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On the Inverse Spectral Problem for the One-dimensional Stark Operator on the Semiaxis

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Abstract. We consider the Stark operator $T=-\frac{d^2}{dx^2}+x+q\left(x\right)$ on the semiaxis $0\leq x<\infty$ with the Dirichlet boundary condition at the origin. The asymptotic behavior of the eigenvalues of this operator is studied. By the method of transformation operators, we study the spectral problem. We give a rigorous derivation of the main integral equation for the inverse problem.

Key Words and Phrases: Stark operator, Dirichlet boundary condition, Airy function, inverse spectral problem, main integral equation.

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1. Introduction

Stark operator is of significant interest for the spectral theory. Many authors have studied various aspects of the direct and inverse spectral problems for this operator (see [2, 3, 4, 5, 6, 7, 10, 11, 12] and the references therein).

Consider a boundary value problem generated on the semiaxis $0\leq x<\infty$ by a differential equation

$$-y''+xy+q\left(x\right)y=\lambda y,\quad \lambda\in C, \tag{1}$$

with a boundary condition

$$y\left(0\right)=0 \tag{2}$$

in the case where the function $q\left(x\right)$ is real and satisfies the conditions

$$q\left(x\right)\in C^{(1)}\left[0,\infty\right),\sigma_j\left(0\right)=\int_0^\infty x^j\left|q\left(x\right)\right|dx<\infty,\, j\leq 5, \tag{3}$$

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with $\sigma_j(x) = \int_x^{\infty} t^j |q(t)|\, dt$, $x \geq 0$, which are assumed to hold throughout the work.

In [12], the inverse scattering problem for the boundary value problem (1)–(2) was studied, where, in addition to (3), the potential $q(x)$ is subject to the additional condition

$$q(x) = o(x), x \to \infty. \tag{4}$$

Moreover, in [12], when solving the inverse problem, a formal derivation of the Gelfand-Levitan-Marchenko type main integral equation was given. In this regard, the question arises of a rigorous substantiation for a derivation of main integral equation.

In the present paper, by the method of transformation operators, we study the spectral problem for the boundary value problem (1), (2) in the class of potentials (3). We obtain a Marchenko-type integral equation for the problem (1)–(2). A rigorous derivation of this equation is given.

2. Preliminary information and asymptotic behavior of eigenvalues

We consider the unperturbed equation

$$-y'' + xy = \lambda y,\ 0 \leq x < \infty,\quad \lambda \in C. \tag{5}$$

It is known [1] that (5) has a solution $f_0(x,\lambda)$ of the form $f_0(x,\lambda) = Ai(x-\lambda)$, where $Ai(z)$ is the Airy function of the first kind. It is clear that the spectrum of the problem (5), (2) is discrete and consists of the roots of the function $f_0(0,\lambda) = Ai(-\lambda)$. The function $Ai(-\lambda)$ has roots $\hat{\lambda}_n, n = 1, 2, ...$, only on the positive semiaxis [1] and the following asymptotic equality is true:

$$\hat{\lambda}_n = g\left(\frac{3\pi(4n-1)}{8}\right), \tag{6}$$

where

$$g(z) \sim z^{\frac{2}{3}}\left(1 + \frac{5}{48}z^{-2} - \frac{5}{36}z^{-4} + \frac{77125}{82944}z^{-6} - \frac{108056875}{6967296}z^{-8} + ...\right), z \to \infty.$$

From the well-known properties of Airy functions it follows that

$$\hat{\alpha}_n \overset{def}{=} \left(\int_0^{\infty} \left|f_0\left(x,\hat{\lambda}_n\right)\right|^2 dx\right)^{\frac{1}{2}} = Ai'\left(-\hat{\lambda}_n\right) = (-1)^{n-1} g_1\left(\frac{3\pi(4n-1)}{8}\right),$$

where

$$g_1(z) \sim \pi^{-\frac{1}{2}} z^{\frac{1}{6}} \left(1 + \frac{5}{48} z^{-2} - \frac{1525}{4608} z^{-4} + \frac{2397875}{663552} z^{-6} ...\right), z \to \infty.$$

Moreover, the system of functions $\left\{ \frac{f_0(x,\hat{\lambda}_n)}{\hat{\alpha}_n} \right\}_{n=0}^{\infty}$ forms as an orthonormal basis for the space $L_2(0,\infty)$, i.e. the following equality holds:

$$\sum_{n=0}^{\infty} \frac{f_0\left(x,\hat{\lambda}_n\right)}{\hat{\alpha}_n} \frac{f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n} = \delta(x-y), \tag{7}$$

where δ is the Dirac delta function.

Now we introduce the solution $f(x,\lambda)$ of the perturbed equation (1) with asymptotics $f(x,\lambda) = f_0(x,\lambda)(1+o(1))$, $x \to \infty$. It follows from [13] that under the conditions (3) such solution exists and admits the following representation by means of a transformation operator:

$$f(x,\lambda) = f_0(x,\lambda) + \int_x^{\infty} A(x,t)\ f_0(t,\lambda)\,dt, \tag{8}$$

where the kernel $K(x,t)$ is a continuous function and satisfies the following relation:

$$|A(x,t)| \leq C\frac{1}{2}\sigma\left(\frac{x+t}{2}\right). \tag{9}$$

Hereinafter, C will denote various constants which do not depend on x, λ and n. By using the well-known properties of transformation operators (see, e.g., [14]) and (7), we find

$$f_0(x,\lambda) = f(x,\lambda) + \int_x^{\infty} \hat{A}(x,t) f(t,\lambda) dt, \tag{10}$$

where the kernel $\hat{A}(x,y)$ satisfies the equation

$$\hat{A}(x,y) + A(x,y) + \int_x^{y} \hat{A}(x,t)\,A(t,y)\,dt = 0. \tag{11}$$

The last equation and (9) imply

$$\left|\hat{A}(x,y)\right| \leq C\sigma_0\left(\frac{x+y}{2}\right). \tag{12}$$

Let us now return to problem (1)-(2). Obviously, under condition (3), differential equation (1) together with boundary condition (2) defines in space $L_2(0,+\infty)$

a self-adjoint operator L_0, which can be obtained by closure of the symmetric operator defined by equation (1) and boundary condition (2) on twice continuously differentiable finite functions (see [15]). Moreover, the spectrum of problem (1), (2) consists of simple eigenvalues $\lambda_n, n = 0, 1, 2, ...$ and coincides with the set of roots of the function $f(0,\lambda)$, i.e., the equalities $f(0,\lambda_n) = 0, n = 0, 1, 2, ...,$ are true. Therefore, the eigenfunctions $\left\{\frac{f(x,\lambda_n)}{\alpha_n}\right\}_{n=0}^{\infty}$ of problem (1), (2), where $\alpha_n = \sqrt{\int_0^\infty |f(x,\lambda_n)|^2 dx}$, form an orthonormal basis for the space $L_2(0,\infty)$, i.e., the following relation is true:

$$\sum_{n=0}^{\infty} \frac{f(x,\lambda_n)}{\alpha_n} \frac{f(y,\lambda_n)}{\alpha_n} = \delta(x-y). \tag{13}$$

Theorem 1. *Under conditions* (3)*, the spectrum of the operator L consists of a sequence of simple real eigenvalues $\lambda_n, n \geq 0$, and the asymptotic formula*

$$\lambda_n = \left(\frac{3\pi(4n-1)}{8}\right)^{\frac{2}{3}} + O\left(n^{-\frac{2}{3}}\right), n \to \infty \tag{14}$$

is valid.

Proof. By virtue of (8) we have

$$f(0,\lambda) = f_0(0,\lambda) + \int_0^\infty A(0,t)\ f_0(t,\lambda)\,dt. \tag{15}$$

From the well-known relation [1]

$$\mathrm{A}i(-z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \sin\left(\zeta + \frac{\pi}{4}\right)\left[1 + O\left(\zeta^{-1}\right)\right], \zeta = \frac{2}{3} z^{\frac{3}{2}},\ z \to +\infty,$$

and the definition of the function $f_0(x,\lambda)$, it follows that

$$f_0(0,\lambda) = 2\lambda^{-\frac{1}{4}} \sin\left(\frac{2}{3}\lambda^{\frac{3}{2}} + \frac{\pi}{4}\right)\left[1 + O\left(\lambda^{-\frac{3}{2}}\right)\right],\ \lambda \to +\infty. \tag{16}$$

On the other hand, as shown in [13], when the condition $Q_4 < \infty$ is satisfied, the relation

$$\int_0^\infty A(0,t)\ f_0(t,\lambda)\,dt = O\left(\lambda^{-\frac{3}{4}}\right),\ \lambda \to +\infty,$$

holds. From the last three relations we find

$$f(0,\lambda) = 2\lambda^{-\frac{1}{4}} \sin\left(\frac{2}{3}\lambda^{\frac{3}{2}} + \frac{\pi}{4}\right) + O\left(\lambda^{-\frac{3}{4}}\right),\ \lambda \to +\infty. \tag{17}$$

Let $\tilde{\lambda}_n = \left(\frac{3\pi(4n-1)}{8}\right)^{\frac{2}{3}}$. Then, for large values of n, by virtue of (17), the function $f\left(0, \tilde{\lambda}_n + \lambda\right)$ takes values of different signs at the ends of the segment $\left[-\tilde{\lambda}_n^{-\frac{1}{2}}, \tilde{\lambda}_n^{-\frac{1}{2}}\right]$. Therefore, there is a point belonging to this segment at which the function $f\left(0, \tilde{\lambda}_n + \lambda\right)$ vanishes. Let $f\left(0, \tilde{\lambda}_n + \delta_n\right) = 0$, $\delta_n \in \left(-\tilde{\lambda}_n^{-\frac{1}{2}}, \tilde{\lambda}_n^{-\frac{1}{2}}\right)$. Then, on the basis of (17) we conclude that

$$\sin\left(\frac{2}{3}\left(\tilde{\lambda}_n + \delta_n\right)^{\frac{3}{2}} + \frac{\pi}{4}\right) = O\left(\tilde{\lambda}_n^{-\frac{1}{2}}\right), \; n \to \infty. \tag{18}$$

Since $\frac{2}{3}\left(\tilde{\lambda}_n + \delta_n\right)^{\frac{3}{2}} = \frac{2}{3}\tilde{\lambda}_n^{\frac{3}{2}}\left(1 + \frac{\delta_n}{\tilde{\lambda}_n}\right)^{\frac{3}{2}} \sim \frac{2}{3}\tilde{\lambda}_n^{\frac{3}{2}}\left(1 + \frac{3}{2}\frac{\delta_n}{\tilde{\lambda}_n}\right) = \frac{2}{3}\tilde{\lambda}_n^{\frac{3}{2}} + \tilde{\lambda}_n^{\frac{1}{2}}\delta_n$, $n \to \infty$, from (18) we get $\sin \tilde{\lambda}_n^{\frac{1}{2}}\delta_n = O\left(\hat{\lambda}_n^{-\frac{1}{2}}\right)$, $n \to \infty$. Whence it follows that $\delta_n = O\left(n^{-\frac{2}{3}}\right)$, $n \to \infty$. Thus, the proof of the theorem is completed. ◄

3. Derivation of the main integral equation of the inverse problem

An important role in the solution of the inverse problem is played by the Marchenko-type integral equation. We put

$$\Phi_N(x,y) = \sum_{n=0}^{N} \frac{f(x,\lambda_n) f(y,\lambda_n)}{\alpha_n^2} - \sum_{n=0}^{N} \frac{f_0\left(x,\hat{\lambda}_n\right) f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n^2}, \tag{19}$$

$$F_N(x,y) = \sum_{n=0}^{N} \frac{f_0(x,\lambda_n) f_0(y,\lambda_n)}{\alpha_n^2} - \sum_{n=0}^{N} \frac{f_0\left(x,\hat{\lambda}_n\right) f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n^2}. \tag{20}$$

From the results of [8, 9] it follows that the sequences $\Phi_N(x,y)$ and $F_N(x,y)$ are uniformly convergent in each finite domain of variation of the variables x and y. Put $\Phi(x,y) = \lim\limits_{N\to\infty} \Phi_N(x,y)$. By using (6), (13), (19) we find that the sequence $\int_x^\infty \Phi_N(x,y)\, g(y)\, dy$ converges to zero in quadratic mean. In what follows, for brevity, we will write this equality in the form

$$\underset{N\to\infty}{l \cdot i \cdot m} \cdot (L_2(x,\infty)) \int_x^\infty \Phi_N(x,y)\, g(y)\, dy = 0.$$

Since $g(y)$ is a function with bounded support, integration in $\int_x^\infty \Phi_N(x,y)\,g(y)\,dy$ is, in fact, between finite limits. Passing to the limit as $N\to\infty$, we obtain

$$\lim_{N\to\infty}\int_x^\infty \Phi_N(x,y)\,g(y)\,dy=\int_x^\infty \Phi(x,y)\,g(y)\,dy=0.$$

Thus, we have proved the identity

$$\int_x^\infty \Phi(x,y)\,g(y)\,dy=0,$$

where $g(y)$ is an arbitrary function with bounded support. Therefore,

$$\Phi(x,y)=\lim_{N\to\infty}\Phi_N(x,y)=0.$$

Let

$$F(x,y)=\sum_{n=0}^{\infty}\left\{\frac{f_0(x,\lambda_n)\,f_0(y,\lambda_n)}{\alpha_n^2}-\frac{f_0\left(x,\hat\lambda_n\right)f_0\left(y,\hat\lambda_n\right)}{\hat\alpha_n^2}\right\}. \tag{21}$$

Theorem 2. *For any fixed $x\ge 0$, the function $A(x,y)$ from representation* (7) *satisfies the integral equation*

$$F(x,y)+A(x,y)+\int_x^\infty A(x,t)\,F(t,y)\,dt=0,\ \ y>x. \tag{22}$$

Equation (22) *is called the main integral equation of the Marchenko-type.*

Proof. Consider representation (10). For $y>x$, it follows from (19) that

$$\sum_{n=0}^{N}\frac{f(x,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}=\sum_{n=0}^{N}\frac{f(x,\lambda_n)}{\alpha_n}\frac{f(y,\lambda_n)}{\alpha_n}+$$

$$+\int_y^\infty \hat A(y,t)\left\{\sum_{n=0}^{N}\frac{f(x,\lambda_n)}{\alpha_n}\frac{f(t,\lambda_n)}{\alpha_n}\right\}dt=$$

$$=\sum_{n=0}^{N}\frac{f_0\left(x,\hat\lambda_n\right)}{\hat\alpha_n}\frac{f_0\left(y,\hat\lambda_n\right)}{\hat\alpha_n}+\Phi_N(x,y)+\int_y^\infty \hat A(y,t)\left\{\sum_{n=0}^{N}\frac{f(x,\lambda_n)}{\alpha_n}\frac{f(t,\lambda_n)}{\alpha_n}\right\}dt.$$

Further, using formulas (7) and (6), we obtain

$$\sum_{n=0}^{N}\frac{f(x,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}=\sum_{n=0}^{N}\frac{f_0(x,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}+$$

$$+\int_x^{\infty} A(x,t)\left\{\sum_{n=0}^{N}\frac{f_0(t,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}\right\}dt=\sum_{n=0}^{N}\frac{f_0\left(x,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\frac{f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n}+$$

$$+\sum_{n=0}^{N}\left\{\frac{f_0(x,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}-\frac{f_0\left(x,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\frac{f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\right\}+$$

$$+\int_x^{\infty} A(x,t)\left\{\sum_{n=0}^{N}\frac{f_0\left(t,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\frac{f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\right\}dt+$$

$$+\int_x^{\infty} A(x,t)\left\{\sum_{n=0}^{N}\left\{\frac{f_0(t,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}-\frac{f_0\left(t,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\frac{f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\right\}\right\}dt=$$

$$=\sum_{n=0}^{N}\frac{f_0\left(x,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\frac{f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n}+F_N(x,y)+\int_x^{\infty} A(x,t)\left\{\sum_{n=0}^{N}\frac{f_0\left(t,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\frac{f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\right\}dt+$$

$$+\int_x^{\infty} A(x,t)\,F_N(t,y)\,dt.$$

Comparing the last two equalities, we get

$$\Phi_N(x,y)+\int_y^{\infty}\hat{A}(y,t)\left\{\sum_{n=0}^{N}\frac{f(x,\lambda_n)}{\alpha_n}\frac{f(t,\lambda_n)}{\alpha_n}\right\}dt=$$

$$=\int_x^{\infty} A(x,t)\left\{\sum_{n=0}^{N}\frac{f_0\left(t,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\frac{f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\right\}dt+$$

$$+\int_x^{\infty} A(x,t)\,F_N(t,y)\,dt+F_N(x,y).$$

Fix x and denote the smooth function with bounded support contained in the interval (x,∞) by $g(y)$. Multiplying both sides of the latter identity by $g(y)$, integrating with respect to y, we obtain

$$\int_x^{\infty}\Phi_N(x,y)\,g(y)\,dy+\int_x^{\infty}\left[\sum_{n=0}^{N}\left(\int_y^{\infty}\hat{A}(y,t)\,\frac{f(t,\lambda_n)}{\alpha_n}dt\right)\frac{f(x,\lambda_n)}{\alpha_n}\right]g(y)\,dy=$$

$$=\int_x^{\infty}F_N(x,y)\,g(y)\,dy+\int_x^{\infty}\left[\sum_{n=0}^{N}\left(\int_y^{\infty}A(x,t)\,\frac{f_0\left(t,\hat{\lambda}_n\right)}{\hat{\alpha}_n}dt\right)\frac{f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\right]g(y)\,dy+$$

$$+\int_x^\infty \left[\int_x^\infty A(x,t)\,F_N(t,y)\,dt\right] g(y)\,dy.$$

Passing to the limit as $N \to \infty$, for each smooth function $g(y)$ with bounded support we obtain

$$\int_x^\infty \Phi(x,y)\,g(y)\,dy + \int_x^\infty \hat{A}(y,x)\,g(y)\,dy =$$

$$\int_x^\infty F(x,y)\,g(y)\,dy + \int_x^\infty A(x,y)\,g(y)\,dy +$$

$$+\lim_{N\to\infty} \int_x^\infty \left[\int_x^\infty A(x,t)\,F_N(t,y)\,dt\right] g(y)\,dy.$$

Since $y > x$, we have $\hat{A}(y,x) = 0$. Moreover, as shown above, the identity $\Phi(x,y) = 0$ is true. Therefore, the relation

$$\int_x^\infty F(x,y)\,g(y)\,dy + \int_x^\infty A(x,y)\,g(y)\,dy +$$

$$+\lim_{N\to\infty} \int_x^\infty \left[\int_x^\infty A(x,t)\,F_N(t,y)\,dt\right] g(y)\,dy = 0 \tag{23}$$

holds. We now show that

$$\lim_{N\to\infty} \int_x^\infty \left[\int_x^\infty A(x,t)\,F_N(t,y)\,dt\right] g(y)\,dy =$$

$$= \int_x^\infty \left[\int_x^\infty A(x,t)\,F(t,y)\,dt\right] g(y)\,dy. \tag{24}$$

Note that for each $b > x$, uniformly with respect to $y \in (x,b)$, the following equality holds:

$$\lim_{N\to\infty} \int_x^b A(x,t)\,F_N(t,y)\,dt = \int_x^b A(x,t)\,F(t,y)\,dt. \tag{25}$$

Further, it follows from the definition of the function $F_N(x,y)$ and (6), (13) that the sequence $\int_x^\infty A(x,t)\,F_N(t,y)\,dt$ converges to the limit $A_0(x,y) = \left(I + \hat{A}\right)\left(I + \hat{A}^*\right) A(x,y) - A(x,y)$ in quadratic mean, where I is a unit operator, and the operator $\hat{A}$ is defined by the formula $\hat{A}h(y) = \int_y^\infty \hat{A}(y,s)\,h(s)\,ds$. From the last relations we find

$$A_0(x,y) = \int_y^\infty \hat{A}(y,t)\,A(x,t)\,dt +$$

$$+\int_x^y \hat{A}(s,y)\,A(x,s)\,ds+\int_y^\infty \hat{A}(y,t)\int_x^t \hat{A}(s,t)\,A(x,s)\,dsdt. \qquad (26)$$

If we consider the function

$$A_b(x,t)=\begin{cases} A(x,t)\,,\ x\le t\le b,\\ 0,\ t>b,\end{cases}$$

then it can be similarly shown, that the equality

$$\underset{N\to\infty}{l\cdot i\cdot m\cdot}\left(L_2(x,b)\right)\int_x^b A(x,t)\,F_N(t,y)\,dt=G(y,b)\,,$$

is valid, where

$$G(y,b)=\int_y^b \hat{A}(y,t)\,A(x,t)\,dt+\int_x^y \hat{A}(s,y)\,A(x,s)\,ds+$$

$$+\int_y^b \hat{A}(y,t)\int_x^t \hat{A}(s,t)\,A(x,s)\,dsdt+\int_b^\infty \hat{A}(y,t)\int_x^b \hat{A}(s,t)\,A(x,s)\,dsdt,$$

for $x\le y\le b$. Further, using (8), (12), (26) and expression for $G(y,b)$, we obtain

$$G(y,b)\to A_0(x,y)\,,b\to\infty.$$

Moreover, the last relation is true uniformly with respect to y. Indeed, due to formulas (8), (12), (26),

$$|G(y,b)-A_0(x,y)|\le C\sigma_0(b)\to 0,b\to\infty.$$

On the other hand, taking into account (25), we obtain

$$\int_x^b A(x,t)\,F(t,y)\,dt=G(y,b)$$

for $y\le b$. Therefore, we get

$$\lim_{b\to\infty}\int_x^b A(x,t)\,F(t,y)\,dt=A_0(x,y)\,,$$

and this equality is true uniformly with respect to y taken from each finite interval (x,a). Thus, we have proved that the improper integral $\int_x^\infty A(x,t)\,F(t,y)\,dt$ converges and the equality

$$\int_x^\infty A(x,t)\,F(t,y)\,dt=A_0(x,y)$$

is true. Taking into account that $\underset{N\to\infty}{l\cdot i\cdot m}\cdot\left(L_2\left(y,\infty\right)\right)\int_x^\infty A\left(x,t\right)F_N\left(t,y\right)dt=A_0\left(x,y\right)$, we have

$$\underset{N\to\infty}{l\cdot i\cdot m}\cdot\left(L_2\left(y,\infty\right)\right)\int_x^\infty A\left(x,t\right)F_N\left(t,y\right)dt=\int_x^\infty A\left(x,t\right)F\left(t,y\right)dt.$$

The last equality implies (24). Since $g\left(x\right)$ is an arbitrary function with bounded support, from (23) we finally obtain equation (22).

The theorem is proved. ◄

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