gamma_\eta) \subseteq \mathcal{I} - core\{\eta\}$.

More generalizations and applications of statistical convergence and recent works on ideal convergence can be found in ([1], [2], [3], [4], [6], [7], [10] [14], [15], [16], [31], [34]) and ([9] , [11], [12], [19], [20], [26], [29], [32], [33], [35]).

Analogous to the Knopp core theorem, the sufficient conditions for

$$\mathcal{K} - core\{\mathcal{T}\eta\} \subseteq \mathcal{I} - core\{\eta\}$$

were obtained in [8] for every bounded complex sequence η; and the necessary and sufficient conditions were given in [25].

In [27] and [30], the necessary and sufficient conditions have been obtained for $\mathcal{T}$ to yield

$$st-core\{\mathcal{T}\eta\} \subseteq \mathcal{K}-core\{\eta\},$$

and moreover

$$st-core\{\mathcal{T}\eta\} \subseteq st-core\{\eta\}.$$

We generalize these results to establish necessary and sufficient conditions to prove some core theorems.

3. Some matrix classes involving the space $\mathcal{I}_c$ and core theorems

We state the following results with slight modifications of some matrix transformation involving the space $\mathcal{I}_c$ due to Kolk [22].

Lemma 3. *Let $\mathcal{I}$ be an admissible ideal satisfying $(\mathcal{AP})$ condition. Then $\mathcal{T} \in (\mathcal{I}_c \cap l_\infty, c)_{reg}$ if and only if*

(i) $\mathcal{T} \in (c,c)_{reg}$;

(ii) $\lim_n \sum_{k\in\mathcal{U}} |t_{nk}| = 0$*, for every* $\mathcal{U} \in \mathcal{I}$.

Lemma 4. *Let $\mathcal{I}$ be an admissible ideal satisfying $(\mathcal{AP})$ condition. Then $\mathcal{T} \in (c, \mathcal{I}_c \cap l_\infty)_{reg}$ if and only if*

(i) $\|\mathcal{T}\| < \infty$;

there exists $\mathcal{N} = \{n_i\}$ such that $\mathcal{N} \in \mathcal{F}(\mathcal{I})$ and

(ii) $\mathcal{I} - \lim\limits_n t_{nk} = \lim\limits_i t_{n_ik} = 0$, $(k \in \mathbb{N})$;

(iii) $\mathcal{I} - \lim_n \sum_k t_{nk} = \lim_i \sum_k t_{n_ik} = 1$.

We need the following lemma which is an $\mathcal{I}-$analogue of the results of Simons [37] (Corollary 12, Theorem 11).

Lemma 5. *Let $\mathcal{I}$ be an admissible ideal satisfying $(\mathcal{AP})$ condition. If $\|\mathcal{T}\| < \infty$ and there exists $\mathcal{N} = \{n_i\}$ such that $\mathcal{N} \in \mathcal{F}(\mathcal{I})$ and $\mathcal{I} - \limsup_n t_{nk} = \limsup_i t_{n_ik} = 0$, then there exists $y \in l_\infty$ such that $\|y\| \leq 1$ and*

$$\limsup_i \sum_k t_{n_ik} y_k = \limsup_i \sum_k |t_{n_ik}|,$$

i.e.

$$\mathcal{I} - \limsup_n \sum_k t_{nk} y_k = \mathcal{I} - \limsup_n \sum_k |t_{nk}|.$$

To prove our theorems we need first to characterize the matrix class $(\mathcal{I}_c \cap l_\infty, \mathcal{I}_c \cap l_\infty)_{reg}$.

Theorem 1. *Let $\mathcal{I}$ be an admissible ideal satisfying $(\mathcal{AP})$ condition. Then $\mathcal{T} \in (\mathcal{I}_c \cap l_\infty, \mathcal{I}_c \cap l_\infty)_{reg}$ if and only if*

(1.1) $\mathcal{T} \in (c, \mathcal{I}_c \cap l_\infty)_{reg}$;

(1.2) there exists $\mathcal{N} = \{n_i\}$ such that $\mathcal{N} \in \mathcal{F}(\mathcal{I})$ and $\lim_i \sum_{k\in\mathcal{U}} |t_{n_i k}| = \mathcal{I}-\lim_n \sum_{k\in\mathcal{U}} |t_{nk}| = 0$, for every $\mathcal{U} \in \mathcal{I}$.

Proof. Let $\mathcal{T} \in (\mathcal{I}_c \cap l_\infty, \mathcal{I}_c \cap l_\infty)_{reg}$ and $\mathcal{I} - \lim \eta = \mathcal{I} - \lim \mathcal{T}\eta = \xi$, say. Since $c \subset \mathcal{I}_c$, we have $\mathcal{T} \in (c, \mathcal{I}_c \cap l_\infty)_{reg}$.

Let $\mathcal{U} \subseteq \mathbb{N}$ be such that $\varnothing \neq \mathcal{U} \in \mathcal{I}$ and let $\eta \in l_\infty$ be defined by

$$\eta_k = \begin{cases} 1 & \text{, if } \; k \in \mathcal{U}, \\ 0 & \text{, otherwise.} \end{cases}$$

Then $\mathcal{I} - \lim \eta = 0$ and so $\mathcal{I} - \lim \mathcal{T}\eta = 0$. Hence $\mathcal{T}$ satisfies the conditions in Lemma 5 and so we have $\mathcal{I}- \lim_n \sum_{k\in\mathcal{U}} |t_{nk}| = 0$, whenever $\mathcal{U} \in \mathcal{I}$.

Conversely. Suppose that (1.1) and (1.2) hold and $\eta \in \mathcal{I}_c \cap l_\infty$ with $\mathcal{I} - \lim \eta = \xi$. Let $\mathcal{U} = \{k : |\eta_k - \xi| \geq \varepsilon\} \in \mathcal{I}$ for $\varepsilon > 0$. We have

$$\mathcal{I} - \lim \mathcal{T}\eta = \mathcal{I} - \lim \left(\sum_k t_{nk} (\eta_k - \xi) + \xi \sum_k t_{nk} \right).$$

Using Lemma 4, we have

$$\mathcal{I} - \lim \mathcal{T}\eta = \mathcal{I} - \lim_n \sum_k t_{nk} (\eta_k - \xi) + \xi. \tag{1}$$

Since

$$\left| \sum_k t_{nk} (\eta_k - \xi) \right| = \left| \sum_{k\in\mathcal{U}} t_{nk} (\eta_k - \xi) + \sum_{k\notin\mathcal{U}} t_{nk} (\eta_k - \xi) \right|$$
$$\leq \|\eta_k - \xi\| \sum_{k\in\mathcal{U}} |t_{nk}| + \epsilon \|\mathcal{T}\|.$$

Since $\|\mathcal{T}\| < \infty$ by condition (1.1), applying condition (1.2), we have

$$\mathcal{I} - \lim_n \sum_k t_{nk} (\eta_k - \xi) = 0.$$

Hence (1) implies that

$$\mathcal{I} - \lim \mathcal{T}\eta = \xi = \mathcal{I} - \lim \eta,$$

i.e. $\mathcal{T} \in (\mathcal{I}_c \cap l_\infty, \mathcal{I}_c \cap l_\infty)_{reg}$, which completes the proof. ◀

Theorem 2. *Let $\mathcal{I}$ be an admissible ideal that satisfies $(\mathcal{AP})$ condition. If $\|\mathcal{T}\| < \infty$, then for every $\eta \in l_\infty$*

$$\mathcal{I} - core\,\{\mathcal{T}\eta\} \subseteq \mathcal{K} - core\,\{\eta\} \tag{2}$$

if and only if the following conditions hold:

(2.1) $\mathcal{T} \in (c, \mathcal{I}_c \cap l_\infty)_{reg}$;

(2.2) $\mathcal{I} - \lim_n \sum_{k\in\mathcal{D}} |t_{nk}| = 1$, *whenever* $\mathcal{D} \in \mathcal{F}(\mathcal{I})$ *and* $\mathbb{N}\backslash\mathcal{D}$ *is finite.*

Proof. Suppose that (2) holds and $\eta \in c$ is such that $\lim_k \eta_k = \xi$. Then

$$\mathcal{I} - core\{\mathcal{T}\eta\} \subseteq \mathcal{K} - core\,\{\eta\} = \{\xi\}\,.$$

Since $\mathcal{T}\eta \in l_\infty$ for $\eta \in l_\infty$, from Lemma 2 (iii), we have $\mathcal{T}\eta$ has at least one $\mathcal{I}-$cluster point. From Remark 4 (c), we have

$$\varnothing \neq \mathcal{I}\,(\Gamma_{\mathcal{T}\eta}) \subseteq \mathcal{I} - core\{\mathcal{T}\eta\} \subseteq \mathcal{K} - core\{\eta\} = \{\xi\}.$$

Hence from Remark 3, $\mathcal{T}\eta$ is $\mathcal{I}-$convergent to ξ, i.e. $\mathcal{T} \in (c, \mathcal{I}_c \cap l_\infty)_{reg}$.
Let $\mathcal{D} \in \mathcal{F}(\mathcal{I})$ be such that $\mathbb{N}\backslash\mathcal{D}$ is finite and define the sequence $\eta = (\eta_k)$ by

$$\eta_k = \begin{cases} 1 & \text{, if } \ k \in \mathcal{D}, \\ 0 & \text{, otherwise.} \end{cases}$$

Then $\lim \eta_k = 1$ and hence we have

$$\varnothing \neq \mathcal{I} - core\{\mathcal{T}\eta\} \subseteq \mathcal{K} - core\{\eta\} = \{1\}.$$

Therefore, 1 is the only cluster point of $\mathcal{T}\eta$. Hence by Remark 3, we have $\mathcal{I} - \lim \mathcal{T}\eta = 1$. Since $\mathcal{I}$ is admissible ideal satisfying $(\mathcal{AP})$ condition, by Lemma 5, we have

$$\mathcal{I} - \lim_n \sum_{k\in\mathcal{D}} |t_{nk}| = 1, \text{ whenever } \mathcal{D} \in \mathcal{F}(\mathcal{I}) \text{ and } \mathbb{N}\backslash\mathcal{D} \text{ is finite.}$$

Conversely, let conditions (2.1) and (2.2) hold and $w \in \mathcal{I} - core\{\mathcal{T}\eta\}$. Then for any $z \in \mathbb{C}$, we have

$$| \, w - z \, | \leq \mathcal{I} - \limsup_n \, | \, z - \mathcal{T}_n(\eta) \, |$$

$$= \mathcal{I} - \limsup_n \, | \, z - \sum_k t_{nk}\eta_k \, |$$

$$\leq \mathcal{I} - \limsup_n \mid \sum_k t_{nk}(z - \eta_k) \mid + \mathcal{I} - \limsup_n \mid z \mid\mid 1 - \sum_k t_{nk} \mid .$$

Using condition (2.2), we have

$$\mid w - z \mid \leq \mathcal{I} - \limsup_n \left| \sum_k t_{nk}(z - \eta_k) \right| . \tag{3}$$

Let $\alpha = \limsup_k \mid z - \eta_k \mid$ and $\mathcal{U} = \{k : \mid z - \eta_k \mid > \alpha + \varepsilon\}$ for $\varepsilon > 0$. Then $\mathcal{U}$ is a finite set and hence $\mathcal{U} \in \mathcal{I}$, so we have

$$\mid \sum_k t_{nk}(z - \eta_k) \mid \leq \sup_k \mid z - \eta_k \mid \sum_{k \in \mathcal{U}} \mid t_{nk} \mid + (\alpha + \varepsilon) \sum_{k \notin \mathcal{U}} \mid t_{nk} \mid .$$

Therefore, by conditions (2.1), (2.2) and Remark 5 (a), we obtain

$$\mathcal{I} - \limsup_n \mid \sum_k t_{nk}(z - \eta_k) \mid \leq \alpha + \varepsilon.$$

Hence (3) implies that

$$\mid w - z \mid \leq \alpha + \varepsilon,$$

and since ε is arbitrary,

$$\mid w - z \mid \leq \alpha = \limsup_k \mid z - \eta_k \mid,$$

i.e. $w \in K_\eta(z)$. Hence $w \in \mathcal{K} - core\{\eta\}$, and so

$$\mathcal{I} - core\{\mathcal{T}\eta\} \subseteq \mathcal{K} - core\{\eta\}.$$

This completes the proof of the theorem. ◄

Remark 5. *(a) If $\mathcal{I} - \lim_n \sum_{k \in \mathcal{D}} |t_{nk}| = 1$ whenever $\mathbb{N} \backslash \mathcal{D}$ is finite, then*

$$\mathcal{I} - \lim_n \sum_{k \in \mathcal{U}} |t_{nk}| = 0, \text{ for any finite set } \mathcal{U}.$$

(b) We can not replace condition (2.2) by

$$\mathcal{I} - \lim_n \sum_{k \in \mathcal{D}} |t_{nk}| = 1, \text{ whenever } \mathcal{D} \in \mathcal{F}(\mathcal{I}). \tag{4}$$

Consider the following example.

Example 1. *Let $\mathcal{I}$ be an admissible ideal that satisfies $(\mathcal{AP})$ condition and let $\mathcal{H} = \{h_i\}_{i=1}^{\infty}$ be any infinite set in $\mathcal{I}$. Define $\mathcal{T} = (t_{nk})$ as*

$$t_{nk} = \begin{cases} 1 & , n \notin \mathcal{H},\ k = \min\{h_i\} > n, \\ 0 & , \textit{otherwise}. \end{cases}$$

Then

$$\sum_k t_{nk} = \begin{cases} 1 & , n \notin \mathcal{H}, \\ 0 & , \textit{otherwise}. \end{cases}$$

It is easy to see that $\mathcal{T}$ is not regular but $\mathcal{T} \in (c, \mathcal{I}_c \cap \ell_\infty)_{reg}$. Further, for any set $\mathcal{D} \in \mathcal{F}(\mathcal{I})$ such that $\mathbb{N} \backslash \mathcal{D}$ is finite we have

$$\mathcal{I} - \lim_n \sum_{k \in \mathcal{D}} |t_{nk}| = 1.$$

So for any $\eta \in l_\infty$, we have

$$\mathcal{I} - core\{\mathcal{T}\eta\} \subseteq \mathcal{K} - core\{\eta\}.$$

Now, let $\mathcal{M} = \mathbb{N} \backslash \mathcal{H}$. Then $\mathcal{M} \in \mathcal{F}(\mathcal{I})$ and we have

$$\sum_{k \in \mathcal{M}} t_{nk} = 0,\ \forall\ n.$$

Hence

$$\mathcal{I} - \lim_n \sum_{k \in \mathcal{M}} |t_{nk}| = 0.$$

Further, for any $\eta \in l_\infty$, e.g. $\eta = (1, 1, \cdots)$, we have $\mathcal{K} - core\{\eta\} = \{1\}$ and

$$\sum_k t_{nk}\eta_k = \begin{cases} 1 & , n \notin \mathcal{H}, \\ 0 & , \textit{otherwise}. \end{cases}$$

Hence we have $\mathcal{I} - core\{\mathcal{T}\eta\} = \mathcal{K} - core\{\eta\} = \{1\}$.
Therefore, we see that (2.2) holds but (4) does not hold.

Theorem 3. *Let $\mathcal{I}$ be an admissible ideal that satisfies $(\mathcal{AP})$ condition. If $\|\mathcal{T}\| < \infty$, then for every $\eta \in l_\infty$*

$$\mathcal{I} - core\{\mathcal{T}\eta\} \subseteq \mathcal{I} - core\{\eta\} \tag{5}$$

if and only if the following conditions hold:

(3.1) $\mathcal{T} \in (\mathcal{I}_c \cap \ell_\infty, \mathcal{I}_c \cap \ell_\infty)_{reg}$;

(3.2) $\mathcal{I}-\lim_n \sum_{k\in\mathcal{D}} |t_{nk}| = 1,$ *whenever* $\mathcal{D}\in\mathcal{F}(\mathcal{I})$.

Proof. Suppose that (5) holds and let $\eta\in\mathcal{I}_c\cap\ell_\infty$ be such that $\mathcal{I}-\lim\eta=\xi$. Then

$$I-core\,\{\mathcal{T}\eta\}\subseteq\mathcal{I}-core\,\{\eta\}=\{\xi\}.$$

Since $\|\mathcal{T}\|<\infty$ implies $\mathcal{T}\eta\in\ell_\infty$ for $\eta\in\ell_\infty$, from Remark 4 (a), we have $\mathcal{I}-core\,\{\mathcal{T}\eta\}\neq\varnothing$. Hence $\mathcal{I}-core\,\{\mathcal{T}\eta\}=\{\xi\}$, i.e. $\mathcal{T}\in(\mathcal{I}_c\cap\ell_\infty,\mathcal{I}_c\cap\ell_\infty)_{reg}$.

Let $\mathcal{D}\subseteq\mathbb{N}$ be such that $\mathcal{D}\in\mathcal{F}(\mathcal{I})$ and let $\chi_{\mathcal{D}}$ be the characteristic function of $\mathcal{D}$ defined as

$$\chi_{\mathcal{D}}(d)=\begin{cases}1 & ,\ \text{if } d\in\mathcal{D}\\ 0 & ,\ \text{otherwise.}\end{cases}$$

Then

$$\mathcal{I}-core\{\chi_{\mathcal{D}}\}=\{1\}.$$

Since $\|\mathcal{T}\|<\infty$ implies $\mathcal{T}\chi_{\mathcal{D}}\in\ell_\infty$ for $\chi_{\mathcal{D}}\in\ell_\infty$, from Lemma 2 (iii) it follows that $\mathcal{T}\chi_{\mathcal{D}}$ has at least one $\mathcal{I}-$cluster point. Therefore, $\mathcal{I}-core\{\mathcal{T}\chi_{\mathcal{D}}\}\neq\emptyset$. Also $\mathcal{I}-core\{\mathcal{T}\chi_{\mathcal{D}}\}=\{1\}$, since $\mathcal{I}-core\{\mathcal{T}\chi_{\mathcal{D}}\}\subseteq\mathcal{I}-core\{\chi_{\mathcal{D}}\}=\{1\}$, hence $\mathcal{I}-\lim\mathcal{T}\chi_{\mathcal{D}}=1$. Using Lemma 5, we have

$$\mathcal{I}-\lim_n\sum_{k\in\mathcal{D}}|t_{nk}|=1,\text{whenever } \mathcal{D}\in\mathcal{F}(\mathcal{I}).$$

Conversely. Let $w\in\mathcal{I}-core\{\mathcal{T}\eta\}$. We proceed along the same lines as in Theorem 2. Then we arrive at

$$|\ w-z\ |\leq\alpha,\text{where } \alpha=\mathcal{I}-\limsup_k\ |\ z-\eta_k\ |,\text{for any } z\in\mathbb{C},$$

by using conditions (3.1), (3.2) and Remark 6. So $w\in B_\eta(z)$. Hence $w\in\mathcal{I}-core\{\eta\}$, i.e.

$$\mathcal{I}-core\{\mathcal{T}\eta\}\subseteq I-core\{\eta\}.$$

This completes the proof. ◄

Remark 6. *If* $\mathcal{I}-\lim_n\sum_{k\in\mathcal{D}}|t_{nk}|=1$ *for any* $\mathcal{D}\in\mathcal{F}(\mathcal{I})$, *then*

$$\mathcal{I}-\limsup_n\sum_{k\in\mathcal{U}}|t_{nk}|=0,\ \textit{whenever}\ \mathcal{U}\in I\ .$$

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