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Some Inequalities-Equalities Concerning Continuous Generalized Fusion Frames in Hilbert Spaces

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Abstract. Continuous generalized fusion frame theory was recently introduced by Rahimi *et al.* [14]. Several equalities and inequalities have been obtained for frame, fusion generalized fusion frame, among others. In the present paper, we continue and extend these results to obtain some important identities and inequalities in the case of continuous generalized fusion frame, Parseval continuous generalized fusion frame, λ-tight continuous generalized fusion frame. Moreover, we obtain some new inequalities for the alternate dual continuous generalized fusion frame. Finally, we obtain frame operator of a pair of Bessel continuous generalized fusion mapping and we derive some results about resolution of identity.

Key Words and Phrases: continuous generalized fusion Frame, Parseval continuous generalized fusion frame, tight continuous generalized fusion frame, resolution of identity.

2010 Mathematics Subject Classifications: 42C15, 42C40, 94A12

1. Introduction

Frames are among the most intensively studied and best understood of all classes of overcomplete bases: one can represent each element in the vector space via a frame. In last few decades this notion has attracted much attention because of its practical applications in many areas such as coding and communications, filter bank theory, and widely used in signal and image processing, among others.

Recently, several generalizations of frames in Hilbert spaces have been proposed. For instance: Fusion frame, g-frame, K-frame, b-frame, for more details see [1, 12, 14, 15].

More recently, Zaghami Farfar *et al.* [17] combined two types of frames which lead to a new concept in the theory of frames $\ll generalized\, fusion\, frame \gg$.

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One of the most important inequalities and useful identities was found by Balan *et al.* [2] as follows:

Theorem 1. *[3] Let $\{f_i\}_{i\in I}$ be a Parceval frame for $\mathcal{H}$. Then*

$$\sum_{i\in J}|\langle f,f_i\rangle|^2-\left\|\sum_{i\in J}\langle f,f_i\rangle f_i\right\|^2=\sum_{i\in J^c}|\langle f,f_i\rangle|^2-\left\|\sum_{i\in J^c}\langle f,f_i\rangle f_i\right\|^2. \tag{1}$$

This is particulary used in the study of signal processing. Inspired by Theorem 1, Zhu and Wu [18] generalized this inequality to an alternate dual frame.

Theorem 2. *[18] Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ and $\{g_i\}_{i\in I}$ be an alternate dual frame of $\{f_i\}_{i\in I}$. Then for any $J\in I$ and $f\in\mathcal{H}$ we have*

$$\left(\sum_{i\in J}\langle f,g_i\rangle\overline{\langle f,f_i\rangle}\right)-\left\|\sum_{i\in J}\langle f,g_i\rangle f_i\right\|^2=\left(\sum_{i\in J^c}\langle f,g_i\rangle\overline{\langle f,f_i\rangle}\right)-\left\|\sum_{i\in J^c}\langle f,g_i\rangle f_i\right\|^2.$$

Later on, this inequality (1) has motivated a large number of authors such as Jian-Zhen Li *et al.* [11] and X.H. Yang *et al.* [16] for various other inequalities related to this inequality, we refer the readers to [2, 3, 4, 5, 9, 10, 17].

Motivated by the aforementioned works, we aim to extend and improve identities and inequalities in the case of continuous generalized fusion frame, Parseval continuous generalized fusion frame, λ-tight continuous generalized fusion frame and continuous generalized fusion pairs.

2. Preliminaries

2.1. Background

Throughout this paper, we adopt the following notations: $\mathcal{H}$ will be a Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$, $Id_{\mathcal{H}}$ the identity operator on $\mathcal{H}$, and $\mathbb{H}$ the collection of all closed subspaces of $\mathcal{H}$. Also, μ is a positive measure and (X,μ) is a measure space. π_V is the orthogonal projection from $\mathcal{H}$ onto a closed subspace $V\subset\mathcal{H}$, and $\omega : X\longrightarrow[0,+\infty)$ will be a measurable mapping such that $\omega\neq 0$ *a.e.*.

Lemma 1. *[13] Let $U\in\mathcal{H}$ be self-adjoint and $T=aU^2+bU+cId_{\mathcal{H}}$ such that $a,b,c\in\mathbb{R}$. Then the following hold:*

i) If $a>0$, then

$$\inf_{\|f\|=1}\langle Tf,f\rangle\geq\frac{4ac-b^2}{4a}.$$

ii) If $a < 0$, then

$$\sup_{\|f\|=1} \langle Tf, f\rangle \leq \frac{4ac - b^2}{4a}.$$

Lemma 2. *[3] If T_1, T_2 are operators on $\mathcal{H}$ satisfying $T_1 + T_2 = Id_{\mathcal{H}}$, then $T_1 + T_2 = T_1^2 + T_2^2$.*

Lemma 3. *[18] Let T_1, T_2 be two linear bounded operators on $\mathcal{H}$ satisfying $T_1 + T_2 = Id_{\mathcal{H}}$. Then $T_1 + T_1^* T_1 = T_2^* - T_2^* T_2$.*

Lemma 4. *[8] Let $V \subseteq \mathcal{H}$ be a closed subspace and T be a linear bounded operator on $\mathcal{H}$. Then*

$$\pi_V T^* = \pi_V T^* \pi_{\overline{TV}}.$$

If T is unitary, i.e., $T^*T = TT^* = Id_{\mathcal{H}}$, then

$$\pi_{\overline{TV}} T = T\pi_V.$$

The following definition is a generalized continuous version of fusion frames proposed and defined by Faroughi and Ahmadi [7]:

Definition 1. *(see [7]) Let $\mathbf{F} : X \to \mathbb{H}$ be such that for each $f \in \mathcal{H}$ the mapping $x \longmapsto \pi_{F(x)}(f)$ is measurable (i.e., is weakly measurable) and let $\{\mathcal{K}_x\}_{x\in X}$ be a collection of Hilbert spaces. For each $x \in X$, suppose that $\Lambda_x \in \mathcal{B}(\mathbf{F}(x), \mathcal{K}_x)$ and put*

$$\Lambda = \{\Lambda_x \in \mathcal{B}(\mathbf{F}(x), \mathcal{K}_x) : \ x \in X\}.$$

Then $(\Lambda, \mathbf{F}, \omega)$ is a continuous g-fusion frame for $\mathcal{H}$ if there exist $0 < A \leq B < \infty$ such that for all $f \in \mathcal{H}$

$$A\|f\|^2 \leq \int_{X_1} \omega^2(x) \|\Lambda_x \pi_{F(x)}(f)\|^2 d\mu(x) \leq B\|f\|^2,$$

where $\pi_{F(x)}$ is the orthogonal projection of $\mathcal{H}$ onto the subspace $F(x)$.

Furthermore, $(\Lambda, \mathbf{F}, \omega)$ is called a tight continuous g-fusion frame for $\mathcal{H}$ if $A = B$, and Parseval if $A = B = 1$, and $(\Lambda, \mathbf{F}, \omega)$ is called a Bessel continuous g-fusion frame for $\mathcal{H}$ if the right inequality holds.

Let $\mathcal{K} = \oplus_{x\in X} \mathcal{K}_x$ and $L^2(X, \mathcal{K})$ be a collection of all measurable functions $\varphi : X \longrightarrow \mathcal{K}$ such that for each $x \in X$ $\varphi(x) \in \mathcal{K}_x$, and

$$\int_X \|\varphi(x)\|^2 d\mu(x) < \infty.$$

The synthesis operator is defined weakly as follows (for more details refer to [7]):

$$T_{\mathbf{F},\Lambda}: \quad L^2(X,\mathcal{K}) \to \mathcal{H},$$
$$\langle T_{\mathbf{F},\Lambda}(\varphi), f\rangle = \int_X \omega(x)\langle \Lambda_x^*(\varphi(x)), h\rangle d\mu(x),$$

where $\varphi \in L^2(X,\mathcal{K})$ and $h \in \mathcal{H}$. It is obvious that $T_{\mathbf{F},\Lambda}$ is linear and by [7, Remark 1.6], $T_{\mathbf{F},\Lambda}$ is a bounded linear operator. Its adjoint, which is called analysis operator, is

$$T^*_{\mathbf{F},\Lambda}: \quad \mathcal{H} \longrightarrow L^2(X,\mathcal{K}),$$
$$T^*_{\mathbf{F},\Lambda} = \omega(.)\Lambda^*_{(.)}\pi_{F(.)}.$$

Assume that

$$S_{\mathbf{F},\Lambda}(f) = T_{\mathbf{F},\Lambda}T^*_{\mathbf{F},\Lambda}(f) = \int_{X_1} \omega^2(x)\pi_{F(x)}\Lambda_x^*\Lambda_x\pi_{F(x)}(f)d\mu(x), \quad f \in \mathcal{H}.$$

Then $S_{\mathbf{F},\Lambda}$ is a bounded, positive, self-adjoint and invertible operator and we have

$$B^{-1}id_{\mathcal{H}} \leq S^{-1}_{\mathbf{F},\Lambda} \leq A^{-1}id_{\mathcal{H}}.$$

So we have the following reconstruction formula for any $f \in \mathcal{H}$,

$$\begin{aligned} f &= \int_X \omega^2(x)\pi_{F(x)}\Lambda_x^*\Lambda_x\pi_{F(x)}S^{-1}_{\mathbf{F},\Lambda}(f)d\mu(x) \\ &= \int_X \omega^2(x)S^{-1}_{\mathbf{F},\Lambda}\pi_{F(x)}\Lambda_x^*\Lambda_x\pi_{F(x)}(f)d\mu(x). \end{aligned} \tag{2}$$

3. Inequalities-equalities for Parseval continuous generalized fusion frame

Let $(\Lambda, \mathbf{F}, \omega)$ be a continuous g-fusion frame for $\mathcal{H}$ with bounds A and B. Denote its canonical dual continuous g-fusion frame by $\tilde{\Lambda} := \Big(S^{-1}_{\mathbf{F},\Lambda}F(x), \Lambda_x\pi_F(x)S^{-1}_{\mathbf{F},\Lambda}, \omega\Big)$. Hence for each $f \in \mathcal{H}$ the reconstruction formula (2) may be written in the form

$$f = \int_X \omega^2(x)\pi_{F(x)}\Lambda_x^*\tilde{\Lambda}_x\pi_{\tilde{F}(x)}(f)d\mu(x) = \int_X \omega^2(x)\pi_{\tilde{F}(x)}\tilde{\Lambda}_x^*\Lambda_x\pi_{F(x)}(f)d\mu(x),$$

where $\tilde{F}(x) := S^{-1}_{\mathbf{F},\Lambda}F(x)$, $\tilde{\Lambda}_x := \Lambda_x\pi_F(x)S^{-1}_{\mathbf{F},\Lambda}$. Thus we obtain

$$\langle S^{-1}_{\mathbf{F},\Lambda}f, f\rangle = \int_{X_1} \omega^2(x)\|\tilde{\Lambda}_x\pi_{\tilde{F}(x)}(f)\|^2 d\mu(x). \tag{3}$$

For any $X_1 \subset X$, we denote $X_1^c = X \setminus X_1$, and we define the following operators:

$$S_{\mathbf{F},\Lambda}^{X_1} f = \int_{X_1} \omega^2(x) \pi_{\mathbf{F}(x)} \Lambda_x^* \tilde{\Lambda}_x \pi_{\tilde{F}(x)}(f) d\mu(x), \quad f \in \mathcal{H},$$

$$\mathcal{M}_{\mathbf{F},\Lambda}^{X_1} f = \int_{X_1} \omega^2(x) \pi_{\mathbf{F}(x)} \Lambda_x^* \Lambda_x \pi_{F(x)}(f) d\mu(x), \quad f \in \mathcal{H}.$$

Obviously, $S_{\mathbf{F},\Lambda} = \mathcal{M}_{\mathbf{F},\Lambda}^{X_1} + \mathcal{M}_{\mathbf{F},\Lambda}^{X_1^c}$, and $\mathcal{M}_{\mathbf{F},\Lambda}^{X_1}$, $\mathcal{M}_{\mathbf{F},\Lambda}^{X_1^c}$ are self-adjoint operators. It is easy to check that $S_{\mathbf{F},\Lambda}^{X_1}$ is a bounded, linear and positive operator. Again, we have

$$S_{\mathbf{F},\Lambda}^{X_1} + S_{\mathbf{F},\Lambda}^{X_1^c} = Id_{\mathcal{H}}.$$

Theorem 3. *Let $f \in \mathcal{H}$. Then*

$$\int_{X_1} \omega^2(x) \langle \tilde{\Lambda}_x \pi_{\tilde{F}(x)}(f), \Lambda_x \pi_{F(x)}(f) \rangle d\mu(x) - \|S_{\mathbf{F},\Lambda}^{X_1} f\|^2 \\ = \int_{X_1^c} \omega^2(x) \overline{\langle \tilde{\Lambda}_x \pi_{\tilde{F}(x)}(f), \Lambda_x \pi_{F(x)}(f) \rangle} d\mu(x) - \|S_{\mathbf{F},\Lambda}^{X_1^c} f\|^2.$$

Proof. For any $f \in \mathcal{H}$, we have

$$\begin{aligned} &\int_{X_1} \omega^2(x) \langle \tilde{\Lambda}_x \pi_{\tilde{F}(x)}(f), \Lambda_x \pi_{F(x)}(f) \rangle d\mu(x) - \|S_{\mathbf{F},\Lambda}^{X_1} f\|^2 \\ &= \langle S_{\mathbf{F},\Lambda}^{X_1} f, f \rangle - \|S_{\mathbf{F},\Lambda}^{X_1} f\|^2 \\ &= \langle S_{\mathbf{F},\Lambda}^{X_1} f, f \rangle - \langle (S_{\mathbf{F},\Lambda}^{X_1})^* S_{\mathbf{F},\Lambda}^{X_1} f, f \rangle \\ &= \langle (Id_{\mathcal{H}} - S_{\mathbf{F},\Lambda}^{X_1})^* S_{\mathbf{F},\Lambda}^{X_1} f, f \rangle = \langle (S_{\mathbf{F},\Lambda}^{X_1^c})^* (Id_{\mathcal{H}} - S_{\mathbf{F},\Lambda}^{X_1^c}) f, f \rangle \\ &= \langle (S_{\mathbf{F},\Lambda}^{X_1^c})^* f, f \rangle - \langle (S_{\mathbf{F},\Lambda}^{X_1^c})^* S_{\mathbf{F},\Lambda}^{X_1^c} f, f \rangle. \end{aligned}$$

This completes the proof. ◄

Furthermore, if we suppose that $(\Lambda, \mathbf{F}, \omega)$ is a Parseval continuous generalized fusion frame, then we can easily obtain the same equality presented in [13] as follows:

Theorem 4. *Assume that $(\Lambda, \mathbf{F}, \omega)$ is a Parseval continuous generalized fusion frame for $\mathcal{H}$. Then for $X_1 \subset X$ and $f \in \mathcal{H}$, the following holds:*

$$\int_{X_1} \omega^2(x) \|\Lambda_x \pi_{F(x)}(f)\|^2 d\mu(x) - \| \int_{X_1} \omega^2(x) \pi_{F(x)} \Lambda_x^* \Lambda_x \pi_{F(x)}(f) d\mu(x) \|^2$$

$$= \int_{X_1^c} \omega^2(x)\|\Lambda_x \pi_{F(x)}(f)\|^2 d\mu(x) - \| \int_{X_1^c} \omega^2(x)\pi_{F(x)}\Lambda_x^*\Lambda_x\pi_{F(x)}(f)d\mu(x)\|^2.$$

Moreover,

$$\int_{X_1^c} \omega^2(x)\|\Lambda_x \pi_{F(x)}(f)\|^2 d\mu(x) - \| \int_{X_1^c} \omega^2(x)\pi_{F(x)}\Lambda_x^*\Lambda_x\pi_{F(x)}(f)d\mu(x)\|^2 \geq \frac{3}{4}\|f\|^2.$$

Proof. Since $(\Lambda, \mathbf{F}, \omega)$ is a Parseval continuous generalized fusion frame for $\mathcal{H}$, and by using the fact that $S_{\mathbf{F},\Lambda}^{X_1}$ and $S_{\mathbf{F},\Lambda}^{X_1^c}$ are commuting, for each $f \in \mathcal{H}$, we have

$$\begin{aligned}&\int_{X_1} \omega^2(x)\|\Lambda \pi_{F(x)}(f)\|^2 d\mu(x) - \| \int_{X_1} \omega^2(x)\pi_{F(x)}\Lambda_x^*\Lambda_x\pi_{F(x)}(f)d\mu(x)\|^2 \\ &= \langle (S_{\mathbf{F},\Lambda}^{X_1^c} + (S_{\mathbf{F},\Lambda}^{X_1^c})^2)f, f\rangle \\ &= \langle (Id_{\mathcal{H}} - S_{\mathbf{F},\Lambda}^{X} + (S_{\mathbf{F},\Lambda}^{X_1^c})^2)f, f\rangle.\end{aligned}$$

By lemma 1 for $a = 1$, $b = -1$ and $c = 1$, the result follows. ◀

Corollary 1. *Let* $(\Lambda, \mathbf{F}, \omega)$ *be a Parseval continuous generalized fusion frame for* $\mathcal{H}$. *Then we have*

$$\begin{aligned}\frac{1}{2}\|f\|^2 \leq \| \int_{X_1} &\omega^2(x)\pi_{F(x)}\Lambda_x^*\Lambda_x\pi_{F(x)}(f)d\mu(x)\|^2 d\mu(x)\|^2 \\ &- \| \int_{X_1^c} \omega^2(x)\pi_{F(x)}\Lambda_x^*\Lambda_x\pi_{F(x)}(f)d\mu(x)\|^2 \leq \frac{3}{2}\|f\|^2,\end{aligned}$$

$$\begin{aligned}\frac{3}{4}\|f\|^2 \leq \int_{X_1} &\omega^2(x)\|\Lambda \pi_{F(x)}(f)\|^2 d\mu(x) \\ &- \| \int_{X_1^c} \omega^2(x)\pi_{F(x)}\Lambda_x^*\Lambda_x\pi_{F(x)}(f)d\mu(x)\|^2 \leq \|f\|^2.\end{aligned}$$

Proof. Observe that

$$\begin{aligned}(S_{\mathbf{F},\Lambda}^{X_1})^2 + (S_{\mathbf{F},\Lambda}^{X_1^c})^2 &= (S_{\mathbf{F},\Lambda}^{X_1})^2 + (S_{\mathbf{F},\Lambda}^{X_1^c})^2 \\ &= 2(S_{\mathbf{F},\Lambda}^{X_1})^2 - 2S_{\mathbf{F},\Lambda}^{X_1} + Id_{\mathcal{H}}.\end{aligned}$$

Applying Lemma 1, we get

$$(S_{\mathbf{F},\Lambda}^{X_1})^2 + (S_{\mathbf{F},\Lambda}^{X_1^c})^2 \geq \frac{1}{2}Id_{\mathcal{H}}.$$

Since $S_{\mathbf{F},\Lambda}^{X_1} - (S_{\mathbf{F},\Lambda}^{X_1})^2 \geq 0$ and

$$\begin{aligned}(S_{\mathbf{F},\Lambda}^{X_1})^2 + (S_{\mathbf{F},\Lambda}^{X_1^c})^2 &= 2(S_{\mathbf{F},\Lambda}^{X_1})^2 - 2S_{\mathbf{F},\Lambda}^{X_1} + Id_{\mathcal{H}} \\ &= Id_{\mathcal{H}} + 2S_{\mathbf{F},\Lambda}^{X_1} - 2(S_{\mathbf{F},\Lambda}^{X_1})^2 + 4((S_{\mathbf{F},\Lambda}^{X_1})^2 - S_{\mathbf{F},\Lambda}^{X_1}),\end{aligned}$$

we have

$$(S_{\mathbf{F},\Lambda}^{X_1})^2 + (S_{\mathbf{F},\Lambda}^{X_1^c})^2 \leq Id_{\mathcal{H}} + 2S_{\mathbf{F},\Lambda}^{X_1} - 2(S_{\mathbf{F},\Lambda}^{X_1})^2.$$

Applying again Lemma 1, we get

$$(S_{\mathbf{F},\Lambda}^{X_1})^2 + (S_{\mathbf{F},\Lambda}^{X_1^c})^2 \leq \frac{3}{2} Id_{\mathcal{H}}.$$

Thus

$$\frac{1}{2} Id_{\mathcal{H}} \leq (S_{\mathbf{F},\Lambda}^{X_1})^2 + (S_{\mathbf{F},\Lambda}^{X_1^c})^2 \leq \frac{3}{2} Id_{\mathcal{H}}.$$

Next, observe that

$$\begin{aligned} S_{\mathbf{F},\Lambda}^{X_1} - (S_{\mathbf{F},\Lambda}^{X_1^c})^2 &= S_{\mathbf{F},\Lambda}^{X_1} - (Id_{\mathcal{H}} - S_{\mathbf{F},\Lambda}^{X_1})^2 \\ &= (S_{\mathbf{F},\Lambda}^{X_1})^2 - S_{\mathbf{F},\Lambda}^{X_1} + Id_{\mathcal{H}}.\end{aligned}$$

Since $S_{\mathbf{F},\Lambda}^{X_1} - (S_{\mathbf{F},\Lambda}^{X_1})^2 \geq 0$ implies

$$\frac{3}{2} Id_{\mathcal{H}} \leq S_{\mathbf{F},\Lambda}^{X_1} + (S_{\mathbf{F},\Lambda}^{X_1})^2 \leq Id_{\mathcal{H}},$$

by Lemma 1, for each $f \in \mathcal{H}$, we get

$$\begin{aligned}&\langle (S_{\mathbf{F},\Lambda}^{X_1} - (S_{\mathbf{F},\Lambda}^{X_1})^2) f, f\rangle = \langle S_{\mathbf{F},\Lambda}^{X_1} f, f\rangle - \langle (S_{\mathbf{F},\Lambda}^{X_1})^2 f, f\rangle \\ &= \int_{X_1} \omega^2(x) \|\Lambda \pi_{F(x)}(f)\|^2 d\mu(x) - \| \int_{X_1^c} \omega^2(x) \pi_{F(x)} \Lambda_x^* \Lambda_x \pi_{F(x)}(f) d\mu(x)\|^2.\end{aligned}$$

The proof is completed. ◄

Corollary 2. *Let $(\Lambda, \mathbf{F}, \omega)$ be a Parseval continuous generalized fusion frame for $\mathcal{H}$. Then*

$$0 \leq S_{\mathbf{F},\Lambda}^{X_1} - (S_{\mathbf{F},\Lambda}^{X_1})^2 \leq \frac{1}{4} Id_{\mathcal{H}}.$$

Proof. Since $S^{X_1}_{\mathbf{F},\Lambda} S^{X_1^c}_{\mathbf{F},\Lambda} = S^{X_1^c}_{\mathbf{F},\Lambda} S^{X_1}_{\mathbf{F},\Lambda}$ and $S^{X_1}_{\mathbf{F},\Lambda}$, $S^{X_1^c}_{\mathbf{F},\Lambda}$ are positive, self-adjoint operators, it follows that $S^{X_1}_{\mathbf{F},\Lambda} S^{X_1^c}_{\mathbf{F},\Lambda}$ is also positive and self-adjoint. Hence, we have

$$0 \leq S^{X_1}_{\mathbf{F},\Lambda} S^{X_1^c}_{\mathbf{F},\Lambda} = S^{X_1}_{\mathbf{F},\Lambda} - (S^{X_1^c}_{\mathbf{F},\Lambda})^2.$$

Applying Lemma 1, we obtain

$$S^{X_1}_{\mathbf{F},\Lambda} - (S^{X_1}_{\mathbf{F},\Lambda})^2 \leq \frac{1}{4} Id_{\mathcal{H}}.$$

This completes the proof. ◄

Since $S_{\mathbf{F},\Lambda}$ (resp. $S^{-1}_{\mathbf{F},\Lambda}$) is a positive operator in $\mathcal{B}(\mathcal{H})$, there exists a unique positive square root $S^{\frac{1}{2}}_{\mathbf{F},\Lambda}$ (resp. $S^{-\frac{1}{2}}_{\mathbf{F},\Lambda}$) which commutes with every operator which commutes with $S_{\mathbf{F},\Lambda}$ (resp. $S^{-1}_{\mathbf{F},\Lambda}$). Therefore, for each $f \in \mathcal{H}$ we have

$$f = S^{-\frac{1}{2}}_{\mathbf{F},\Lambda} S_{\mathbf{F},\Lambda} S^{-\frac{1}{2}}_{\mathbf{F},\Lambda} f = \int_{X_1} \omega^2(x) S^{-\frac{1}{2}}_{\mathbf{F},\Lambda} \pi_{F(x)} \Lambda^*_x \Lambda_x \pi_{F(x)} S^{-\frac{1}{2}}_{\mathbf{F},\Lambda} f \, d\mu(x)$$

and thus, by Lemma 4, we have

$$\begin{aligned}
\|f\|^2 &= \langle \int_{X_1} \omega^2(x) S^{-\frac{1}{2}}_{\mathbf{F},\Lambda} \pi_{F(x)} \Lambda^*_x \Lambda_x \pi_{F(x)} S^{-\frac{1}{2}}_{\mathbf{F},\Lambda} f \, d\mu(x), f \rangle \\
&= \int_{X_1} \omega^2(x) \left\| \Lambda_x \pi_{F(x)} S^{-\frac{1}{2}}_{\mathbf{F},\Lambda} f \right\|^2 d\mu(x) \\
&= \int_{X_1} \omega^2(x) \left\| \Lambda_x \pi_{F(x)} S^{-\frac{1}{2}}_{\mathbf{F},\Lambda} \pi_{S^{-\frac{1}{2}}_{\mathbf{F},\Lambda} F(x)} f \right\|^2 d\mu(x).
\end{aligned}$$

This means that $(S^{-\frac{1}{2}}_{\mathbf{F},\Lambda} F(x), \Lambda_x \pi_{F(x)} S^{-\frac{1}{2}}_{\mathbf{F},\Lambda}, \omega)$ is a Parseval continuous generalized fusion frame. Hence we have the following theorem:

Theorem 5. *Let* $(\Lambda, \mathbf{F}, \omega)$ *be a Parseval continuous generalized fusion frame for* $\mathcal{H}$*. Then*

$$\begin{aligned}
&\int_{X_1} \omega^2(x) \|\Lambda_x \pi_{F(x)}(f)\|^2 d\mu(x) - \|S^{-\frac{1}{2}}_{\mathbf{F},\Lambda} S^{X_1}_{\mathbf{F},\Lambda} f\|^2 \\
&\qquad = \int_{X_1^c} \omega^2(x) \|\Lambda_x \pi_{F(x)}(f)\|^2 d\mu(x) - \|S^{-\frac{1}{2}}_{\mathbf{F},\Lambda} S^{X_1^c}_{\mathbf{F},\Lambda} f\|^2.
\end{aligned}$$

Proof. Assume that $\chi_x := \Lambda_x \pi_{F(x)} S^{-\frac{1}{2}}_{\mathbf{F},\Lambda}$ and $V(x) := S^{\frac{1}{2}}_{\mathbf{F},\Lambda} F(x)$. Then by the previous result that $(S^{-\frac{1}{2}}_{\mathbf{F},\Lambda} F(x), \Lambda_x \pi_{F(x)} S^{-\frac{1}{2}}_{\mathbf{F},\Lambda}, \omega)$ is a Parseval continuous generalized fusion frame, and Corollary 4, we get

$$\int_{X_1} \omega^2(x) \left\|\chi_x \pi_{V(x)} f\right\|^2 d\mu(x) + \left\|\int_{X_1} \omega^2(x) \pi_{V(x)} \chi_x^* \chi_x \pi_{V(x)} f d\mu(x)\right\|^2$$
$$\int_{X_1^c} \omega^2(x) \left\|\chi_x \pi_{V(x)} f\right\|^2 d\mu(x) + \left\|\int_{X_1^c} \omega^2(x) \pi_{V(x)} \chi_x^* \chi_x \pi_{V(x)} f d\mu(x)\right\|^2 .$$

Moreover, we have

$$\begin{aligned}
&\int_{X_1} \omega^2(x) \pi_{V(x)} \chi_x^* \chi_x \pi_{V(x)} f d\mu(x) \\
&= \int_{X_1} \omega^2(x) \left(\chi_x \pi_{V(x)}\right)^* \chi_x \pi_{V(x)} f d\mu(x) \\
&= \int_{X_1} \omega^2(x) \left(\Lambda_x \pi_{F(x)} S_{\mathbf{F},\Lambda}^{-\frac{1}{2}} \pi_{V(x)}\right)^* \Lambda_x \pi_{F(x)} S_{\Lambda}^{-\frac{1}{2}} \pi_{V(x)} f d\mu(x) \\
&= \int_{X_1} \omega^2(x) S_{\mathbf{F},\Lambda}^{-\frac{1}{2}} \pi_{V(x)} \Lambda_x^* \Lambda_x \pi_{V(x)} S_{\mathbf{F},\Lambda}^{-\frac{1}{2}} f d\mu(x) \\
&= S_{\mathbf{F},\Lambda}^{-\frac{1}{2}} S_{\mathbf{F},\Lambda} S_{\mathbf{F},\Lambda}^{-\frac{1}{2}} f.
\end{aligned}$$

Now, replacing f by $S_{\mathbf{F},\Lambda}^{\frac{1}{2}} f$, we complete the proof. ◀

Corollary 3. *Let $(\Lambda, \mathbf{F}, \omega)$ be a Parseval continuous generalized fusion frame for $\mathcal{H}$. Then*

$$0 \leq S_{\mathbf{F},\Lambda}^{X_1} - S_{\mathbf{F},\Lambda}^{X_1} S_{\mathbf{F},\Lambda}^{-1} S_{\mathbf{F},\Lambda}^{X_1} \leq \frac{1}{4} S_{\mathbf{F},\Lambda}.$$

Proof. In the proof of Theorem 5, we have

$$\int_{X_1} \omega^2(x) \pi_{V(x)} \chi_x^* \chi_x \pi_{V(x)} f d\mu(x) = S_{\mathbf{F},\Lambda}^{-\frac{1}{2}} S_{\mathbf{F},\Lambda} S_{\mathbf{F},\Lambda}^{-\frac{1}{2}} f.$$

By Corollary 1, we get

$$0 \leq \int_{X_1} \omega^2(x) \pi_{V(x)} \chi_x^* \chi_x \pi_{V(x)} f d\mu(x) - \left(\int_{X_1} \omega^2(x) \pi_{V(x)} \chi_x^* \chi_x \pi_{V(x)} f d\mu(x)\right)^2 \leq \frac{1}{4} Id_{\mathcal{H}}.$$

Therefore, we have

$$0 \leq S_{\mathbf{F},\Lambda}^{-\frac{1}{2}} \left(S_{\mathbf{F},\Lambda}^{X_1} - S_{\mathbf{F},\Lambda}^{X_1} S_{\mathbf{F},\Lambda}^{-1} S_{\mathbf{F},\Lambda}^{X_1}\right) S_{\mathbf{F},\Lambda}^{-\frac{1}{2}} \leq \frac{1}{4} Id_{\mathcal{H}}.$$

This completes the proof. ◀

Corollary 4. *Suppose that $(\Lambda, \mathbf{F}, \omega)$ is a continuous generalized fusion frame for $\mathcal{H}$ with continuous g-fusion frame operator $S_{\mathbf{F},\Lambda}$. If $X_1 \subseteq X$ and $f \in \mathcal{H}$, then we have*

$$\int_{X_1} \omega^2(x)\|\Lambda_x \pi_{F(x)}(f)\|^2 d\mu(x) - \|S_{\mathbf{F},\Lambda}^{-\frac{1}{2}} S_{\mathbf{F},\Lambda}^{X_1^c} f\|^2 \geq \frac{3}{4}\left\|S_{\mathbf{F},\Lambda}^{-1}\right\|^{-1}\|f\|^2.$$

Proof. By Theorems 5 and 4, we can write

$$\begin{aligned}
&\int_{X_1} \omega^2(x)\left\|\Lambda_x \pi_{F(x)}(f)\right\|^2 d\mu(x) + \left\|S_{\mathbf{F},\Lambda}^{-\frac{1}{2}} S_{\mathbf{F},\Lambda}^{X_1^c} f\right\| \\
&= \int_{X_1} \omega^2(x)\left\|\chi_x \pi_{V(x)} S_{\mathbf{F},\Lambda}^{-\frac{1}{2}} f\right\|^2 d\mu(x) + \left\|\int_{X_1^c} \omega^2(x) \pi_{V(x)} \chi_x^* \chi_x f \pi_{V(x)} S_{\mathbf{F},\Lambda}^{\frac{1}{2}} f d\mu(x)\right\|^2 \\
&\geq \frac{3}{4}\left\|S_{\mathbf{F},\Lambda}^{\frac{1}{2}} f\right\|^2 = \frac{3}{4}\langle S_{\mathbf{F},\Lambda} f, f\rangle \\
&\geq \frac{3}{4}\left\|S_{\mathbf{F},\Lambda}^{-1}\right\|^{-1}\|f\|^2.
\end{aligned}$$

This completes the proof. ◀

Theorem 6. *Let $(\Lambda, \mathbf{F}, \omega)$ be a continuous generalized fusion frame for $\mathcal{H}$. Then for $X_1 \subset X$ and for each $f \in \mathcal{H}$, we have*

$$\begin{aligned}
&\int_{X_1} \omega^2(x)\|\Lambda_x \pi_{F(x)}(f)\|^2 d\mu(x) - \int_{X_1} \omega^2(x)\|\tilde{\Lambda}_x \pi_{\tilde{F}(x)} \mathcal{M}_{\mathbf{F},\Lambda}^{X_1}(f)\|^2 d\mu(x) \\
&\qquad = \int_{X_1^c} \omega^2(x)\|\Lambda_x \pi_{F(x)}(f)\|^2 d\mu(x) - \int_{X_1^c} \omega^2(x)\|\tilde{\Lambda}_x \pi_{\tilde{F}(x)} \mathcal{M}_{\mathbf{F},\Lambda}^{X_1^c} f\|^2 d\mu(x),
\end{aligned}$$

where

$$\mathcal{M}_{\mathbf{F},\Lambda}^{X_1} f = \int_{X_1} \omega^2(x) \pi_{F(x)} \Lambda_x^* \Lambda_x \pi_{F(x)} f d\mu(x).$$

Proof. Assume that $S_{\mathbf{F},\Lambda}$ is a continuous generalized fusion frame for $(\Lambda, \mathbf{F}, \omega)$. By the definition of $S_{\mathbf{F},\Lambda}$, it is clear that $\mathcal{M}_{\mathbf{F},\Lambda}^{X_1} + \mathcal{M}_{\mathbf{F},\Lambda}^{X_1^c} = S_{\mathbf{F},\Lambda}$. It follows that $S_{\mathbf{F},\Lambda}^{-1}\mathcal{M}_{\mathbf{F},\Lambda}^{X_1} + S_{\mathbf{F},\Lambda}^{-1}\mathcal{M}_{\mathbf{F},\Lambda}^{X_1^c} = Id_{\mathcal{H}}$. Hence, by applying Lemma 2 to the operators $S_{\mathbf{F},\Lambda}^{-1}\mathcal{M}_{\mathbf{F},\Lambda}^{X_1}$ and $S_{\mathbf{F},\Lambda}^{-1}\mathcal{M}_{\mathbf{F},\Lambda}^{X_1^c}$, we get

$$S_{\mathbf{F},\Lambda}^{-1}\mathcal{M}_{\mathbf{F},\Lambda}^{X_1} - S_{\mathbf{F},\Lambda}^{-1}\mathcal{M}_{\mathbf{F},\Lambda}^{X_1} = \left(S_{\mathbf{F},\Lambda}^{-1}\mathcal{M}_{\mathbf{F},\Lambda}^{X_1}\right)^2 - \left(S_{\mathbf{F},\Lambda}^{-1}\mathcal{M}_{\mathbf{F},\Lambda}^{X_1^c}\right)^2.$$

Thus, for each $f, g \in \mathcal{H}$, we obtain

$$\begin{aligned}&\left\langle S_{\mathbf{F},\Lambda}^{-1}\mathcal{M}_{\mathbf{F},\Lambda}^{X_1}f, g\right\rangle - \left\langle S_{\mathbf{F},\Lambda}^{-1}\mathcal{M}_{\mathbf{F},\Lambda}^{X_1}S_{\mathbf{F},\Lambda}^{-1}\mathcal{M}_{\mathbf{F},\Lambda}^{X_1}f, g\right\rangle\\ &= \left\langle S_{\mathbf{F},\Lambda}^{-1}\mathcal{M}_{\mathbf{F},\Lambda}^{X_1^c}f, g\right\rangle - \left\langle S_{\mathbf{F},\Lambda}^{-1}\mathcal{M}_{\mathbf{F},\Lambda}^{X_1^c}S_{\mathbf{F},\Lambda}^{-1}\mathcal{M}_{\mathbf{F},\Lambda}^{X_1^c}f, g\right\rangle.\end{aligned}$$

Letting $g = S_{\mathbf{F},\Lambda}f$, we get

$$\left\langle \mathcal{M}_{\mathbf{F},\Lambda}^{X_1^c}f, f\right\rangle - \left\langle S_{\mathbf{F},\Lambda}^{-1}\mathcal{M}_{\mathbf{F},\Lambda}^{X_1}f, \mathcal{M}_{\mathbf{F},\Lambda}f\right\rangle = \left\langle \mathcal{M}_{\mathbf{F},\Lambda}^{X_1^c}f, f\right\rangle - \left\langle S_{\mathbf{F},\Lambda}^{-1}\mathcal{M}_{\mathbf{F},\Lambda}^{X_1^c}f, \mathcal{M}_{\mathbf{F},\Lambda}^{X_1^c}f\right\rangle.$$

Finally, by (3), we complete the proof. ◄

Notice that $(\frac{1}{\sqrt{\lambda}}\Lambda, \mathbf{F}, \omega)$ is a Parseval continuous generalized fusion frame if $(\Lambda, \mathbf{F}, \omega)$ is a λ-tight continuous generalized fusion frame for $\mathcal{H}$.

Corollary 5. *Let $(\Lambda, \mathbf{F}, \omega)$ be a λ-tight continuous generalized fusion frame for $\mathcal{H}$. Then for $X_1 \subset X$ and $f \in \mathcal{H}$, the following hold:*

$$0 \le \lambda \int_{X_1} \omega^2(x)\|\Lambda\pi_{F(x)}f\|^2 d\mu(x) - \|\int_{X_1} \omega^2(x)\pi_{F(x)}\Lambda_x^*\Lambda_x\pi_{F(x)}f d\mu(x)\|^2 \le \frac{\lambda^2}{4}\|f\|^2,$$

$$\begin{aligned}\frac{\lambda^2}{2}\|f\|^2 \le \|\int_{X_1} &\omega^2(x)\pi_{F(x)}\Lambda_x^*\Lambda_x\pi_{F(x)}f d\mu(x)\|^2 d\mu(x)\|^2\\ &- \|\int_{X_1^c} \omega^2(x)\pi_{F(x)}\Lambda_x^*\Lambda_x\pi_{F(x)}f d\mu(x)\|^2 \le \frac{3\lambda^2}{2}\|f\|^2,\end{aligned}$$

$$\begin{aligned}\frac{3\lambda^2}{2}\|f\|^2 \le \lambda \int_{X_1} &\omega^2(x)\|\Lambda_x\pi_{F(x)}f\|^2 d\mu(x) - \|\int_{X_1^c} \omega^2(x)\pi_{F(x)}\Lambda_x^*\Lambda_x\pi_{F(x)}f d\mu(x)\|^2\\ &\le \lambda^2\|f\|^2.\end{aligned}$$

Next we discuss equality for tight continuous generalized fusion frames. First, we define two operators $S_{\mathbf{F},\Lambda}^1$, $S_{\mathbf{F},\Lambda}^2$ as follows:

$$\begin{aligned}S_{\mathbf{F},\Lambda}^1 :\ \mathcal{H} \longrightarrow \mathcal{H}, \quad S_{\mathbf{F},\Lambda}^1 f &= \int_X a_x\omega^2(x)\pi_{F(x)}\Lambda_x^*\Lambda_x\pi_{F(x)}(f)d\mu(x), \quad f \in \mathcal{H},\\ S_{\mathbf{F},\Lambda}^2 :\ \mathcal{H} \longrightarrow \mathcal{H}, \quad S_{\mathbf{F},\Lambda}^2 f &= \int_X (1-a_x)\omega^2(x)\pi_{F(x)}\Lambda_x^*\Lambda_x\pi_{F(x)}(f)d\mu(x), \quad f \in \mathcal{H},\end{aligned}$$

where $(\Lambda, \mathbf{F}, \omega)$ is a Bessel continuous generalized fusion frame for $\mathcal{H}$ and $\{a_x : x \in X\} \in l^\infty(X)$ such that $l^\infty(X) = \{\{a_x : x \in X\} : \sup_{x\in X}|a_x| < \infty\}$.

Proposition 1. *Let $(\Lambda, \mathbf{F}, \omega)$ be a continuous g-fusion frame for $\mathcal{H}$ with bound B. Then $S^1_{\mathbf{F},\Lambda}$ and $S^2_{\mathbf{F},\Lambda}$ are bounded linear operators and*

$$\begin{aligned}(S^1_{\mathbf{F},\Lambda})^* f &= \int_X \overline{a}_x \omega^2(x) \pi_{F(x)} \Lambda_x^* \Lambda_x \pi_{F(x)}(f) d\mu(x), \quad f \in \mathcal{H},\\ (S^2_{\mathbf{F},\Lambda})^* f &= \int_X (1-\overline{a}_x) \omega^2(x) \pi_{F(x)} \Lambda_x^* \Lambda_x \pi_{F(x)}(f) d\mu(x), \quad f \in \mathcal{H}.\end{aligned}$$

Proof. For $f \in \mathcal{H}$ and $X_1 \subset X$, we have

$$\begin{aligned}&\left\| \int_X a_x \omega^2(x) \pi_{F(x)} \Lambda_x^* \Lambda_x \pi_{F(x)}(f) d\mu(x) \right\|\\ &= \sup_{g \in H, \|g\|=1} \left| \left\langle \int_X a_x \omega^2(x) \pi_{F(x)} \Lambda_x^* \Lambda_x \pi_{F(x)}(f) d\mu(x), g \right\rangle \right|\\ &= \sup_{g \in H, \|g\|=1} \left| \int_{X_1} \omega^2(x) \left\langle \Lambda_x \pi_{F(x)} f, \bar{a}_x \Lambda_x \pi_{F(x)} g \right\rangle d\mu(x) \right|\\ &\leq \sup_{g \in H, \|g\|=1} \left(\int_{X_1} \omega^2(x) \left\| \Lambda_x \pi_{F(x)}(f) \right\|^2 d\mu(x) \right)^{\frac{1}{2}} \left(\int_{X_1} v^2(x) \left\| \bar{a}_x \Lambda_x \pi_{F(x)}(g) \right\|^2 d\mu(x) \right)^{\frac{1}{2}}\\ &\leq BM_a \|f\|,\end{aligned}$$

where $M_a = \sup_{x \in X} |a_x|$ and $\bar{a}_x$ is the conjugate of a_x. This implies that $S^1_{\mathbf{F},\Lambda}$ is well-defined and $\left\| S^1_{\mathbf{F},\Lambda} f \right\| \leq BM_a \|f\|$. Therefore, $S^1_{\mathbf{F},\Lambda}$ is a bounded linear operator. Now let us compute its adjoint

$$\begin{aligned}\left\langle f, \left(S^1_{\mathbf{F},\Lambda}\right)^* (g) \right\rangle = \left\langle S^1_{\mathbf{F},\Lambda} f, g \right\rangle &= \left\langle \int_X a_x \omega^2(x) \Lambda_x \pi_{F(x)} f d\mu(x), g \right\rangle\\ &= \left\langle f, \int_X \overline{a}_x \omega^2(x) \pi_{F(x)} \Lambda_x^* \Lambda_x \pi_{F(x)}(g) d\mu(x) \right\rangle d\mu(x).\end{aligned}$$

Similarly, we can show that $S^2_{\mathbf{F},\Lambda}$ is a bounded linear operator and its adjoint is

$$(S^2_{\mathbf{F},\Lambda})^* f = \int_X (1-\overline{a}_x) \omega^2(x) \pi_{F(x)} \Lambda_x^* \Lambda_x \pi_{F(x)}(f) d\mu(x), \quad f \in \mathcal{H}.$$

This completes the proof. ◄

Theorem 7. *Let $(\Lambda, \mathbf{F}, \omega)$ be a λ-tight continuous generalized fusion frame for $\mathcal{H}$. Then for $f \in \mathcal{H}$ and $\{a_x : \quad x \in X\} \in l^\infty(X)$, we have*

$$\lambda \int_X a_x \omega^2(x) \|\Lambda \pi_{F(x)}(f)\|^2 d\mu(x) - \| \int_X (1-a_x) \omega^2(x) \pi_{F(x)} \Lambda_x^* \Lambda_x \pi_{F(x)}(f) d\mu(x) \|^2$$

$$= \lambda \int_X (1-\overline{a}_x)\omega^2(x)\|\Lambda\pi_{F(x)}(f)\|^2 d\mu(x) - \|\int_X a_x\omega^2(x)\pi_{F(x)}\Lambda_x^*\Lambda_x\pi_{F(x)}(f)d\mu(x)\|^2,$$

where $\overline{a}_x$ is the conjugate of a_x.

Proof. Since $(\Lambda, \mathbf{F}, \omega)$ is a λ-tight continuous generalized fusion frame for $\mathcal{H}$, by Proposition 1, $S^1_{\mathbf{F},\Lambda}$ and $S^2_{\mathbf{F},\Lambda}$ are well defined. In particular, for each $f \in \mathcal{H}$, we have

$$S^1_{\mathbf{F},\Lambda}f + S^2_{\mathbf{F},\Lambda}f = \int_X \omega^2(x)\pi_{F(x)}\Lambda_x^*\Lambda_x\pi_{F(x)}(f)d\mu(x).$$

So $\lambda^{-1}S^1_F + \lambda^{-1}S^2_{\mathbf{F},\Lambda} = Id_{\mathcal{H}}$. Now if we suppose that $Q_1 = \lambda^{-1}S^1_{\mathbf{F},\Lambda}$ and $Q_2 = \lambda^{-1}S^2_F$, then we have

$$\begin{aligned} Q_1 + Q_2^*Q_2 &= Q_1 + (I_H - Q_1)^*(I_H - Q_1) \\ &= Q_1 + (I_H - Q_1^*)(I_H - Q_1) \\ &= Q_1 + I_H - Q_1 - Q_1^* + Q_1^*Q_1 \\ &= I_H - Q_1^* + Q_1^*Q_1 \\ &= Q_2^* + Q_1^*Q_1 \end{aligned}$$

and thus

$$\lambda S^1_{\mathbf{F},\Lambda} + \left(S^2_{\mathbf{F},\Lambda}\right)^* S^2_{\mathbf{F},\Lambda} = \lambda S^2_{\mathbf{F},\Lambda} + \left(S^1_{\mathbf{F},\Lambda}\right)^* S^1_{\mathbf{F},\Lambda}.$$

Hence for $h \in \mathcal{H}$, we get

$$\begin{aligned} &\lambda \int_X a_x v^2(x)\left\|\pi_{F(x)}(h)\right\|^2 d\mu(x) + \left\|\int_X (1-a_x)\, v^2(x)\pi_{F(x)}(h)d\mu(x)\right\|^2 \\ &= \left\langle \lambda S^1_{\mathbf{F},\Lambda}h, h\right\rangle + \left\langle \left(S^2_{\mathbf{F},\Lambda}\right)^* S^2_{\mathbf{F},\Lambda}h, h\right\rangle \\ &= \left\langle \left(\lambda S^1_{\mathbf{F},\Lambda} + \left(S^2_{\mathbf{F},\Lambda}\right)^* S^2_{\mathbf{F},\Lambda}\right)h, h\right\rangle \\ &= \left\langle \left(\lambda S^2_{\mathbf{F},\Lambda} + \left(S^1_{\mathbf{F},\Lambda}\right)^* S^1_{\mathbf{F},\Lambda}\right)h, h\right\rangle \\ &= \left\langle \lambda\left(S^2_{\mathbf{F},\Lambda}\right)^* h, h\right\rangle + \left\langle \left(S^1_{\mathbf{F},\Lambda}\right)^* S^1_{\mathbf{F},\Lambda}h, h\right\rangle \\ &= \left\langle h, S^2_{\mathbf{F},\Lambda}h\right\rangle + \left\|S^1_{\mathbf{F},\Lambda}h\right\|^2. \end{aligned}$$

This completes the proof. ◄

Furthermore, by Theorems 4 and 5, we immediately obtain the following result.

Corollary 6. *Let $(\Lambda, \mathbf{F}, \omega)$ be a λ-tight continuous generalized fusion frame for $\mathcal{H}$. Then for $f \in \mathcal{H}$ and $\{a_x : x \in X\} \in l^\infty(X)$, we have*

$$\begin{aligned}
&\lambda \int_X a_x \omega^2(x) \|\Lambda \pi_{F(x)}(f)\|^2 d\mu(x) - \| \int_X (1-a_x)\omega^2(x)\pi_{F(x)}\Lambda_x^* \Lambda_x \pi_{F(x)}(f) d\mu(x)\|^2 \\
&= \lambda \int_X (1-a_x)\omega^2(x)\|\Lambda \pi_{F(x)}(f)\|^2 d\mu(x) \\
&-\| \int_X a_x \omega^2(x) \pi_{F(x)} \Lambda_x^* \Lambda_x \pi_{F(x)}(f) d\mu(x)\|^2 \geq \frac{3}{4}\|f\|^2.
\end{aligned}$$

4. Inequalities-equalities for continuous generalized fusion pairs

By Lemma 1, we have

$$\pi_{F(x)} S_{\mathbf{F},\Lambda}^{-1} = \pi_{F(x)} S_{\mathbf{F},\Lambda}^{-1} \pi_{S_{\mathbf{F},\Lambda}^{-1} F(x)},$$

which implies that

$$S_{\mathbf{F},\Lambda}^{-1} \pi_{F(x)} = \pi_{S_{\mathbf{F},\Lambda}^{-1} F(x)} S_{\mathbf{F},\Lambda}^{-1} \pi_{F(x)}.$$

Moreover, (2) also can be rewritten as

$$f = \int_X \omega^2(x) \pi_{S_{\mathbf{F},\Lambda}^{-1} F(x)} (S_{\mathbf{F},\Lambda}^{-1} \pi_{F(x)} \Lambda_x^* S_{\mathbf{F},\Lambda}) S_{\mathbf{F},\Lambda}^{-1} \Lambda_x \pi_{F(x)} f d\mu(x), \quad f \in \mathcal{H}.$$

Now, we introduce the following definition.

Definition 2. *Let $\mathcal{V} = (\Lambda, \mathbf{F}, \omega)$ be a continuous generalized fusion frame with bounds A, B and $S_{\mathbf{F},\Lambda}$ be a frame operator. We consider also $\mathcal{W} = (\Gamma, \mathbf{G}, \nu)$ as a Bessel continuous generalized fusion mapping. We say that $\mathcal{W}$ is an alternate dual of $\mathcal{V}$ if we have*

$$f = \int_X \omega(x)\nu(x)\pi_{G(x)} \Gamma_x^* S_{\mathbf{F},\Lambda}^{-1} \Lambda_x \pi_{F(x)} f d\mu(x), \quad f \in \mathcal{H}. \tag{4}$$

Proposition 2. *The alternate dual of continuous generalized fusion frame of $\mathcal{V}$ is a continuous generalized fusion frame.*

Proof. By (4), for each $f \in \mathcal{H}$, we get

$$\begin{aligned}
\|f\|^2 &= \int_X \omega(x)\nu(x)\langle \pi_{G(x)} \Gamma_x^* S_{\mathbf{F},\Lambda}^{-1} \Lambda_x \pi_{F(x)} f, f\rangle d\mu(x) \\
&\leq \int_X \omega(x)\nu(x) \langle S_{\mathbf{F},\Lambda}^{-1} \Lambda_x \pi_{F(x)} f, \Gamma_x \pi_{G(x)} f\rangle d\mu(x)
\end{aligned}$$

$$
\begin{aligned}
&= \int_X \omega(x)\nu(x)\|S_{\mathbf{F},\Lambda}^{-1}\Lambda_x\pi_{F(x)}f\|\|\Gamma_x\pi_{G(x)}f\|d\mu(x) \\
&\leq \left(\int_X \omega(x)^2\|S_{\mathbf{F},\Lambda}^{-1}\Lambda_x\pi_{F(x)}f\|^2 d\mu(x)\right)^{1/2} \left(\int_X \nu(x)^2\|\Gamma_x\pi_{G(x)}f\|^2 d\mu(x)\right)^{1/2} \\
&\leq \|S_{\mathbf{F},\Lambda}^{-1}\|\sqrt{B}\left(\int_X \nu(x)^2\|\Gamma_x\pi_{G(x)}f\|^2\right)^{1/2},
\end{aligned}
$$

where B is the upper bound of $\mathcal{V}$. ◄

Theorem 8. *Assume that $(\Lambda, \mathbf{F}, \omega)$ is a continuous g-fusion frame for $\mathcal{H}$ with the continuous g-fusion frame operator $S_{\mathbf{F},\Lambda}$, $\mathcal{W} = (\Gamma, \mathbf{G}, \nu)$ is an alternate dual continuous g-fusion frame of $\mathcal{V} = (\Lambda, \mathbf{F}, \omega)$. Then for any $X_1 \subset X$ and for each $f \in \mathcal{H}$,*

$$
\begin{aligned}
&\int_{X_1} \omega(x)\nu(x)\langle S_{\mathbf{F},\Lambda}^{-1}\Lambda_x\pi_{F(x)}f, \Gamma_x\pi_{G(x)}f\rangle d\mu(x) \\
&- \left\|\int_{X_1} \omega(x)\nu(x)\pi_{G(x)}\Gamma_x^* S_{\mathbf{F},\Lambda}^{-1}\Lambda_x\pi_{F(x)}f d\mu(x)\right\|^2 \\
&= \int_{X_1^c} \omega(x)\nu(x)\overline{\langle S_{\mathbf{F},\Lambda}^{-1}\Lambda_x\pi_{F(x)}f, \Gamma_x\pi_{G(x)}f\rangle} d\mu(x) \\
&- \left\|\int_{X_1^c} \omega(x)\nu(x)\pi_{G(x)}\Gamma_x^* S_{\mathbf{F},\Lambda}^{-1}\Lambda_x\pi_{F(x)}f d\mu(x)\right\|^2.
\end{aligned}
$$

Proof. For each $X_1 \subset X$, let us consider a bounded linear operator $\mathcal{T}_{\mathbf{FG},\Lambda\Gamma}$ as follows:

$$
\mathcal{T}_{\mathbf{FG},\Lambda\Gamma}^{X_1} f = \int_{X_1} \omega(x)\nu(x)\pi_{G(x)}\Gamma_x^* S_{\mathbf{F},\Lambda}^{-1}\Lambda_x\pi_{F(x)}f d\mu(x), \quad f \in \mathcal{H}.
$$

It is clear that $\mathcal{T}_{\mathbf{FG},\Lambda\Gamma}^{X_1} + \mathcal{T}_{\mathbf{FG},\Lambda\Gamma}^{X_1^c} = Id_{\mathcal{H}}$. Applying Lemma 3, we have

$$
\begin{aligned}
&\int_{X_1} \omega(x)\nu(x)\langle S_{\mathbf{F},\Lambda}^{-1}\Lambda_x\pi_{F(x)}f, \Gamma_x\pi_{G(x)}f\rangle d\mu(x) \\
&- \left\|\int_{X_1} \omega(x)\nu(x)\pi_{G(x)}\Gamma_x^* S_{\mathbf{F},\Lambda}^{-1}\Lambda_x\pi_{F(x)}f d\mu(x)\right\|^2 \\
&= \int_{X_1} \omega(x)\nu(x)\langle S_{\mathbf{F},\Lambda}^{-1}\Lambda_x\pi_{F(x)}f, \Gamma_x\pi_{G(x)}f\rangle d\mu(x) - \langle \mathcal{T}_{\mathbf{FG},\Lambda\Gamma}^{X_1} f, \mathcal{T}_{\mathbf{FG},\Lambda\Gamma}^{X_1} f\rangle \\
&= \langle \mathcal{T}_{\mathbf{FG},\Lambda\Gamma}^{X_1} f, f\rangle + \langle (\mathcal{T}_{\mathbf{FG},\Lambda\Gamma}^{X_1})^* \mathcal{T}_{\mathbf{FG},\Lambda\Gamma}^{X_1} f, f\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \langle (\mathcal{T}^{X_1^c}_{\mathbf{FG},\Lambda\Gamma})^* f, f\rangle + \langle \mathcal{T}^{X_1^c}_{\mathbf{FG},\Lambda\Gamma} f, \mathcal{T}^{X_1^c}_{\mathbf{FG},\Lambda\Gamma} f\rangle \\
&= \langle f, \int_{X_1^c} \omega(x)\nu(x)\pi_{G(x)}\Gamma_x^* S^{-1}_{\mathbf{F},\Lambda}\Lambda_x\pi_{F(x)} f d\mu(x)\rangle \\
&- \left\| \int_{X_1} \omega(x)\nu(x)\pi_{G(x)}\Gamma_x^* S^{-1}_{\mathbf{F},\Lambda}\Lambda_x\pi_{F(x)} f d\mu(x)\right\|^2 \\
&= \int_{X_1^c} \omega(x)\nu(x)\langle \Gamma_x\pi_{G(x)} f, S^{-1}_{\mathbf{F},\Lambda}\Lambda_x\pi_{F(x)} f\rangle d\mu(x) \\
&- \left\| \int_{X_1^c} \omega(x)\nu(x)\pi_{G(x)}\Gamma_x^* S^{-1}_{\mathbf{F},\Lambda}\Lambda_x\pi_{F(x)} f d\mu(x)\right\|^2 .
\end{aligned}
$$

This completes the proof. ◄

In the case of Parseval fusion frame, the previous equality can have a special form as follows:

Corollary 7. *Let $(\Lambda, \mathbf{F}, \omega)$ be a Parseval continuous g-fusion frame for $\mathcal{H}$ with the continuous g-fusion frame operator $S_{\mathbf{F},\Lambda} = id_{\mathcal{H}}$, and $(\Gamma, \mathbf{G}, \nu)$ be an alternate dual continuous g-fusion frame of $(\Lambda, \mathbf{F}, \omega)$. Then for any $X_1 \subset X$ and for each $f \in \mathcal{H}$,*

$$
\begin{aligned}
&\int_{X_1} \omega(x)\nu(x)\langle \Lambda_x\pi_{F(x)}, \Gamma_x\pi_{G(x)}(f)\rangle d\mu(x) - \| \int_{X_1} \omega(x)\nu(x)\pi_{G(x)}\Gamma_x^*\Lambda_x\pi_{F(x)} f d\mu(x)\|^2 \\
&= \int_{X_1^c} \omega(x)\nu(x)\langle \Lambda_x\pi_{F(x)}, \Gamma_x\pi_{G(x)}(f)\rangle d\mu(x) - \left\| \int_{X_1^c} \omega(x)\nu(x)\pi_{G(x)}\Gamma_x^*\Lambda_x\pi_{F(x)} f d\mu(x)\right\|^2 .
\end{aligned}
$$

5. Frame operator of a pair of Bessel continuous generalized fusion mappings

Now, let us consider two Bessel continuous generalized fusion mappings: $\mathcal{V} = (\Lambda, \mathbf{F}, \omega)$ with Bessel bound B_1 and $\mathcal{W} = (\Gamma, \mathbf{G}, \nu)$ with Bessel bound B_2. We define the operator

$$
S_{\mathbf{FG},\Lambda\Gamma}(f) = \int_X \omega(x)\nu(x)\pi_{F(x)}\Lambda_x^*\Gamma_x\pi_{G(x)}(f) d\mu(x), \quad f \in \mathcal{H}.
$$

For all $f, g \in \mathcal{H}$, we have

$$
\langle S_{\mathbf{FG},\Lambda\Gamma} f, g\rangle = \int_X \omega(x)\nu(x)\langle \Gamma_x\pi_{G(x)} f, \Lambda_x\pi_{F(x)} g\rangle d\mu(x).
$$

Furthermore, by using Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&|\langle S_{\mathbf{FG},\Lambda\Gamma} f, g\rangle| \qquad (5)\\
&\leq \left(\int_X \omega^2(x)\|\Gamma_x \pi_{G(x)} f\|^2 d\mu(x)\right)^{1/2} \left(\int_X \nu^2(x)\|\Lambda_x \pi_{F(x)} g\|^2 d\mu(x)\right)^{1/2}.
\end{aligned}$$

From (5), it follows that

$$|\langle S_{\mathbf{FG},\Lambda\Gamma} f, g\rangle| \leq \sqrt{B_1}\sqrt{B_2}\|g\|\|f\|.$$

Hence, $S_{\mathbf{FG},\Lambda\Gamma}$ is a bounded operator and we have

$$\|S_{\mathbf{FG},\Lambda\Gamma}\| \leq \sqrt{B_1}\sqrt{B_2}.$$

From (5), we obtain

$$\|S_{\mathbf{FG},\Lambda\Gamma} f\| \leq \sqrt{B_1}\left(\int_X \nu^2(x)\|\Lambda_x \pi_{F(x)} g\|^2 d\mu(x)\right)^{1/2} \qquad (6)$$

and

$$\|(S_{\mathbf{FG},\Lambda\Gamma})^* f\| \leq \sqrt{B_2}\left(\int_X \omega^2(x)\|\Gamma_x \pi_{G(x)} f\|^2 d\mu(x)\right)^{1/2}.$$

Moreover, from the adjointability of the operator $S_{\mathbf{FG},\Lambda\Gamma}$, we get

$$\langle S_{\mathbf{FG},\Lambda\Gamma} f, g\rangle = \int_X \omega(x)\nu(x)\langle f, \pi_{F(x)}\Gamma_x^*\Lambda_x\pi_{G(x)} g\rangle d\mu(x).$$

Hence, $S^*_{\mathbf{FG},\Lambda\Gamma} = S_{\mathbf{GF},\Gamma\Lambda}$.

Theorem 9. *The following assertions are equivalent:*

(i) $S_{\mathbf{FG},\Lambda\Gamma}$ *is bounded below;*

(ii) There exists $K \in \mathcal{B}(\mathcal{H})$ *such that* $\{T_x\}_{x\in X}$ *is a resolution of identity, where*

$$T_x = \omega(x)\nu(x)K\pi_{F(x)}\Lambda_x^*\Gamma_x\pi_{G(x)}, \quad x \in X.$$

If one of the above conditions is satisfied, then $\mathcal{W}$ *is a continuous generalized fusion frame.*

Proof. $(i) \Rightarrow (ii)$ It is obvious.

$(ii) \Rightarrow (i)$ If (ii) holds, then for $f, g \in \mathcal{H}$, we have

$$\begin{aligned}\langle KS_{\mathbf{FG},\Lambda\Gamma} f, g\rangle = \langle S_{\mathbf{FG},\Lambda\Gamma} f, K^* g\rangle &= \int_X \omega(x)\nu(x) \left\langle f, \left(K\pi_{F(x)}\Lambda_x^*\Gamma_x\pi_{G(x)}\right)^* g\right\rangle d\mu(x) \\ &= \langle f, g\rangle,\end{aligned}$$

which implies that $I_{\mathcal{H}} = KS_{\mathbf{FG},\Lambda\Gamma}$. Thus $S_{\mathbf{FG},\Lambda\Gamma}$ is bounded below.

Now, if $S_{\mathbf{FG},\Lambda\Gamma}$ is bounded below, from (6) it follows that $\mathcal{G}$ is a continuous fusion frame. So

$$f = \omega(x)\nu(x)K\pi_{F(x)}\Lambda_x^*\Gamma_x\pi_{G(x)} f d\mu(x), \quad x \in X.$$

Hence

$$f = K\left(\int_X \omega(x)\nu(x)\pi_{F(x)}\Lambda_x^*\Gamma_x\pi_{G(x)} f d\mu(x)\right), \quad x \in X.$$

This completes the proof. ◄

Corollary 8. *The following assertions are equivalent:*

i) $S_{\mathbf{FG},\Lambda\Gamma}$ *is an invertible operator;*

ii) There exists $K \in \mathcal{B}(\mathcal{H})$ *invertible such that* $\{T_x\}_{x\in X}$ *is a resolution of identity, where*

$$T_x = \omega(x)\nu(x)K\pi_{F(x)}\Lambda_x^*\Gamma_x\pi_{G(x)}, \quad x \in X$$

is a resolution of identity.

If one of the above conditions is satisfied, then $\mathcal{V}$ *and* $\mathcal{W}$ *are continuous generalized fusion frames.*

Theorem 10. *Assume that there exist* $\lambda_1 < 1$ *and* $\lambda_2 > -1$ *such that*

$$\begin{aligned}&\left\| f - \int_X \omega(x)\nu(x)\pi_{F(x)}\Lambda_x^*\Gamma_x\pi_{G(x)}(f) d\mu(x)\right\| \\ &\leq \lambda_1\|f\| + \lambda_2 \left\| \int_X \omega(x)\nu(x)\pi_{F(x)}\Lambda_x^*\Gamma_x\pi_{G(x)}(f) d\mu(x)\right\|,\end{aligned}$$

for all $f \in \mathcal{H}$. *Then* $\mathcal{W}$ *is a continuous generalized fusion frame and*

$$\left(\frac{1-\lambda_1}{1+\lambda_2}\right)^2 \frac{1}{B_1}\|f\|^2 \leq \int_X \omega^2(x)\|\Gamma_x\pi_{G(x)} f\|^2 d\mu(x), \quad f \in \mathcal{H}.$$

Proof. Since $S_{\mathbf{FG},\Lambda\Gamma}f = \int_X \omega(x)\nu(x)\pi_{F(x)}\Lambda_x^*\Gamma_x\pi_{G(x)}(f)d\mu(x)$, we have

$$\|f - S_{\mathbf{FG},\Lambda\Gamma}f\| \leq \lambda_1\|f\| + \lambda_2\|S_{\mathbf{FG},\Lambda\Gamma}f\|.$$

Since

$$\|f - S_{\mathbf{FG},\Lambda\Gamma}f\| \geq | \, \|f\| - \|S_{\mathbf{FG},\Lambda\Gamma}f\| \, |,$$

$$\lambda_1\|f\| + \lambda_2\|S_{\mathbf{FG},\Lambda\Gamma}f\| \geq |\|f\| - \|S_{\mathbf{FG},\Lambda\Gamma}f\|$$

and thus

$$\|S_{\mathbf{FG},\Lambda\Gamma}f\| \geq \frac{1-\lambda_1}{1+\lambda_2}\|f\|.$$

Hence we obtain

$$\int_X \omega^2(x)\|\Gamma_x\pi_{G(x)}f\|^2 d\mu(x) \geq \left(\frac{1-\lambda_1}{1+\lambda_2}\right)^2 \frac{1}{B_1}\|f\|^2.$$

This completes the proof. ◄

In particular, if we take $\lambda_2 = 0$, then in this case we have obviously a stronger result.

Corollary 9. *Assume that there exists* $\lambda \in [0,1)$ *such that*

$$\left\| f - \int_X \omega(x)\nu(x)\pi_{F(x)}\Lambda_x^*\Gamma_x\pi_{G(x)}(f)d\mu(x)\right\| \leq \lambda\|f\|, \quad f \in \mathcal{H}. \tag{7}$$

Then $\mathcal{V}$ *and* $\mathcal{W}$ *are continuous generalized fusion frames and the following estimates hold:*

$$\begin{aligned} \int_X \nu(x)^2(x)\|\Gamma_x\pi_{G(x)}f\|^2 d\mu(x) &\geq \frac{(1-\lambda)^2}{B_1}\|f\|^2, \\ \int_X \omega^2(x)\|\Lambda_x\pi_{F(x)}f\|^2 d\mu(x) &\geq \frac{(1-\lambda)^2}{B_2}\|f\|^2 \end{aligned}$$

for all $f \in \mathcal{H}$.

Proof. Under the assumption (7), for each $f \in \mathcal{H}$, we have

$$\begin{aligned} \|f - S_{\mathbf{FG},\Lambda\Gamma}f\| &= \|(I_{\mathcal{H}} - S_{\mathbf{FG},\Lambda\Gamma})^* f\| \leq \|(I_{\mathcal{H}} - S_{\mathbf{FG},\Lambda\Gamma})^*\| \, \|f\| \\ &\leq \lambda \, \|f\| \, . \end{aligned}$$

Hence applying Theorem 10, we get the result. ◄

6. Conclusion

In this paper, we have established some equalities and inequalities for continuous generalized fusion frame, Parseval continuous generalized fusion frame, alternate dual continuous generalized fusion frame, which generalize some remarkable and existing results which have been obtained before.

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