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On the Weak Solvability of Dirichlet Problem for a Fractional Order Degenerate Elliptic Equation

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Abstract. In this paper, we study the weak solvability of nonhomogeneous Dirichlet problem for a degenerate fractional order elliptic equation: $(-\Delta)^{\frac{\alpha}{2}}\left(\omega(x)(-\Delta)^{\frac{\alpha}{2}}u\right) = f(x), \quad x\in\Omega\subset R^n, \quad \alpha\in(0;1)\ u|_{R^n\setminus\Omega} = \varphi(x)$. For that a sufficient condition is found on the data of problem such as $\Omega,\ \alpha,\ n,$ the weight function $\omega:\ \mathbb{R}^n\to[0,\ \infty)$ and the functions $f,\ \varphi$. Weighted fractional order Sobolev-Poincare type inequality and Lax-Milgram principle are used.

Key Words and Phrases: fractional Laplacian, Lax-Milgram principle, fractional order Sobolev spaces, Sobolev-Poincare inequality, weight, degenerate elliptic equations.

2010 Mathematics Subject Classifications: 39A06, 39A14, 39A70, 26D10, 47G20

1. Introduction

This paper deals with the weak solvability of Dirichlet problem for the degenerate fractional order elliptic equation

$$(-\Delta)^{\frac{\alpha}{2}}\left(\omega(x)(-\Delta)^{\frac{\alpha}{2}}u\right) = f(x), \quad x\in\Omega\subset R^n, \quad \alpha\in(0;1) \tag{1}$$

$$u|_{R^n\setminus\Omega} = \varphi(x), \tag{2}$$

where Δ is Laplace's operator, and $(-\Delta)^{\alpha/2}$ is a Laplace operator of fractional order $\alpha/2$ in the sense below (see, e.g. [16, Section 3]):

$$(-\Delta)^{\alpha/2}u(x) = C(n,\ \alpha)\lim_{\varepsilon\to+0}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x)-u(y)}{|x-y|^{n+\alpha}}dy, \quad x\in\mathbb{R}^n\ ,$$

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with

$$C(n,\alpha) = \pi^{-(\alpha+n/2)} \frac{\Gamma((\alpha+n)/2)}{\Gamma(-\alpha/2)}, \ \alpha \in (0,1),$$

which is a nonlocal operator, the kind of well-known Laplace operator

$$-\Delta u(x) = C(n) \lim_{\varepsilon\to+0} \frac{1}{\varepsilon^{n+2}} \int_{B_\varepsilon(x)} (u(x) - u(y))\, dy, \quad x \in \mathbb{R}^n \ ,$$

with

$$C(n) = \pi^{-n/2} \Gamma\left(\frac{n}{2}\right) n(n+1).$$

In case of no degeneration ($\omega \equiv const$), the problem (1)-(2) has been well-studied (see, e.g., [4, 7, 17, 20]). For a survey of the fractional Sobolev spaces, the corresponding nonlocal equations and their applications we refer to [8, 15, 16].

The fractional Sobolev spaces have been a classical topic, and some important books ([9, 12, 19]) treat this topic in detail; the wide bibliography is given in [16]. For applications see, e.g., [6]. Though the method used here to proof the existence and uniqueness of the solution to the problem (1)-(2), that is the Lax-Milgram principle, is not distinguished by its originality, its use is fraught with many non-trivial problems. This is caused by the extension and trace problems on weighted fractional Sobolev spaces (see [3]), approximation problems with smooth functions (see [5]). This paper aims to draw attention to these actual problems.

2. Weighted Sobolev spaces of fractional order

Let $1 \le p < \infty$ and $\alpha \in (0,\ 1)$. Let $\Omega \subset \mathbb{R}^n$ be an open domain. Let $g(x) \in \mathrm{Lip}\ \left(\overline{\Omega}\right)$ be a Lipschitz continuous function. Denote by $W^\alpha_{p,\ \omega}(\Omega)$ a closure of $\mathrm{Lip}\ \left(\overline{\Omega}\right)$ with respect to the norm

$$\|g\|_{W^\alpha_{p,\omega}(\Omega)} = \|g\|_{L^p(\Omega)} + \left(\iint_{\Omega\times\Omega} \frac{|g(x) - g(y)|^p}{|x-y|^{n+p\alpha}}\, \omega(x)\, dxdy \right)^{1/p}. \tag{3}$$

Also, we denote by $\dot{W}^\alpha_{p,\omega}(\Omega)$ a Sobolev space obtained from the closure of $\overline{\mathrm{Lip}_0\ (\Omega)}$ with respect to the norm (3).

Denote by $\hat{W}^\alpha_{p,\ \omega}(\Omega)$ a closure with respect to the norm (3) of the functions with finite norm (3). Evidently, $W^\alpha_{p,\ \omega}(\Omega) \subset \hat{W}^\alpha_{p,\ \omega}(\Omega)$.

On the relation $W^\alpha_{p,\omega}(\Omega) = \hat{W}^\alpha_{p,\omega}(\Omega)$ for these spaces without weights ($\omega \equiv$ *const*) , i.e. possibility of smooth approximation on the Lipschitz domains, see [2], and [5] for other domains in terms of Assouad dimension.

For these spaces, their trace analogues and interpolation inequalities in non-weight cases $(\omega \equiv const)$ we refer to [10, 12, 18, 20, 21] (see also, [1, 13, 14] for some weight cases).

Denote the trace space of $W^{\alpha}_{p,\omega}(\Omega)$ by $\daleth r\left(W^{\alpha}_{p,\omega}(\Omega)\right)$ (probably it would be better to use "extension" instead of "trace"). This trace space consists of functions $\varphi: \mathbb{R}^n \backslash \Omega \to \mathbb{R}$ such that there exists a bounded extension operator $T: \varphi \to \Phi$ from $W^{\alpha}_{p,\,\omega}(\mathbb{R}^n \backslash \Omega)$ to $W^{\alpha}_{p,\,\omega}(\mathbb{R}^n)$ such that $\Phi = \varphi$ on $\mathbb{R}^n \backslash \Omega$.

Also, we denote by $W^{\alpha-1/p}_{p,\omega}(\partial\Omega)$ a closure of Lip $(\overline{\partial\Omega})$ with respect to the norm

$$\|u\|_{W^{\alpha-\frac{1}{p}}_{p,\omega}(\partial\Omega)} :=$$

$$\left(\int_{\partial\Omega} |u(x)|^p ds_x\right)^{1/p} + \left(\iint_{\partial\Omega\times\partial\Omega} \frac{[\varphi(x)-\varphi(y)]^p}{|x-y|^{n-1+p(\alpha-1/p)}}\ \omega(x)\ ds_x ds_y\right)^{1/p},$$

where ds_x is an element of $(n-1)-$dimensional Hausdorff measure on the surface $\partial\Omega$. Note that $\daleth r\left(W^{\alpha}_{p,\,\omega}(\Omega)\right) \equiv W^{\alpha-1/p}_{p,\,\omega}(\partial\Omega)$ in the classical nonfractional case (i.e. for the positive integer α and $\omega \equiv const$). We are not aware of the related results concerning the case of fractional α, e.g. $\frac{1}{p} < \alpha < 1,\ \ p > 1,$ not to mention weight cases. Also we do not know if there exists an extension $G \in W^{\alpha}_{p,\,\omega}(\mathbb{R}^n)$ of the function $g \in W^{\alpha}_{p,\,\omega}(\Omega)$ to the whole space such that $g \to G$ is a bounded operator, and what condition on Ω is sufficient for that.

Conjecture 1. *We conjecture that the equality* $\daleth r\left(W^{\alpha}_{p,\,\omega}(\Omega)\right) \equiv W^{\alpha}_{p,\,\omega}(\mathbb{R}^n \ \backslash\Omega)$ *holds for the Lipshitsz domain* Ω *in the cases of* $p > 1,\ \ \frac{1}{p} < \alpha < 1$ *if the weight function* ω *satisfies some Muckenhoupt type conditions.*

Now let us consider the problem (1)-(2) for $p = 2,\ \ 1/2 < \alpha < 1$. We introduce the substitution $z{=}u - \Phi$ in order to solve the problem (1)-(2), where Φ is an extension of the given function $\varphi: \mathbb{R}^n \backslash \Omega \to \mathbb{R}$ as $\Phi \in W^{\alpha}_{2,\,\omega}(\mathbb{R}^n)$, i.e. there is a bounded operator

$$T:\ \varphi \in W^{\alpha}_{2,\,\omega}(\mathbb{R}^n \backslash \Omega) \to \Phi \in W^{\alpha}_{2,\,\omega}(\mathbb{R}^n) \tag{4}$$

such that $\Phi{\equiv}\varphi$ on $\mathbb{R}^n \backslash \Omega$ and $\Phi{=}\,\mathrm{T}\varphi \in W^{\alpha}_{2,\,\omega}(\mathbb{R}^n)$. In our notations this is expressed as $\varphi \in \daleth r\left(W^{\alpha}_{p,\,\omega}(\Omega)\right)$.

Problem 1. *Find the necessary and sufficient conditions on domain* Ω*, the fractional parameter* $0 < \alpha < 1$*, on* $p > 1$ *and the weight function* ω *which provide the existence of the extension* (4) *and its inverse.*

Problem 2. *We are also not aware of sufficient (necessary) conditions which provide the existence of the extension* (4) *or its inverse in weighted cases* $\omega \neq const$ *and* $p \neq 2$.

Consider the functions

$$f \in L^2(\Omega) \text{ and } \varphi \in \urcorner r\left(W^{\alpha}_{2,\omega}(\Omega)\right)$$

Definition 1. *By the solution* $u(x)$ *of the problem* (1)-(2)*, we mean a function* $u \in W^{\alpha}_{2,\,\omega}(\mathbb{R}^n)$ *such that* $u - \Phi \in \dot{W}^{\alpha}_{2,\,\omega}(\Omega)$ *and*

$$\iint_{\Omega\times\Omega} \frac{[u(x) - u(y)]\;[g(x) - g(y)]}{|x-y|^{n+2\alpha}}\,\omega(x)\,dxdy = \int_{\Omega} f(x)\,g(x)\,dx \quad (5)$$

for all test functions $g \in \dot{W}^{\alpha}_{2,\,\omega}(\Omega)$.

In other words, we are going to solve the problem (1)-(2) for $\varphi \in \urcorner r\left(W^{\alpha}_{2,\omega}(\Omega)\right)$ and $f(x) \in L_2(\Omega)$, which means that there exists a bounded extension of $\varphi : \mathbb{R}^n\backslash\Omega \to \mathbb{R}$ from $\varphi \in W^{\alpha}_{2,\omega}(\mathbb{R}^n\backslash\Omega)$ to $\Phi \in W^{\alpha}_{2,\,\omega}(\mathbb{R}^n)$ such that $\Phi\equiv\varphi$ on $\mathbb{R}^n\backslash\Omega$. Applying the substitution $z = u - \Phi$, we get the relation

$$(-\Delta)^{\alpha/2}\left(\omega(x)\,(-\Delta)^{\alpha/2} z\right) = f(x) + F\,, \quad (6)$$

where

$$F = (-\Delta)^{\frac{\alpha}{2}}\left(\omega(x)\,(-\Delta)^{\frac{\alpha}{2}}\,\Phi\right) \quad \text{and} \quad z|_{R^n\backslash\Omega} = 0\,. \quad (7)$$

Now, the solution of the problem (6)-(7) is a function $z \in \dot{W}^{\alpha}_{2,\,\omega}(\Omega)$ satisfying for $\forall g \in \dot{W}^{\alpha}_{2,\,\omega}(\Omega)$ the integral identity

$$\iint_{\Omega\times\Omega} \frac{[z(x) - z(y)]\;[g(x) - g(y)]}{|x-y|^{n+2\alpha}}\,\omega(x)\,dxdy = \int_{\Omega} f(x)\,g(x)\,dx -$$

$$- \iint_{\Omega\times\Omega} \frac{[\Phi(x) - \Phi(y)]\,[g(x) - g(y)]}{|x-y|^{n+2\alpha}}\,\omega(x)\,dxdy\,. \quad (8)$$

To find a function $z(x)$ satisfying (8) for all $g \in \dot{W}^{\alpha}_{2,\,\omega}(\Omega)$, we apply the Lax-Milgram principle [6]. For that, let us consider the bilinear form

$$B(z,\; g) = \iint_{\Omega\times\Omega} \frac{[z(x) - z(y)]\;[g(x) - g(y)]}{|x-y|^{n+2\alpha}}\,\omega(x)\,dxdy \quad (9)$$

and show that it is bounded and coercive on a Hilbert space. For that, we use the space $\dot{W}^{\alpha}_{2,\omega}(\Omega)$ obtained from the closure of $\mathrm{Lip}_0(\Omega)$ with respect to the norm of space $\dot{W}^{\alpha}_{2,\omega}(\Omega)$. Define the inner product space on $\mathrm{Lip}_0(\Omega)$ as follows:

$$(z,\ g)_H = \iint_{\Omega\times\Omega} \frac{[z\,(x) - z(y)]\ \ [g\,(x) - g(y)]}{|x-y|^{n+2\alpha}}\ \omega\,(x)\,dxdy + \int_\Omega z\,(x)\,g\,(x)\,dx. \tag{10}$$

Evidently, (10) satisfies the properties of inner product. Hence it is a norm. Denote by H its closure with respect to the norm $\|z\|_H = ((z,\ z)_H)^{1/2}$. The space H is a Hilbert space. We will show that the bilinear form $B(z,\ g)$ acts boundedly and coercively in H. Let us note that $H = \dot{W}^{\alpha}_{2,\omega}(\Omega)$, i.e.

$$\|g\|_H = \|g\|_{L^2(\Omega)} + \left(\iint_{\Omega\times\Omega} \frac{|g\,(x) - g\,(y)|^2}{|x-y|^{n+2\alpha}}\ \omega\,(x)\,dxdy\right)^{1/2}.$$

We need the following inequality from [11, 12] (see also [14]):

$$\|g\|^2_{L^2(\Omega)}\ \le c\left(\iint_{\Omega\times\Omega} \frac{[g\,(x) - g\,(y)]^2}{|x-y|^{n+2\alpha}}\ \omega\,(x)\,dxdy\right)^{1/2}, \tag{11}$$

where $g \in H$, the weight function $\omega :\ \ \mathbb{R}^n\ \to (0,\ \infty)$ is measurable, takes finite positive values a.e. and is such that for all cubes $Q \subset \mathbb{R}^n$ centered on Ω with the edge less than d_Q (diameter of cube) the following inequality holds:

$$\int_Q \omega^{-1}\,(x)\,dx \le c|Q|^{\frac{n-2\alpha}{n}}, \tag{12}$$

where $0 < \alpha < \min\ \{n/2,\ 1\}$. Hence it easily follows that the same inequality holds for all functions $g \in \dot{W}^{\alpha}_{2,\ \omega}(\Omega)$, too. Note that more general result was proved in the cited works: for the exponents $\ 1 \le p \le q < \infty,\ \ \alpha \in (0,\ 1)\ $ the inequality

$$\|g\|_{L_{q,v}(\Omega)} \le C_0 A\left(\iint_{\Omega\times\Omega} \frac{[g\,(x) - g\,(y)]^p}{|x-y|^{n+p\alpha}}\ K\,(x,\ y)\ \ dxdy\right)^{1/p} \tag{13}$$

was proved for all $g \in \dot{W}^{\alpha}_{2,\ \omega}(\Omega)$ under the sufficient condition on the kernel function:

$$\frac{1}{|Q|}\left[\iint_Q K\,(x\ y)^{1-p'} v(y)^{p'} dxdy\right]^{1/p'} \le A\ \left[\int_Q v\,(x)\,dx\right]^{1/q'} \tag{14}$$

for all cubes $Q \subset \mathbb{R}^n$ centered on Ω with the edge less than $\ d_Q\ $.

Consider the kernel function $K\,(x,\ y) = \omega\,(x)\,/|x-y|^{n+p\alpha}$ with $p = 2,\ \ q = 2$. Then

$$\frac{1}{|Q|}\left[\iint_{Q\cap Q} \omega^{-1}(x)|x-y|^{n+2\alpha} dxdy\right]^{1/2} \le c\ |Q|^{1/2}.$$

Therefore, this condition becomes

$$\iint_{Q\cap Q} \omega^{-1}(x)|x-y|^{n+2\alpha} dxdy \le c\ |Q|^3,$$

and hence

$$\omega^{-1}\left(Q\right) \le c\ |Q|^{1-2\alpha/n}, \tag{15}$$

which yields the inequality with more specified constant:

$$\|u\|_{L_2(\Omega)} \le C_0 \left[\frac{\omega^{-1}\left(Q_0\right)}{|Q_0|^{1-\frac{2\alpha}{n}}}\right]^{\frac{1}{2}} \left(\iint_{\Omega\times\Omega} \frac{\left[u\left(x\right)-u\left(y\right)\right]^2}{|x-y|^{n+2\alpha}}\ \omega\left(x\right)dxdy\right)^{\frac{1}{2}} \tag{16}$$

for all functions $u \in \dot{W}^{\alpha}_{2,\ \omega}\left(\Omega\right)$, where Q_0 is a minimal cube such that $\Omega \subset Q_0$.

The boundedness of bilinear form follows from (16) using Hölder's inequality,

$$|B\left(z,\ v\right)| \ = \left|\iint_{\Omega\times\Omega} \frac{\left[z\left(x\right)-z(y)\right]\ \left[v\left(x\right)-v(y)\right]}{|x-y|^{n+2\alpha}}\ \omega\left(x\right)dxdy\right|$$

$$\le \left[\iint_{\Omega\times\Omega} \frac{\left[z\left(x\right)-z\left(y\right)\right]^2}{|x-y|^{n+2\alpha}}\ \omega\left(x\right)dxdy\right]^{\frac{1}{2}} \cdot \left[\iint_{\Omega\times\Omega} \frac{\left[v\left(x\right)-v\left(y\right)\right]^2}{|x-y|^{n+2\alpha}}\ \omega\left(x\right)dxdy\right]^{\frac{1}{2}}$$

$$\le \|z\|_H\ \|v\|_H\ \ . \tag{17}$$

The coercivity of bilinear form $B(z,\ g)$ follows from (16) and the inequality:

$$B\left(z,\ z\right) = \iint_{\Omega\times\Omega} \frac{\left[z\left(x\right)-z\left(y\right)\right]^2}{|x-y|^{n+2\alpha}}\ \omega\left(x\right)dxdy \le$$

$$\le \frac{1}{2}\iint_{\Omega\times\Omega} \frac{\left[z\left(x\right)-z\left(y\right)\right]^2}{|x-y|^{n+2\alpha}}\ \omega\left(x\right)dxdy + \frac{1}{2}\iint_{\Omega\times\Omega} \frac{\left[z\left(x\right)-z\left(y\right)\right]^2}{|x-y|^{n+2\alpha}}\ \omega\left(x\right)dxdy.$$

Now, using the inequality (16) we have

$$B\left(z,\ z\right) \ge\ \frac{1}{2}\ \iint_{\Omega\times\Omega} \frac{\left[z\left(x\right)-z\left(y\right)\right]^2}{|x-y|^{n+2\alpha}}\ \omega\left(x\right)dxdy + \frac{1}{2C}\|z\|^2_{L^2(\Omega)} \ge c_1\|z\|^2_H\ , \tag{18}$$

where $c_1 = min\left\{\frac{1}{2},\ \frac{1}{2C}\right\},\ C$ is from the inequality (16) and does not depend on $z(x)$.

We have to solve the problem

$$(-\Delta)^{\frac{\alpha}{2}}\ \left(\omega\left(x\right)(-\Delta)^{\frac{\alpha}{2}}z\right) = f\left(x\right)+F, \tag{19}$$

$$z|_{R^n\setminus\Omega} = 0, \tag{20}$$

where F is defined by (7). Substituting (7) into (19), we obtain the relation

$$\iint_{\Omega\times\Omega} \frac{[z(x) - z(y)]\ [g(x) - g(y)]}{|x-y|^{n+2\alpha}}\, \omega(x)\, dxdy = \int_\Omega f(x)\, g(x)\, dx-$$

$$-\iint_{\Omega\times\Omega} \frac{[\Phi(x) - \Phi(y)]\, [g(x) - g(y)]}{|x-y|^{n+2\alpha}}\, \omega(x)\, dxdy, \tag{21}$$

which is suitable for applying the Lax-Milgram principle.

Now, after proving that the bilinear form is bounded and coercive, it remains to show that the right-hand side of (21) is a bounded functional on H. To prove this, we use Holder's inequality to obtain

$$\left|\int_\Omega f(x)\, g(x)\, dx\right| \le \|f\|_{L^2(\Omega)} \times\ \|g\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}\ \|g\|_H$$

and

$$\left|\iint_{\Omega\times\Omega} \frac{[\Phi(x) - \Phi(y)]\, [g(x) - g(y)]}{|x-y|^{n+2\alpha}}\, \omega(x)\, dxdy\right| \le \|\Phi\|_H \times\ \|g\|_H.$$

Therefore, the right-hand side of (21) is a bounded linear functional on H. Then, by the Lax-Milgram principle, there exists a unique solution $z \in H$ of the problem (19)-(20).

Therefore, assuming $f \in L_2(\Omega)$ and $\varphi \in \urcorner r\left(W^\alpha_{2,\omega}\ (\Omega)\right)$ (the class of functions $\varphi : \mathbb{R}^n\setminus\Omega \to \mathbb{R}$ for which there exists an extention to the function $\Phi \in W^\alpha_{2,\ \omega}(\mathbb{R}^n)$ such that $\Phi = \varphi$ on Ω), we see that the right-hand side of (8) is a bounded functional on H. Therefore, we can use the Lax-Milgram principle. Applying it, we get a unique function $z \in \dot{W}^\alpha_{2,\omega}(\Omega)$ satisfying the identity (8). In other words, we get a solution of the problem (6)-(7).

Now, substituting the found function $z \in \dot{W}^\alpha_{2,\omega}(\Omega)$ into (1)-(2) and using the presentation u=$z + \Phi$, we get

$$\iint_{\Omega\times\Omega} \frac{[u(x) - \Phi(x) - u(y) + \Phi(y)]\, [g(x) - g(y)]}{|x-y|^{n+2\alpha}}\, \omega(x)\, dxdy = \int_\Omega f(x)\, g(x)\, dx$$

$$-\iint_{\Omega\times\Omega} \frac{[\Phi(x) - \Phi(y)]\, [g(x) - g(y)]}{|x-y|^{n+2\alpha}}\, \omega(x)\, dxdy.$$

After some simplifying, we arrive at the conclusion that there exists a unique $u \in W^{\alpha}_{2,\ \omega}(\mathbb{R}^n)$ such that $u - \Phi \in \dot{W}^{\alpha}_{2,\ \omega}(\Omega)$ and

$$\iint_{\Omega\times\Omega} \frac{[u(x) - u(y)]\ [g(x) - g(y)]}{|x - y|^{n+2\alpha}}\ \omega(x)\, dxdy = \int_{\Omega} f(x)\, g(x)\, dx \qquad (22)$$

for all $g \in \dot{W}^{\alpha}_{2,\ \omega}(\Omega)$, i.e. $u(x)$ is a unique solution of the problem (1)-(2).

So we have proved the following main result of this work:

Theorem 1. *Let* $f \in L_2(\Omega)$, $\varphi \in \neg r\left(W^{\alpha}_{2,\omega}(\Omega)\right)$, $0 < \alpha < 1$, $p > 1$, *and the positive measurable function* $\omega :\ \mathbb{R}^n \to (0,\ \infty)$ *satisfy* (15). *Then, for any pair of functions* $(f,\ \varphi)$ *there exists a unique weak solution* $u \in W^{\alpha}_{2,\ \omega}(\mathbb{R}^n)$, *which solves the problem* (1)-(2) *in the sense of* (5).

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