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On Solvability of a Mixed Problem for a Class of Equations That Change Type

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Abstract. In [6, 7], it was shown that mixed problems can be both ill-posed for Petrovsky well-conditioned equations and well-posed for ill-conditioned equations. In the present paper we study the existence and uniqueness of the solution of a mixed problem for a class of equations with complex-valued coefficients that behave as parabolic ones, despite the fact that they can change over "time" from parabolic type to Schrödinger type, or even to antiparabolic type.

Key Words and Phrases: fundamental solution, asymptotics, analytic function, continuous differentiation, asymptotic formula, parabolic equation, spectral problem, Cauchy problem, operator.

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1. Introduction and problem statement

Unique solvability and well-posedness of linear mixed problems for Petrovsky parabolic equations and systems were considered by many authors [1, 10, 11, 12, 13, 14], etc. Depending on their statement, such problems can be solved by the methods of Fourier separation of variables, Laplace transform, heat potential, freezing of coefficients, a priori estimates, contour integral, residue, finite-difference, etc. It is known that the Fourier method is ineffective for mixed problems generating non-self-adjoint spectral problem, and there arise more difficulties when the latter has multiple eigenvalues; The thermal potential method is inapplicable when boundary conditions contain a higher order time derivative. The Laplace transform method does not work when the corresponding spectral problem is almost regular and its eigenvalues have increasing real parts. The residue method and the contour integral method developed by M.L. Rasulov became effective also for the problems with the above properties [1, 8]. In this

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paper, these two methods are applied to study the solvability of one-dimensional mixed problem with new, previously unexplored features. It is known that the equation of the form

$$\frac{\partial U}{\partial t} = a\,(t,\, x)\,\frac{\partial^2 U}{\partial x^2} \tag{1}$$

is called Petrovsky parabolic (uniformly parabolic) equation in some domain Q, if $Re\ a\,(t,x) > 0 \quad (Re\, a\,(t,x) \geq \delta > 0)$, for all $(t,x) \in Q$.

Mixed problems for the equations of the form (1) were studied only subject to their parabolicity [15, 16, 17] or when they are Schrödinger [12, 18, 19], i.e. when

$$Re\, a\,(t,x) = 0. \tag{2}$$

At the same time, it is known that [19] if the equation (1) is anti-parabolic (i.e. if $Re\, a\,(t,x) < 0$, for $(t,x) \in Q$), then, with the right hand sides of initial-boundary conditions having only finite smoothness, the mixed problem for it is not well-posed.

Definition 1. *Considering equation* (1) *in some domain* $Q_T = \{(t,x) : 0 < t < T \leq \infty,\ 0 < x < 1\}$, *we call it generalized parabolic in this domain if* $Re \int_0^t a\,(\tau,x)\; d\tau > 0$ *for all* $(t,x) \in Q_T$.

Note that an equation that is parabolic in the domain Q_T is generalized parabolic as well. However, not every generalized parabolic equation is parabolic: a generalized parabolic equation, being parabolic to a certain moment of time $t_0 > 0$, can become Schrödinger or even anti-parabolic type. In [6, 7], it is shown that mixed problems can be both ill-posed for Petrovsky well-conditioned equations and well-posed for ill-conditioned equations.

2. Problem solution

We study the solvability of the mixed problem

$$M\left(t, \frac{\partial}{\partial t}\right) U = L\left(x, \frac{\partial}{\partial x}\right) U, 0 < t < T,\ \ 0 < x < 1, \tag{3}$$

$$U(0,x) = \varphi(x), \tag{4}$$

$$U(t,0) = U(t,1) = 0, \tag{5}$$

where $M\left(t, \frac{\partial}{\partial t}\right) = \frac{1}{P(t)} \frac{\partial}{\partial t}$, $L\left(x, \frac{\partial}{\partial x}\right) = \frac{1}{(x+b)^2} \cdot \frac{\partial^2}{\partial x^2}\,, b = b_1 + ib_2,\ P\,(t) = p_1\,(t) + ip_2\,(t)$ are complex-valued functions, $p_j\,(t) \in C[0,\,1]\ \ (j = 1,\, 2)\,,\ P_1\,(t) \neq 0\,, \varphi\,(x)$ is a given function, and $U\,(x)$ is a sought-for function.

It is known that [14] equation (1) is said to be Petrovsky parabolic in the domain $D = \{(t, x) : 0 \leq t \leq T, 0 \leq x \leq 1\}$ of the space (t, x) if for any point $(t, x) \in D$ the real part of the root γ of the characteristic equation

$$\frac{1}{P(t)}\gamma - \frac{1}{(x+b)^2}\sigma^2 = 0$$

satisfies the inequality

$$Re\gamma\,(t,\,x,\,\sigma) < 0$$

for any real $\sigma \neq 0$.

The final solvability conditions will be

1^0. $\int_0^t p_1(\tau)\,d\tau > 0,\ b_1 > 0,\ b_2 > 0,;$

2^0. $Reb^2 + \omega(T)\,Imb^2 > 0$, if $Im\left[\overline{p} \cdot \int_0^t p(\tau)\,d\tau\right] \geq 0$ and $Reb^2 + \omega(T)\,Imb^2 < 0$ if $Im\left[\overline{p} \cdot \int_0^t p(\tau)\,d\tau\right] < 0$, where $\omega(t) = \int_0^t p_2(\tau)\,d\tau \cdot \left(\int_0^t p_1(\tau)\,d\tau\right)^{-1}$

3^0. $\varphi(x) \in C^2[0,\,1], \varphi(0) = \varphi(1) = 0$.

It is easy to see that even if the inequalities $p_1(t) > 0,\ b_1 > 0,\ b_2 > 0$ are fulfilled, the equation (3) is a Petrovsky equation only when

$$Im\left[\overline{p} \cdot (p'(t))\right] \leq 0,\ \ Reb^2 + r(T)\,Imb^2 > 0 \tag{6}$$

or

$$Im\left[\overline{p} \cdot (p'(t))\right] > 0,\ \ Reb^2 + r(0)\,Imb^2 > 0, \tag{7}$$

where $r(t) = p_2(t)\,(p_1(t))^{-1}$.

Note that, for example, for the equation

$$(x+1+i)^2\frac{\partial u}{\partial t} = \left(2(\alpha_1 - i)\,t + \beta_1 + \frac{3T}{2}i\right)\frac{\partial^2 u}{\partial x^2},\ \ \alpha_1 > 0,\ \ \beta_1 > 0$$

the conditions 1^0, 2^0 are fulfilled, but due to the violation of the second of the inequalities (6) it is not Petrovsky parabolic in $[0,\,T] \times [0,\,1]$. It is easy to see that this equation is not even Shilov parabolic. Furthermore, in a part of the considered rectangle it is anti-parabolic (for example, for $\alpha_1 = \beta_1 = T = 1$ in the domain $\frac{1}{2}\left(2 - x^2\right)^{-1}\left(x^2 + 5x + 3\right) < t \leq 1,\ \ 0 \leq x \leq \frac{\sqrt{37}-5}{6}$ (see the shaded area in Figure 1).

The spectral problem corresponding to (3)-(5) has the following form:

$$y'' - \lambda^2(x+b)^2\,y = -\varphi(x)\,(x+b)^2, \tag{8}$$

$$y(0) = y(1) = 0 \tag{9}$$

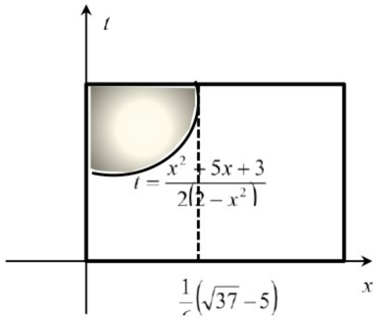

Figure 1:

Also note [1, 2] that the arguments of the roots $\pm(x+b)$ of the Birkhoff characteristic equation are not constant in $[0,\ 1]$.

The Green function $G(x,\ \xi,\ \lambda)$ of the problem (8),(9) is analytic everywhere on the complex λ plane except for a countable set of values $\lambda = \lambda_k$ $(k = 0, \pm 1, \pm 2, ...)$, that are its poles and for which the following asymptotic representation [3] is valid:

$$\lambda_k = \frac{\pi k \sqrt{-1}}{1+2b} + O\left(\frac{1}{k}\right),\ (|k| \to \infty). \tag{10}$$

Let

$$S_i = \left\{\lambda \backslash\ \ Re(\lambda b) \cdot Re(\lambda(1+b)) \le 0,\ (-1)^i Re\lambda > 0\right\};\ \ (i = 1, 2)$$

$$S_i = \left\{\lambda \backslash\ \ Re(\lambda b) < 0,\ (-1)^i Re(\lambda(1+b)) \le 0\right\};\ \ (i = 3, 4)$$

$$\chi(\lambda) = -(Re\lambda)^{-1} \cdot Re\lambda b,\ \ (\lambda \in S_i,\ i = 1,\ 2).$$

Obviously, $0 \le \chi(\lambda) \le 1$, for $\lambda \in S_i,\ (i = 1,\ 2)$.

It is seen from the asymptotic representation (10) that distinct poles λ_k lie in the sectors $\lambda \in S_i,\ (i = 1,\ 2)$. Only finitely many of them can get in $\lambda \in S_i,\ (i = 3,\ 4)$.

The following estimates for the derivatives of the Green function were obtained in [4, 5]:

$$\left|\frac{\partial^k G(x,\ \xi,\ \lambda)}{\partial x^k}\right| \le c\,|\lambda|^{k-1},\ k = 0, 1, 2;\ \lambda \in S_3 \bigcup S_4,\ |\lambda| > R, \tag{$*$}$$

$$\left|\frac{\partial^k G(x,\ \xi,\ \lambda)}{\partial x^k}\right| \le c e^{(-1)^i \chi_0^2(\lambda) Re\lambda},\ k = 0, 1, 2;\ \lambda \in S_i,\ |\lambda| > R,\ (i = 1, 2),$$

that are valid outside δ vicinities of poles, where R is a sufficiently large, and δ is a sufficiently small positive number with $\chi_0(\lambda) = \min(\chi(\lambda);\ 1 - \chi(\lambda))$.

The following theorem is valid:

Theorem 1. *Let the conditions $1^0, 2^0, 3^0$ be fulfilled. Then the problem (3)-(5) has a classic solution $U(t, x) \in C^{1,2}((0;T] \times [0;1]) \bigcap C([0;T] \times [0;1])$ representable by the following formula for $t > 0$:*

$$U(t, x) = \frac{1}{\pi i} \int \lambda e^{\lambda^2 \int_0^t P(\tau) d\tau} \cdot y(x, \lambda) \, d\lambda, \qquad (**)$$

where

$$\Gamma = \bigcup_{j=1}^{3} \Gamma_j,$$

$$\Gamma_j = \{\lambda : \lambda = r(1 + p_j), r \geq R\} \ (j = 1, 2),$$

$$\Gamma_3 = \{\lambda : \lambda = R(1 + i\eta), p_1 \leq \eta \leq p_2\},$$

$$y(x, \lambda) = \int_0^1 G(x, \xi, \lambda)(\xi + b)^2 \varphi(\xi) \, d\xi,$$

$$p_j = K_j(t_j) + (-1)^j \delta, \ K_j(t_j) = -\omega(t) + (-1)^j \sqrt{\omega^2(t) + 1}, \ (j = 1, 2), \qquad (11)$$

$t_1 = 0, \ t_2 = T$ if $Im \left[\overline{p} \cdot \int_0^t p(\tau) d\tau\right] \geq 0$ and $t_1 = T, \ t_2 = 0$ if $Im \left[\overline{p} \cdot \int_0^t p(\tau) d\tau\right] < 0$, R is a sufficiently large, and δ is a sufficiently small positive number.

At first we prove some lemmas:

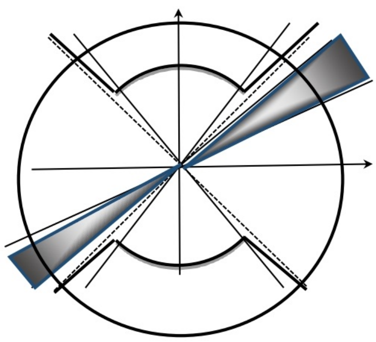

Figure 2:

Lemma 1. *Let $\int_0^t p_1(\tau) d\tau > 0$. Then for $t \in [t_0, T]$ (for $\forall \ t_0 \in (0, T)$), on the rays $\lambda = r(1 + ip_j)$ $(r \geq 0, \ j = 1, 2)$ the estimate of the following form is valid:*

$$Re\left(\lambda^2 \int_0^t p(\tau) d\tau\right) \leq -\varepsilon |\lambda|^2, \qquad (12)$$

where $\varepsilon > 0$.

Proof. First of all, note that there exists a number $\delta_1 > 0$ such that for $t \in [t_0, T]$

$$\int_0^t p_1(\tau)\, d\tau > \delta_1. \tag{13}$$

From the obvious equality

$$Re\left(\lambda^2 \int_0^t p(\tau)\, d\tau\right) = -\int_0^t p_1(\tau)\, d\tau \left[(\lambda_2 - K_1(t)\lambda_1)(\lambda_2 - K_2(t)\lambda_1)\right] =$$

$$= -\int_0^t p_1(\tau)\, d\tau \prod_{m=1}^{2} \left[Im\lambda - K_m(t)\, Re\lambda\right] \tag{14}$$

for $\lambda = r(1 + ip_j)$ $(r \geq 0,\ j = 1, 2)$, we obtain

$$Re\left(\lambda^2 \int_0^t p(\tau)\, d\tau\right) = -r^2 \int_0^t p_1(\tau)\, d\tau \left[p_j - K_m(t)\right] =$$

$$= -r^2 \int_0^t p_1(\tau)\, d\tau \prod_{m=1}^{2} \left[K_j(t_j) + (-1)^j \delta - K_m(t)\right]. \tag{15}$$

But from the representations of the functions $K_m(t)$ (see (11), it follows that if $Im\left[\overline{p} \cdot \int_0^t p(\tau)\, d\tau\right] > 0$, then $K'_m(t) > 0$, consequently, $K_m(0) \leq K_m(t) \leq K_m(T)$). Then

$$K_1(t_1) - \delta - K_m(t) = K_1(0) - \delta - K_m(t) \leq K_1(0) - \delta - K_m(0) \leq -\delta,\ (m = 1, 2)$$

$$K_2(t_2) + \delta - K_m(t) = K_1(T) + \delta - K_m(t) \geq K_2(T) + \delta - K_m(T) \geq \delta.\ (m = 1, 2) \tag{16}$$

By (13), (16), from (15) we have:

$$Re\left(\lambda^2 \int_0^t p(\tau)\, d\tau\right) \leq -\delta_1 \delta^2 r^2 \leq -\frac{\delta_1 \delta^2}{\max\limits_j \sqrt{1 + p_j^2}} \cdot |\lambda|^2. \tag{17}$$

But if $Im\left[\overline{p} \cdot \int_0^t p(\tau)\, d\tau\right] \leq 0$, then $K'_m(t) \leq 0$, consequently $K_m(T) \leq K_m(t) \leq K_m(0)$). Then

$$K_1(t_1) - \delta - K_m(t) = K_1(T) - \delta - K_m(t) \leq K_1(T) - \delta - K_m(T) \leq -\delta,\ (m = 1, 2)$$

$$K_2(t_2) + \delta - K_m(t) \geq K_2(0) + \delta - K_m(0) \geq \delta,\ (m = 1, 2). \tag{18}$$

From (13),(15), (18) we also obtain the estimate of the form (17). The lemma is proved. ◄

Lemma 2. *Let $\int_0^t p_1(\tau)\,d\tau > 0$. Then for $t \in [t_0, T]$ (for $\forall\ t_0 \in (0,T)$) and all λ from the sectors*

$$\sum{}_1 = \{\lambda : \ \arg(1+ip_2) \le \arg\lambda \le \pi + \arg(1+ip_1)\},$$
$$\sum{}_2 = \{\lambda : \ \arg(1+ip_2) - \pi \le \arg\lambda \le \arg(1+ip_1)\}.$$

the estimate of the form (12) *is valid.*

Proof. Denote $\rho = |\lambda|$, $\theta = \arg\lambda$, $\theta_j = \arg(1+ip_j)$. Then

$$Re\left(\lambda^2 \int_0^t p(\tau)\,d\tau\right) = \rho^2 w(\theta,\, t),$$

where $w(\theta,\, t) = Ree^{2i\theta}\left(\int_0^t p(\tau)\,d\tau\right)$ and, according to Lemma 1, these exists $\varepsilon > 0$ such that

$$w(\theta_j,\, t) \le -\varepsilon \quad (j = 1,\ 2),$$

for $t \in [t_0,\, T]$, $(t_0 \in (0,T))$. Obviously

$$w(\theta_1 + \pi,\, t) = w(\theta_1,\, t) \le -\varepsilon\ ,$$

$$w(\theta_2 - \pi,\, t) = w(\theta_2,\, t) \le -\varepsilon,$$

and therefore we should prove non-existence of the zeros of the function $w(\theta,\, t)$ inside the segments $[\theta_2,\ \theta_1+\pi]$ and $[\theta_2-\pi,\ \theta_1]$. But since this function is negative at the ends of these segments, it can have either multiple zero or at least two different zeros inside each of them.

If

$$w(\theta_0,\, t) = \frac{dw(\theta_0,\, t)}{d\theta_0} = 0,$$

for $\theta_0 \in (\theta_2,\ \theta_1+\pi)$ (or for $\theta_0 \in (\theta_2-\pi,\ \theta_1)$), then we have

$$Ree^{2i\theta}\left(\int_0^t p(\tau)\,d\tau\right) = 0, Re2ie^{2i\theta_0}\left(\int_0^t p(\tau)\,d\tau\right) =$$

$$= -2Ime^{2i\theta_0}\left(\int_0^t p(\tau)\,d\tau\right) = 0.$$

Consequently, $e^{2i\theta_0}\left(\int_0^t p(\tau)\,d\tau\right) = 0$, which is impossible by virtue of the condition $\int_0^t p_1(\tau)\,d\tau > 0$.

Let us consider the second case. Let

$$w(\theta_0',\, t) = w(\theta_0'',\, t) = 0, \quad (\theta_0' < \theta_0''),$$

where $\theta_0', \theta_0'' \in (\theta_2,\ \theta_1+\pi)$ (or $\theta_0', \theta_0^{''} \in (\theta_2-\pi,\ \theta_1)$). It is easy to see that the function $w(\theta,\ t)$ is the solution of the differential equation

$$\frac{d^2w}{d\theta^2}+4w=0$$

and, consequently, the distance between two neighboring zeros of any solution of this equation equals $\frac{\pi}{2}$. Then it is clear that

$$\frac{\pi}{2} \le \theta_0''-\theta_0' < \pi+\theta_1-\theta_2,$$

whence we have

$$\theta_2-\theta_1 < \frac{\pi}{2}. \tag{19}$$

On the other hand, $\theta_2-\theta_1$ is an angle between the vectors $\{1,\ p_1\}$ and $\{1,\ p_2\}$, whose scalar product equals

$$f(\delta)=1+p_1p_2=1+[K_1(t_1)-\delta]\,[K_2(t_2)+\delta].$$

The function $f(\delta)$ is a decreasing function:

$$f'(\delta)=-2\delta+K_1(t_1)-K_2(t_2)<0.$$

Therefore, we have

$$f(\delta)<f(0)=1+K_1(t_1)K_2(t_2)=$$

$$=\left\{\begin{array}{ll} 1+K_1(0)K_2(T), & if\ , Im\left[\overline{p}\cdot\int_0^t p\,(\tau)\,d\tau\right]>0, \\ 1+K_1(T)K_2(0), & if\ \ Im\left[\overline{p}\cdot\int_0^t p\,(\tau)\,d\tau\right]\le 0, \end{array}\right. \tag{20}$$

for $\delta>0$. As the functions $K_j\,(t)$ are increasing for $Im\overline{\alpha}\beta>0$, and non-increasing for $Im\left[\overline{p}\cdot\int_0^t p\,(\tau)\,d\tau\right]\le 0$, and furthermore $K_1(0)K_2(0)=-1$, from (17) we obtain:

$$f(\delta)<f(0)\le 1+K_1(0)K_2(0)=0.$$

Negativeness of the scalar product $f(\delta)$ means that the angle $\theta_2-\theta_1$ (for $\delta>0$) is an obtuse angle that contradicts inequality (19).

The lemma is proved. ◄

Lemma 3. *Let conditions* 1^0,2^0 *be fulfilled. Then the numbers* δ *and* R *(in the definition of the contour* Γ*) can be chosen so that*

$$\Gamma\cap S_j=\emptyset \quad (j=1,\,2) \tag{21}$$

and the domain

$$R_\delta = \{\lambda : \quad \lambda = r(1+i\eta), \ \ r \geq R, \ \ p_1 \leq \eta \leq p_2\} \tag{22}$$

does not contain the poles λ_k of the Green function $G(x, \xi, \lambda)$.

Proof. From the definition of domains S_j $(j = 1,\, 2)$ it is seen that to prove (21) we should study the sign of the function

$$J(\lambda) = Re\lambda bRe\lambda(1+b) \tag{23}$$

for $\lambda \in \Gamma$. Assuming in (23) $\lambda = r\,(1+ip_j) \quad (r \geq R)$, we obtain:

$$I_j\,(\delta) = J\,[r(1+ip_j)] = r^2(b_1 - b_2 p_j)(1 + b_1 - b_2 p_j) =$$

$$= r^2 \left[\left(b_1 - b_2 p_j + \frac{1}{2} \right)^2 - \frac{1}{4} \right]. \tag{24}$$

It is seen from (23), (24) that

$$I_j\,(0) = r^2 \left[\left(b_1 - b_2 K_j(t_j) + \frac{1}{2} \right)^2 - \frac{1}{4} \right]$$

Therefore, if

$$b_1 - b_2 K_j(t_j) \notin [-1, \ 0],$$

then $I_j\,(0) > 0$. But then one can find $\delta_0 > 0$ such that for $\delta \in (0, \ \delta_0)$ there will be $I_j\,(\delta) > 0$, which can not be true for the points in the sectors S_1 and S_2. Therefore, assume that

$$-1 \leq b_1 - b_2 K_j(t_j) \leq 0.$$

From condition 1^0 and definition of the function $K_1(t)$ it follows that this inequality is impossible for $j = 1$. Consequently, let

$$-1 \leq b_1 - b_2 K_2(t_2) \leq 0,$$

whence we have:

$$b_1 - b_2\omega(t_2) \leq b_2\sqrt{\omega^2(t_2)+1} \leq b_1 + 1 - b_2\omega(t_2). \tag{25}$$

Let us consider two possible cases:

1) $b_1 - b_2\omega(t_2) \geq 0$;

2) $b_1 - b_2\omega(t_2) < 0$.

In the first case, from (25) we obtain

$$Reb^2 - \omega(t_2)Imb^2 \le 0.$$

In the second case we have $\omega(t_2) > \frac{b_1}{b_2}$, consequently

$$Reb^2 - \omega(t_2)Imb^2 < Reb^2 - 2b_1^2 = -|b|^2 < 0.$$

Both of these inequalities are impossible due to the conditions 1^0,2^0 and definitions of the function $\omega(t)$ and the number t_2.

We now assume in (23) $\lambda = R(1 + i\eta)$ $(p_1 \le \eta \le p_2)$:

$$I(\eta) = J[R(1 + i\eta)] = R^2(b_1 - b_2\eta)(b_1 + 1 - b_2\eta) = R^2\left[\left(b_1 - b_2\eta + \frac{1}{2}\right)^2 - \frac{1}{4}\right].$$

For $\eta = p_j$, we have established above that there exists $\delta > 0$ such that $I(p_j) > 0$ $(j = 1,\ 2)$. Therefore, it suffices to consider only the stationary point

$$\eta_0 = \frac{1}{b_2}\left(b_1 + \frac{1}{2}\right).$$

But since for the function

$$g(\delta) = \frac{1}{b_2}\left(b_1 + \frac{1}{2}\right) - p_2$$

we have

$$\begin{aligned} g(0) &= \frac{1}{b_2}\left(b_1 + \frac{1}{2}\right) - K_2(t_2) > \frac{1}{b_2}\left(b_1 - b_2K_2(t_2)\right) = \\ &= \frac{1}{b_2}\left(b_1 - b_2\omega(t_2) - b_{\_2}\sqrt{\omega^2(t_2) + 1}\right) = \\ &= \frac{Reb^2 - \omega(t_2)Imb^2}{b_2[b_1 - b_2\omega(t_2) + b_{\_2}\sqrt{\omega^2(t_2) + 1}]} > 0, \end{aligned}$$

we can choose $\delta_0 > 0$ such that $g(\delta) > 0$ for $\delta \in (0,\ \delta_0)$. This means that the stationary point η_0 lies outside the segment $[p_1,\ p_2]$, and $I(\eta) > 0$ for $\eta \in [p_1,\ p_2]$. Thus, the first statement of the lemma is proved. From this statement we have the inclusion

$$R_\delta \subset \left(S_3 \bigcup S_4\right), \tag{26}$$

whence, due to the fact that the sectors S_3 and S_4 can contain only finitely many poles λ_k, it follows that for sufficiently large $R > 0$ the following relation is valid:

$$\{\lambda_k\} \bigcap R_\delta = \emptyset.$$

The lemma is proved. ◀

Finally, considering that R, δ (in the definition of the contour Γ) were chosen to meet the requirements of Lemma 3, let us prove the theorem using Lemmas 1-3.

Denote $\Gamma^- = \bigcup\limits_{j=1}^{3} \Gamma_j^-$, where

$$\Gamma_j^- = \{\lambda : \lambda = -r\,(1 + p_j)\,,\, r \geq R\} \;\; (j = 1, 2)$$

$$\Gamma_3^- = \{\lambda : \lambda = -R\,(1 + i\eta)\,,\, p_1 \leq \eta \leq p_2\}\,.$$

Let us agree to consider the directions $\Gamma_1 \to \Gamma_3 \to \Gamma_2$ and $\Gamma_1^- \to \Gamma_3^- \to \Gamma_2^-$ positive on Γ and Γ^-. Let us choose a number n_0 such that

$$n_0 > \frac{2\pi R}{|1+2b|}\sqrt{1+\max_j p_j^2}$$

and denote by $\{r_n\}$ the sequence of numbers

$$r_n = \frac{(4n + 4n_0 + 1)\,\pi}{2\,|1+2b|} \quad (n = 0, 1, ...). \tag{27}$$

Due to the choice of number n_0, the circles

$$O_n = \left\{\lambda : \quad \lambda = r_n e^{i\theta}\,, \;\; (0 \leq \theta \leq 2\pi)\right\}$$

intersect Γ and Γ^- only at the points lying on $\Gamma_j^{\pm}$ $(j = 1,\, 2)$, and, moreover,

$$a_{jn}^{\pm} = \Gamma_j^{\pm} \bigcap O_n = \pm \frac{r_n}{\sqrt{1+p_j^2}}\,(1 + ip_j) = \pm r_n e^{i\theta_j}.$$

Furthermore, from (27) we see that for sufficiently large $R > 0$

$$\left|r_n e^{i\theta} - \lambda_k\right| \geq \frac{\pi}{4\,|1+2b|} \quad (\pm k,\, n = 0, 1, ...; 0 \leq \theta \leq 2\pi).$$

Let us consider the following arcs of the circle O_n:

$$\overset{\cup}{a_{1n}^+ a_{2n}^+} = \left\{\lambda : \;\; \lambda = r_n e^{i\theta}, \;\; \theta_1 \leq \theta \leq \theta_2\right\},$$

$$\overset{\cup}{a_{2n}^+ a_{1n}^-} = \left\{\lambda : \;\; \lambda = r_n e^{i\theta}, \;\; \theta_2 \leq \theta \leq \theta_1 + \pi\right\},$$

$$\overset{\cup}{a_{2n}^{-}a_{1n}^{+}} = \left\{\lambda : \quad \lambda = r_n e^{i\theta}, \ \ \theta_2 + \pi \le \theta \le \theta_1 + 2\pi\right\}.$$

Denote by Ω_n and Ω_n^+ the closed contours

$$\Omega_n = \Gamma^{n,+} \bigcup \overset{\cup}{a_{2n}^{+}a_{1n}^{-}} \bigcup \Gamma^{n,-} \bigcup \overset{\cup}{a_{2n}^{-}a_{1n}^{+}},$$

$$\Omega_n^+ = \Gamma^{n,+} \bigcup \overset{\cup}{a_{2n}^{+}a_{1n}^{+}},$$

where

$$\Gamma^{n,\pm} = \{\pm\lambda : \ \ \lambda \in \Gamma, \ \ |\lambda| \le r_n\}.$$

Formally passing to the limit as $x \to +0$, $x \to 1-0$ under the integral sign in

$$U\,(t,\,x) = -\frac{1}{\pi\, i}\int_{\Gamma} \lambda\, e^{\lambda^2 \int_0^t p(\tau)d\tau} d\lambda \int_0^1 G(x,\,\xi,\,\lambda)(\xi+b)^2\,\varphi\,(\xi)\,d\xi, \tag{28}$$

and using the properties of the Green function, we obtain:

$$U(t,0) = 0, \ U(t,1) = 0, \tag{29}$$

for $t \in (0,\,T]$.

Formally, taking the operation $\frac{\partial}{\partial t}$, $\frac{\partial^2}{\partial x^2}$ under the integral sign in (28), we find:

$$(x+b)^2\, U_t - P\,(t)\ U_{xx} = \frac{1}{\pi i} P\,(t) \cdot \ \ (x+b)^2\,\varphi(x) \int_{\Gamma} \lambda e^{\lambda^2 \int_0^t p(\tau)d\tau} d\lambda \tag{30}$$

for $(t,x) \in (0,\,T] \times [0,\ 1]$.

Note that due to the condition 3^0 we have

$$\int_0^1 G(x,\xi,\lambda)(\xi+b)^2\varphi(\xi)d\xi = \frac{\varphi(x)}{\lambda^2} + \frac{1}{\lambda^2}\int_0^1 G(x,\xi,\lambda)\varphi''(\xi)d\xi.$$

Rewrite formula (28) in the form

$$U(t,x) = \ U_1(t,x) + U_2(t,x), \tag{31}$$

where

$$U_1(t,x) = \frac{1}{\pi i}\varphi(x)\int_{\Gamma} \frac{1}{\lambda}\, e^{\lambda^2 \int_0^t P(\tau)d\tau} d\lambda, \tag{32}$$

$$U_2(t,x) = \frac{1}{\pi i}\int_{\Gamma} \frac{1}{\lambda}\ e^{\lambda^2 \int_0^t P(\tau)d\tau} d\lambda \int_0^1 G(x,\xi,\lambda)\varphi''(\xi)d\xi, \tag{33}$$

$$U_2(0,x)=\frac{1}{\pi i}\int_\Gamma \frac{1}{\lambda}\,d\lambda\int_0^1 G(x,\xi,\lambda)\varphi''(\xi)d\xi. \tag{34}$$

Let us calculate the integrals over Γ on the right-hand sides of formulas (30), (32), (34). Let

$$\gamma_k\left(\Gamma\right)=\int_\Gamma \lambda^{2k-1}\,e^{\lambda^2\int_0^t P(\tau)d\tau}d\lambda \quad (k=0,\ 1). \tag{35}$$

Obviously,

$$\gamma_k\left(\Gamma\right)=\lim_{n\to\infty}\gamma_k\left(\Gamma^{n,+}\right)=\frac{1}{2}\cdot\lim_{n\to\infty}\left[\gamma_k\left(\Gamma^{n,+}\right)+\gamma_k\left(\Gamma^{n,-}\right)\right].$$

On the other hand, using Lemma 2 we have

$$\lim_{n\to\infty}\gamma_k\left(\overset{\bigcup}{a_{2n}^+a_{1n}^-}\right)=0,\ \lim_{n\to\infty}\gamma_k\left(\overset{\bigcup}{a_{2n}^-a_{1n}^+}\right)=0. \tag{36}$$

Consequently,

$$\gamma_k\left(\Gamma\right)=\frac{1}{2}\lim_{n\to\infty}\left[\gamma_k\left(\Gamma^{n,+}\right)+\gamma_k\left(\overset{\bigcup}{a_{2n}^+a_{1n}^-}\right)+\gamma_k\left(\Gamma^{n,-}\right)+\gamma_k\left(\overset{\bigcup}{a_{2n}^-a_{1n}^+}\right)\right]=$$

$$=\frac{1}{2}\cdot\lim_{n\to\infty}\gamma_k\left(\Omega_n\right), \tag{37}$$

for $t>0$ and $k=0,\,1$. But $\gamma_k\left(\Omega_n\right)$ is an integral of the function $\lambda^{2k-1}e^{\lambda^2 t(\alpha t+\beta)}$ over the closed contour Ω_n. Therefore

$$\gamma_k\left(\Omega_n\right)=\begin{cases}2\pi i, & for\ \ k=0\\ 0, & for\ \ k=1.\end{cases}$$

Then, allowing for formulas (30), (32), (35), (37), we deduce:

$$\left(x+b\right)^2U_t-P\left(t\right)\ U_{xx}=0 \tag{38}$$

and

$$U_1(t,x)=\varphi(x), \tag{39}$$

for $(t,x)\in(0,\,T]\times[0,\ 1]$.

Considering the estimates $(*)$ for the Green function of the spectral problem (8)-(9) and $\overset{\bigcup}{a_{2n}^+a_{1n}^-}\subset R_\delta\subset(S_3\bigcup S_4)$ (see Lemma 3), we have :

$$\lim_{n\to\infty}\frac{1}{\pi i}\int_{\overset{\bigcup}{a_{2n}^+a_{1n}^-}}\frac{1}{\lambda}\,d\lambda\int_0^1 G(x,\xi,\lambda)\varphi''(\xi)d\xi=0,$$

uniformly with respect to $x \in [0,\ 1]$. Consequently, from (34) we have :

$$U_2\left(0,x\right)=\frac{1}{\pi i}\lim_{n\to\infty}\int_{\Gamma^{n,+}\bigcup a_{2n}^{+}a_{1n}^{-}}\frac{1}{\lambda}\,d\lambda\int_0^1 G(x,\xi,\lambda)\varphi''(\xi)d\xi, \tag{40}$$

i.e.

$$U_2\left(0,x\right)=0. \tag{41}$$

Using the equalities (31), (39), (41), we have:

$$\lim_{t\to+0}U\left(t,x\right)=\lim_{t\to+0}\left[U_1\left(t,x\right)+U_2\left(t,x\right)\right]=$$

$$=\lim_{t\to+0}\left[\varphi\left(x\right)+U_2\left(t,x\right)\right]=\varphi\left(x\right)+U_2\left(0,x\right)=\varphi\left(x\right). \tag{42}$$

The theorem is proved. ◄

Thus, the function $U\left(t,x\right)$, for $t>0,\ 0\le x\le 1$ defined by the formula (**), belongs to the space $C^{1,2}\left((0,\ T]\times[0,\ 1]\right)$ (see (28)), satisfies the equation (3) for $0<t\le T,\ 0\le x\le 1$ (see (35)) boundary conditions (5) for $0<t\le T$ (see (29)), and (42) is valid for it with $0\le x\le 1$. So it is clear that if we define this function for $t=0,\ 0\le x\le 1$ by the equality $U\left(0,x\right)=\varphi\left(x\right)$, then it will be an element of the space $C^{1,2}\left((0,\ T]\times[0,\ 1]\right)\bigcap C\left([0,\ T]\times[0,\ 1]\right)$ satisfying the equalities (3) for $0<t\le T,\ 0\le x\le 1$, (4) for $0\le x\le 1$ and (5) for $0\le t\le T$ (for $t=0$ by virtue of the condition $\varphi\left(0\right)=\varphi\left(1\right)=0$).

The theorem is proved. ◄

Remark 1. The proved theorem covers a class that contains not only parabolic equations, but also some types of non-parabolic ones. For example, equation (3) satisfies the conditions of this theorem, but it is not parabolic.

Remark 2. To prove the solvability of problem (3)-(5), condition 1 is not necessary. It was introduced for clarity only. In fact, it would be possible to single out all possible cases of solvability only provided $|2b+1|>1$, needed to use the appropriate spectral theory [3].

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