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Weighted Sobolev-Morrey Regularity of Solutions to Variational Inequalities

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Abstract. We establish a global generalized weighted Sobolev-Morrey $W^1M^{p,\varphi}_w$-regularity for solutions to variational inequalities and obstacle problems for divergence form elliptic systems with measurable coefficients in bounded non-smooth domains.

Key Words and Phrases: elliptic obstacle problem, generalized weighted Morrey estimates, measurable coefficients, Reifenberg flat domain, small BMO.

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1. Introduction and main result

Obstacle problems are a classical topic in the regularity theory of partial differential equations. They arise naturally in the classical elasticity theory as one of the simplest unilateral problems in the mechanics of elastic membranes. Applications of obstacle problems include fluid filtration in porous media, constrained heating, elasto-plasticity, stopping time optimal control problem for Brownian motion, phase transitions, groundwater hydrology, financial mathematics, etc. We refer to [8, 11, 15, 16, 23, 29, 35] for a further discussion on the obstacle problems and their applications.

Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$. Given a vector-valued function

$$\psi = (\psi^1, \dots, \psi^m) \in H^1(\Omega, \mathbb{R}^m) \quad \text{and} \quad \psi^i \leq 0 \quad \text{on} \quad \partial\Omega, \quad i = 1, \dots, m, \tag{1}$$

define the admissible set for the test functions:

$$\mathcal{A} = \{\phi = (\phi^1, \dots, \phi^m) \in H^1_0(\Omega, \mathbb{R}^m) : \phi^i \geq \psi^i \quad \text{a.e. in} \quad \Omega, \quad i = 1, \dots, m\}.$$

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Hereafter we adopt the standard summation convention on the repeated indices, with $1 \le \alpha, \beta \le n$ and $1 \le i, j \le m$, where $m \ge 2$.

We are interested in functions $u : \Omega \to \mathbb{R}^m$ lying in $\mathcal{A}$ such that

$$\int_\Omega A_{ij}^{\alpha\beta}(x) D_\beta u^j \cdot D_\alpha(\phi^i - u^i) dx \ge \int_\Omega f_i^\alpha \cdot D_\alpha(\phi^i - u^i) dx \tag{2}$$

for all $\phi \in \mathcal{A}$, where $F = \{f_i^\alpha\} \in L^2(\Omega, \mathbb{R}^{mn})$. Such a function u is called a weak solution to the variational inequality (2).

Throughout this article, the tensor coefficients $A_{ij}^{\alpha\beta} : \mathbb{R}^n \to \mathbb{R}^{mn\times mn}$ are assumed to be uniformly elliptic and uniformly bounded, namely, we suppose that there exist positive constants λ and Λ such that

$$\lambda|\xi|^2 \le A_{ij}^{\alpha\beta}(x)\xi_\alpha^i \xi_\beta^j \quad \text{and} \quad \|A_{ij}^{\alpha\beta}\|_{L^\infty(\mathbb{R}^n, \mathbb{R}^{mn\times mn})} \le \Lambda \tag{3}$$

for all matrices $\xi \in \mathbb{R}^{mn}$ and for almost every point $x \in \mathbb{R}^n$.

According to classical theory of the variational inequalities ([14, 23]), there exists a unique weak solution $u \in \mathcal{A}$ of (2) satisfying the estimate

$$\|Du\|_{L^2(\Omega,\mathbb{R}^{nm})} \le c\Big(\|F\|_{L^2(\Omega,\mathbb{R}^{nm})} + \|D\psi\|_{L^2(\Omega,\mathbb{R}^{nm})}\Big)$$

with a positive constant c depending only on λ, Λ, m, and the Lebesgue measure $|\Omega|$ of the underlying domain Ω.

The classical Morrey spaces $L^{p,\lambda}$ were introduced by Morrey [28] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. The first author, Mizuhara and Nakai [17, 27, 30] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see also [1, 18, 37]). Komori and Shirai [24] defined weighted Morrey spaces $L^{p,\kappa}(w)$. In [19], the first author defined the generalized weighted Morrey spaces $M_w^{p,\varphi}(\mathbb{R}^n)$, which could be viewed as extension of both $M^{p,\varphi}(\mathbb{R}^n)$ and $L^{p,\kappa}(w)$, and proved the boundedness of the classical operators and their commutators in $M_w^{p,\varphi}$ (see also [12, 21, 22, 31]).

The main goal of this article is to derive regularity estimates for the weak solution to the variational inequality (2) for divergence form elliptic systems with measurable coefficients in non-smooth domains in the framework of $M_w^{p,\varphi}(\Omega)$. In that sense, they provide a natural extension of the results in [2, 5, 7, 9, 25]. We are dealing with differential operators having only measurable coefficients. More precisely, the associated coefficients are only measurable in one variable, allowing this way quite arbitrary discontinuities in that direction, while the coefficients are averaged in the sense of small BMO with respect to the remaining $n-1$ variables. This is a typical situation closely related to the equilibrium equations of linearly elastic laminates and composite materials which have been widely applied to various fields, see [10, 26]. Regarding the non-smooth domains considered here, we suppose these have boundaries which are flat in the sense of Reifenberg [34]. This means that the boundary is well approximated by hyperplanes at each point and at each scale, and is a sort of minimal regularity of the boundary guaranteeing the main results of the geometric analysis continue to hold true in the non-smooth domain considered. For instance, C^1-smooth or Lipschitz continuous boundaries belong to that category, but the class of Reifenberg flat domains extends beyond these common examples and contains domains with rough fractal boundaries such as the von Koch snowflake.

Under additional regularity assumptions on the coefficients in (2) and a suitable geometric condition on the boundary of Ω, we will show that for all $p \in (1, \infty)$

$$|Du|^2 \in M_w^{p,\varphi}(\Omega)$$

provided

$$w \in A_p, \quad |F|^2 \in M_w^{p,\varphi}(\Omega) \ \text{ and } \ |D\psi|^2 \in M_w^{p,\varphi}(\Omega).$$

In order to state the additional hypotheses on $A_{ij}^{\alpha\beta}$ and $\partial\Omega$, we need to introduce the following notations:

(1) The open ball in $\mathbb{R}^n$ centered at a point y and of radius $r > 0$:

$$B \equiv B_r(x) = \{y \in \mathbb{R}^n : \ |x - y| < r\}$$

with Lebesgue measure $|B_r| = c(n)r^n$. For each $x \in \Omega$ we write

$$\Omega_r = B_r(x) \cap \Omega, \ 2B_r = B_{2r}(x) \text{ and } \complement(2B_r) = \mathbb{R}^n \setminus 2B_r.$$

The open ball in $\mathbb{R}^{n-1}$ centered at $y' = (y_1, \ldots, y_{n-1})$ and of radius $r > 0$:

$$B_r'(y) = \{x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : |x' - y'| < r\}$$

with $|B_r'| = c(n)r^{n-1}$.

(2) The elliptic cylinder in $\mathbb{R}^n$ centered at $y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and of size $r > 0$ is defined by

$$Q_r(y) = B_r'(y') \times (y_n - r, y_n + r)$$

with $|\mathcal{C}_r| = c(n)r^n$. If the center is the origin $0 = (0', 0)$, then we denote $Q_r(0) = B_r'(0') \times (-r, r)$ by $Q_r = B_r' \times (-r, r)$ for the sake of simplicity.

(3) For each fixed $x_n \in \mathbb{R}$ and for each non empty bounded subset E' of $\mathbb{R}^{n-1}$, the integral average of a function $g(\cdot, x_n)$ over E' is denoted by

$$\bar{g}_{E'}(x_n) = \fint_{E'} g(x', x_n)dx' = \frac{1}{|E'|}\int_{E'} g(x', x_n)dx',$$

where $|E'|$ stands for the $(n-1)$-dimensional Lebesgue measure of E'.

The main assumptions on the data of Problem (2) are given in the next definition.

Definition 1. *We say that* $(A_{ij}^{\alpha\beta}, \Omega)$ *is* (δ, R)*-vanishing of codimension one if for every point* $y \in \Omega$ *and for every number* $r \in (0, 3R]$ *with*

$$dist(y, \partial\Omega) = \min_{x\in\partial\Omega} dist(y, x) > \sqrt{2}r,$$

there exists a coordinate system depending on y *and* r *with variables still denoted by* $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$*, so that in this new coordinate system* y *is the origin and*

$$\fint_{Q_{\sqrt{2}r}} \left|A_{ij}^{\alpha\beta}(x', x_n) - \overline{A_{ij}^{\alpha\beta}}_{B'_{\sqrt{2}r}}(x_n)\right|^2 dx \leq \delta^2.$$

Further on, for every point $y \in \Omega$ and for every number $r \in (0, 3R]$ with

$$dist(y, \partial\Omega) = \min_{x\in\partial\Omega} dist(y, x) = dist(y, x_0) \leq \sqrt{2}r,$$

for some $x_0 \in \partial\Omega$, there exists a coordinate system depending on y and r, whose variables we still denote by $x = (x', x_n)$, so that in this new coordinate system x_0 is the origin,

$$Q_{3r} \cap \{(x', x_n) : x_n > 6r\delta\} \subset Q_{3r} \cap \Omega \subset Q_{3r} \cap \{(x', x_n) : x_n > -6r\delta\} \qquad (4)$$

and

$$\fint_{Q^+_{3r}} |A_{ij}^{\alpha\beta}(x', x_n) - \overline{A_{ij}^{\alpha\beta}}_{B'_{3r}}(x_n)|^2 dx \leq \delta^2.$$

Some remarks in order to clarify the notion just introduced. If $(A_{ij}^{\alpha\beta}, \Omega)$ is (δ, R) -vanishing of *codimension 1*, then, for each point and for each small scale, there is a coordinate system such that the coefficients have *small bounded mean oscillation* (briefly BMO) in the x'-directions with *no regularity* required with respect to x_n, that is, the coefficients can be only *measurable* in x_n.

For what concerns the condition (4), it means that $\partial\Omega$ satisfies the so-called (δ, R)-Reifenberg flat condition (see [34, 38]). Moreover, it implies (cf. [6, 32, 33]) that there is a constant $\tau = \tau(\delta, n, \partial\Omega) > 0$ such that

$$\tau|Q_r(y_0)| \leq |Q_r(y_0) \cap \Omega| \leq (1-\tau)|Q_r(y_0)|$$

for each cylinder $Q_r(y_0)$ with $r > 0$ and $y_0 \in \partial\Omega$.

The constant δ will be determined later to belong to $(0, 1/8)$. Here we would like to emphasize only that δ is invariant under a scaling (see Lemma 11 below). Moreover, by means of the scaling invariant property of Problem (2), the constant R can be any constant greater than or equal to 1.

Finally, the numbers $\sqrt{2}r$ and $\sqrt{3}r$ are selected artificially, since we need to take the size of a cylinder $Q_r(y)$ large enough to contain its rotation in any direction.

We give our main result in the following theorem.

Theorem 2. *Assume that inequalities* (3) *are satisfied and* $u \in H_0^1(\Omega, \mathbb{R}^n)$ *is a weak solution to the variational inequality* (2). *For any given* $p \in (1, \infty)$, *let* $w \in A_p$, $|F|^2 \in M_w^{p,\varphi}(\Omega)$ *and* $|D\psi|^2 \in M_w^{p,\varphi}(\Omega)$. *Then there exists a constant* $\delta = \delta(\lambda, \Lambda, m, n, p, [w]_p, \Omega) > 0$ *such that if* $(A_{ij}^{\alpha\beta}, \Omega)$ *is* (δ, R)*-vanishing of codimension* 1, *then* $|Du|^2 \in M_w^{p,\varphi}(\Omega)$ *with the estimate*

$$\||Du|^2\|_{M_w^{p,\varphi}(\Omega)} \le C\Big(\||F|^2\|_{M_w^{p,\varphi}(\Omega)} + \||D\psi|^2\|_{M_w^{p,\varphi}(\Omega)}\Big), \tag{5}$$

where C *is a positive constant depending only on* λ, Λ, m, n, p, $[w]_p$ *and* Ω.

This paper is organized as follows. In section 2, we present some auxiliary results related to the Hardy-Littlewood maximal function, measure theory and the Krylov-Safonov type covering argument. In section 3, we prove the global generalized weighted Sobolev-Morrey $W^1M_w^{p,\varphi}$-regularity for solutions to variational inequalities and obstacle problems for divergence form elliptic systems with measurable coefficients in bounded non-smooth domains (Theorem 2).

2. Preliminaries

In this section, we present several preliminary results to be used for the rest of this article. Firstly, let us recall the definition of the Muckenhoupt classes A_p weights. A positive locally integrable function w on $\mathbb{R}^n$ is said to be a weight. The weight $w = w(x)$ belongs to the Muckenhoupt class $A_p(\mathbb{R}^n)$, $1 < p < \infty$, if

$$[w]_p := \sup \Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{\frac{-1}{p-1}}dx\Big)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. A typical example of Muckenhoupt class A_p is given by the function

$$w_s(x) = |x|^s, \quad x \in \mathbb{R}^n,$$

and it is easy to see that $w_s \in A_p$ when $-n < s < n(p-1)$.

Later on, for any bounded measurable set $E \subset \mathbb{R}^n$ and a weight w, we define the weighted Lebesgue measure $w(E)$ by

$$w(E) = \int_E w(x)dx.$$

The next lemma plays an important role in what follows and the corresponding proof can be found in [36, 39].

Lemma 3. *Let $w \in A_p(\mathbb{R}^n)$ for some $1 < p < \infty$. Then there exist positive constants c_1 and $k \in (0,1)$, depending only on n, p and $[w]_p$, such that*

$$\frac{1}{c_1}\Big(\frac{|E|}{|Q|}\Big)^p \le \frac{w(E)}{w(Q)} \le c_1\Big(\frac{|E|}{|Q|}\Big)^k \tag{6}$$

for every cube $Q \subset \mathbb{R}^n$ and every measurable subset E of Q.

It is worth noting that this result relies on a reverse Holder type inequality. Moreover, (6) implies that a weight $w \in A_p$ has the doubling property, that is,

$$w(2Q) \le c_2 w(Q), \quad c_2 = c_2(n, p, [w]_p) > 0.$$

Given a weight $w \in A_p$ for some $1 < p < \infty$, the weighted Lebesgue space $L^p_w(\Omega)$ is defined as the collection of all measurable functions $g : \Omega \to \mathbb{R}$ satisfying

$$\|g\|_{L^p_w(\Omega)} = \Big(\int_\Omega |g(x)|^p w(x)dx\Big)^{\frac{1}{p}} < \infty.$$

Definition 4. *Let Ω be an open domain in $\mathbb{R}^n$ and $p \in (1,\infty)$. A function $f \in L^p_w(\Omega)$, $w \in A_p$, belongs to the generalized weighted Morrey space $M^{p,\varphi}_w(\Omega)$ if the following norm is finite:*

$$\|f\|_{M^{p,\varphi}_w(\Omega)} = \sup_{x\in\Omega, r>0} \frac{1}{\varphi(x,r)}\Big(\frac{1}{w(B_r(x))}\int_{\Omega_r} |f(x)|^p w(x)dx\Big)^{\frac{1}{p}} < \infty, \tag{7}$$

where φ is a measurable non-negative function defined on Ω (see [19]).

If $w \equiv 1$, then $M^{p,\varphi}_w(\Omega) \equiv M^{p,\phi}(\Omega)$ with $\phi(B_r(x)) = \varphi(B_r(x))^p r^n$. If $\varphi \equiv r^{(\lambda-n)/p}$ and $w \equiv 1$, then $M^{p,\varphi}_w(\Omega) \equiv L^{p,\lambda}(\Omega)$, $\lambda \in (0,n)$. If $\varphi \equiv w^{-1/p}$, then $M^{p,\varphi}_w(\Omega) \equiv L^p_w(\Omega)$.

The main components of our approach are the Hardy-Littlewood maximal operator and the Krylov-Safonov type covering lemma. In the following, we give the definition of the Hardy-Littlewood maximal operator $\mathcal{M}$.

Definition 5. *Given a locally integrable function g defined in $\mathbb{R}^n$, the maximal function $\mathcal{M}g$ of g is*

$$(\mathcal{M}g)(x) = \sup_{r>0} \fint_{Q_r(x)} |g(y)|dy = \sup_{r>0} \frac{1}{|Q_r(x)|} \int_{Q_r(x)} |g(y)|dy.$$

If g is defined on a bounded subset of $\mathbb{R}^n$, then

$$\mathcal{M}g = \mathcal{M}\overline{g},$$

where $\overline{g}$ is the zero extension of g from the bounded set to $\mathbb{R}^n$.

The *weak type estimate*

$$|\{x \in \mathbb{R}^n : (\mathcal{M}g)(x) > \lambda\}| \le \frac{c(n)}{\lambda} \int_{\mathbb{R}^n} |g(x)|dx$$

is well known for the maximal operator $\mathcal{M}$.

Lemma 6. *(see [36, 39]). Suppose that $w \in A_p$ for some $1 < p < \infty$ and $g \in L^p_w(\mathbb{R}^n)$. Then $\mathcal{M}g \in L^p_w(\mathbb{R}^n)$ and there is a constant $c_3 = c_3(n, p, [w]_p) > 0$ such that*

$$\frac{1}{c_3} \|g\|_{L^p_w(\mathbb{R}^n)} \le \|\mathcal{M}g\|_{L^p_w(\mathbb{R}^n)} \le c_3 \|g\|_{L^p_w(\mathbb{R}^n)}. \tag{8}$$

In some content, the condition $w \in A_p$ is necessary and sufficient for the validity of the inequality (8). So, we expect that the Muckenhoupt A_p class is optimal, in terms of weights, for the *Calderón-Zygmund type estimates* here obtained.

In [19], the following maximal inequality in weighted generalized Morrey spaces $M^{p,\varphi}_w$ under quite general condition on the pair (φ, w) was proved.

Theorem 7. *[19] Let $w \in A_p$, $p \in (1, \infty)$ and (φ_1, φ_2) be a couple of non-negative measurable functions defined on $\mathbb{R}^n \times \mathbb{R}_+$. Assume that there is a positive constant c_1 independent on y and r such that*

$$\sup_{r<s<\infty} \frac{\operatorname*{ess\,inf}_{s<\sigma<\infty} \varphi_1(B_\sigma(y)) w\big(B_\sigma(y)\big)^{\frac{1}{p}}}{w\big(B_s(y)\big)^{\frac{1}{p}}} \le c_1 \, \varphi_2(B_r(y)) \, . \tag{9}$$

Then the operator $\mathcal{M}$ is bounded from M^{p,φ_1}_w to M^{p,φ_2}_w and

$$\|\mathcal{M}f\|_{M^{p,\varphi_2}_w(\mathbb{R}^n)} \le c \|f\|_{M^{p,\varphi_1}_w(\mathbb{R}^n)} \, .$$

Corollary 8. *(Maximal inequality) [19] Let* $w \in A_p$*,* $p \in (1,\infty)$ *and* φ *satisfy*

$$\sup_{r<s<\infty} \frac{\operatorname{ess\,inf}_{s<\sigma<\infty} \varphi(B_\sigma(y)) w\big(B_\sigma(y)\big)^{\frac{1}{p}}}{w\big(B_s(y)\big)^{\frac{1}{p}}} \le c_1\, \varphi(B_r(y)) \tag{10}$$

with c_1 *independent of* r *and* y*. Then there is a constant* $c_p > 0$ *such that*

$$\|f\|_{M^{p,\varphi}_w(\mathbb{R}^n)} \le \|\mathcal{M}f\|_{M^{p,\varphi}_w(\mathbb{R}^n)} \le c_p \|f\|_{M^{p,\varphi}_w(\mathbb{R}^n)}, \quad \forall\ f \in M^{p,\varphi}_w(\mathbb{R}^n)\,.$$

Impose in addition a kind of monotonicity condition on φ, precisely,

$$\varphi(B_r(y))^p w\big(B_r(y)\big) \le \varphi(B_s(z))^p w\big(B_s(z)\big) \ \text{ for all } \ B_r(y) \subset B_s(z)\,. \tag{11}$$

This implies that for a given $\Omega \subset \mathbb{R}^n$, the inequality

$$\sup_{\substack{y\in\Omega\\ r>0}} \frac{w\big(B_r(y)\cap\Omega\big)}{\varphi\big(B_r(y)\big)^p\ w\big(B_r(y)\big)} \le c_2 \tag{12}$$

holds with $c_2 = c_2(n,q,\kappa,\varphi,w,\Omega)$ (see [20]).

We will need also the following standard measure theory results regarding weighted spaces.

Lemma 9. *(see [13]). Let* $f \in L_1(\Omega)$ *be a nonnegative function,* w *be an* A_q*-weight,* $q \in (1,\infty)$*,* φ *be a weight satisfying* (9)*, and* $\theta > 0$ *and* $\lambda > 1$ *be constants. Then* $f \in M^{q,\varphi}_w(\Omega)$ *if and only if*

$$S := \sup_{y\in\Omega,\ r>0} \sum_{k\ge1} \frac{\lambda^{kq} w(\{x\in\Omega_r : f(x) > \theta\lambda^k\})}{\varphi(B_r(y))^q w(B_r(y))} < \infty.$$

Moreover,

$$\frac{1}{c_4} S \le \|f\|^q_{M^{q,\varphi}_w(\Omega)} \le c_4(1+S),$$

for some universal constant $c = c(\theta,\lambda,q,k,\varphi,w,\Omega)$*.*

Lemma 10. *(see [5]). Let* Ω *be a bounded and* $(\delta,1)$*-Reifenberg flat set in* $\mathbb{R}^n$ *and let* C *and* D *be measurable sets such that* $C \subset D \subset \Omega$*. Let* $w \in A_p$ *for some* $p \in (1,\infty)$ *and suppose there exists a small constant* $\epsilon \in (0,1)$ *such that*

$$w\Big(C \cap Q_1(y)\Big) < \epsilon w(Q_1(y))$$

for each $y \in \Omega$*. Assume further that for each* $y \in \Omega$ *and* $r \in (0,1)$ *one has*

$$Q_r(y)\cap\Omega \subset D \quad \textit{whenever} \quad w(C\cap Q_r(y)) \ge \epsilon w(Q_r(y)).$$

Then

$$w(C) \leq c_5 \epsilon w(D)$$

with a constant c_5 depending only on δ, n, p, and $[w]_p$. Moreover, taking δ in the range $(0, 1/8)$, the constant c_5 may be bounded by a uniform constant independent of δ.

We will use the fact that the obstacle problem here considered is invariant under scaling and normalization in the proof of our main theorem. The corresponding properties follow by straightforward computations.

Lemma 11. *[5] Let $u \in \mathcal{A}$ be the weak solution to the variational inequality (2). Assume that $(A_{ij}^{\alpha\beta}, \Omega)$ is (δ, R)-vanishing of codimension 1. Fix $M > 1$ and $0 < \rho < 1$, and define the rescaled maps*

$$\widetilde{A_{ij}^{\alpha\beta}}(x) = A_{ij}^{\alpha\beta}(\rho x), \quad \tilde{u}(x) = \frac{u(\rho x)}{M\rho}, \quad \tilde{F}(x) = \frac{F(\rho x)}{M}, \quad \tilde{\psi}(x) = \frac{\psi(\rho x)}{M\rho},$$

and the set $\widetilde{\Omega} = \left\{ \frac{1}{\rho} x : x \in \Omega \right\}$.

Then

(1) *$\widetilde{A_{ij}^{\alpha\beta}}$ satisfies the basic condition (1) with the same constants λ and Λ.*

(2) *$(\widetilde{A_{ij}^{\alpha\beta}}, \widetilde{\Omega})$ is $\left(\delta, \frac{R}{\rho}\right)$-vanishing of codimension 1.*

(3) *$\tilde{u} \in \tilde{\mathcal{A}} = \left\{ \phi \in H_0^1(\widetilde{\Omega}, \mathbb{R}^m) : \phi^i \geq \tilde{\psi}^i \ \text{ a.e. in } \ \widetilde{\Omega} \ \text{ for each } \ i = 1, \ldots, m \right\}$ is the weak solution to the variational inequality*

$$\int_{\widetilde{\Omega}} \widetilde{A_{ij}^{\alpha\beta}}(x) D_\beta \tilde{u}^j \cdot D_\alpha(\tilde{\phi}^i - \tilde{u}^i) dx \geq \int_{\widetilde{\Omega}} \tilde{f}_i^\alpha \cdot D_\alpha(\tilde{\phi}^i - \tilde{u}^i) dx, \quad \forall \tilde{\phi} \in \tilde{\mathcal{A}}.$$

3. Weighted Sobolev-Morrey $W^1 M_w^{p,\varphi}$ estimates

In this section, we will obtain the optimal weighted Sobolev-Morrey $W^1 M_w^{p,\varphi}$ regularity for the weak solution to the variational inequality (2) based on Lemma 10. So, let $u \in \mathcal{A}$ be the weak solution to (2) and, for the fixed $p \in (1, \infty)$, let w, F and ψ satisfy

$$w \in A_p, \quad |F|^2 \in M_w^{p,\varphi}(\Omega) \quad \text{and} \quad |D\psi|^2 \in M_w^{p,\varphi}(\Omega).$$

In what follows, we will use the letter c to denote a constant that can be explicitly computed in terms of known quantities such as λ, Λ, m, n, p, and $[w]_p$.

Now, in order to apply Lemma 10 to our situation, we need the following result.

Lemma 12. *[5] There exists a constant $N = N(\lambda, \Lambda, m, n) > 1$ such that for each $0 < \epsilon < 1$ one can select a small $\delta = \delta(\epsilon, \lambda, \Lambda, m, n) > 0$ such that if $(A_{ij}^{\alpha\beta}, \Omega)$ is (δ, R)-vanishing of codimension 1 and if $Q_r(y)$, with $y \in \Omega$ and $r > 0$, satisfies*

$$w\Big(\{x \in \Omega : \mathcal{M}(|Du|^2) > N^2\} \cap Q_r(y)\Big) \geq \epsilon w(Q_r(y)) \tag{13}$$

for such a small δ, then

$$\begin{aligned}\Omega_r(y) \subset \{x \in \Omega : \mathcal{M}(|Du|^2) > 1\} \cup \{x \in \Omega : \mathcal{M}(|F|^2) > \delta^2\} \\ \cup \{x \in \Omega : \mathcal{M}(|D\psi|^2) > \delta^2\},\end{aligned} \tag{14}$$

where $\Omega_r(y) = Q_r(y) \cap \Omega$.

Fix now $\epsilon > 0$ and take δ and N as given in Lemma 12. Based on Lemma 10, we will obtain a power decay of

$$w(\{x \in \Omega : \mathcal{M}(|Du|^2) > N^2\}).$$

Lemma 13. *[5] Suppose that $(A_{ij}^{\alpha\beta}, \Omega)$ is (δ, R)-vanishing of codimension 1 and set $\epsilon_* = c_5\epsilon$ with c_5 given by Lemma 10.*

Then for each positive integer k, we have

$$\begin{aligned}w(\{x \in \Omega : \mathcal{M}(|Du|^2) > N^{2k}\}) \leq \epsilon_*^k w(\{x \in \Omega : \mathcal{M}(|Du|^2) > 1\}) \\ + \sum_{i=1}^{k} \epsilon_*^i w(\{x \in \Omega : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\}) \\ + \sum_{i=1}^{k} \epsilon_*^i w(\{x \in \Omega : \mathcal{M}(|D\psi|^2) > \delta^2 N^{2(k-i)}\}).\end{aligned}$$

Now we can give the proof of Theorem 2.

Proof of Theorem 2. We apply Lemma 9 with $g = \mathcal{M}(|Du|^2)$, $\theta = N$ and $\mu = 1$. Thus,

$$\begin{aligned}&\sum_{k=1}^{\infty} N^{kp} w(\{x \in \Omega : \mathcal{M}(|Du|^2) > N^{2k}\}) \\ &\overset{Lemma\,13}{\leq} \sum_{k=1}^{\infty} N^{kp} \epsilon_*^k w(\{x \in \Omega : \mathcal{M}(|Du|^2) > 1\}) \\ &+ \sum_{k=1}^{\infty} N^{kp} \sum_{k=1}^{k} \epsilon_*^i w(\{x \in \Omega : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\})\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{\infty} N^{kp} \sum_{k=1}^{k} \epsilon_*^i w(\{x\in\Omega : \mathcal{M}(|D\psi|^2) > \delta^2 N^{2(k-i)}\}) \\
&\leq \sum_{k=1}^{\infty} (N^p \epsilon_*)^k w(\Omega) \\
&+\sum_{k=1}^{\infty} (N^p \epsilon_*)^i \Big(\sum_{k=i}^{\infty} N^{(k-i)p} w\big(\{x\in\Omega : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\}\big)\Big) \\
&+\sum_{k=1}^{\infty} (N^p \epsilon_*)^i \Big(\sum_{k=i}^{\infty} N^{(k-i)p} w\big(\{x\in\Omega : \mathcal{M}(|D\psi|^2) > \delta^2 N^{2(k-i)}\}\big)\Big) \\
&\overset{Lemma\,9}{\leq} C\Big(w(\Omega) + \|\mathcal{M}(|F|^2)\|^p_{M_w^{p,\varphi}(\Omega)} + \|\mathcal{M}(|D\psi|^2)\|^p_{M_w^{p,\varphi}(\Omega)}\Big) \sum_{k=1}^{\infty} (N^p\epsilon_*)^k \\
&\overset{Lemma\,6}{\leq} C\Big(w(\Omega) + \||F|^2\|^p_{M_w^{p,\varphi}(\Omega)} + \||D\psi|^2\|^p_{M_w^{p,\varphi}(\Omega)}\Big) \sum_{k=1}^{\infty} (N^p\epsilon_*)^k
\end{aligned}
$$

for some universal constant $C = C(\delta, \lambda, \Lambda, m, n, p, [w]_p) > 0$.

Select now a small enough $\epsilon > 0$ in order to have $N^p \epsilon_* < 1$. In view of Lemma 12, we can find a small constant $\delta = \delta(\lambda, \Lambda, m, n, p, [w]_p)$ such that

$$
\begin{aligned}
&\sum_{k=1}^{\infty} N^{kp} w(\{x\in\Omega : \mathcal{M}(|Du|^2) > N^{2k}\}) \\
&\leq C\Big(w(\Omega) + \||F|^2\|^p_{M_w^{p,\varphi}(\Omega)} + \||D\psi|^2\|^p_{M_w^{p,\varphi}(\Omega)}\Big)
\end{aligned}
$$

whenever $(A_{ij}^{\alpha\beta}, \Omega)$ is (δ, R)-vanishing of codimension 1 with the fixed small δ. Therefore, it follows from Lemmas 6 and 9 that

$$
\||Du|^2\|^p_{M_w^{p,\varphi}(\Omega)} \leq C\Big(w(\Omega) + \||F|^2\|^p_{M_w^{p,\varphi}(\Omega)} + \||D\psi|^2\|^p_{M_w^{p,\varphi}(\Omega)}\Big),
$$

which implies the desired estimate (5) by virtue of the Banach inverse mapping theorem or after normalization (see [3]). This completes the proof of the theorem. ◀

Remark 14. *Note that Theorem 2 is proved in [4] for the weighted Sobolev $W^{1,p}$ case.*

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