Azerbaijan Journal of Mathematics
V. 14, No 2, 2024, July
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2024.2.181

Convergence of Iterates of Normal Operators in L^2 Spaces

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Abstract. Let (Ω, Σ, m) be a measure space with m being an σ-finite positive measure and let N be a normal operator on $L^2(\Omega, \Sigma, m)$. In this note, we study strong and almost everywhere convergences of the sequences $\{\phi(N)^n f\}_{n\in\mathbb{N}}$ in $L^2(\Omega, \Sigma, m)$ spaces, where ϕ is a continuous function on the spectrum of N.

Key Words and Phrases: L^2-space, normal operator, norm convergence, almost everywhere convergence.

2010 Mathematics Subject Classifications: 47D03, 46J05, 43A65

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1. Introduction

In this note, we present some results concerning strong and almost everywhere convergences of iterates of normal operators in L^2 spaces. For related results see [1, 2, 4, 5, 9, 10].

Let X be a complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on X. An operator $T \in B(X)$ is said to be *mean ergodic* if the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} T^i x \ \text{ exists in norm for every } x \in X.$$

If T is mean ergodic, then

$$P_T x := \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} T^i x \, (x \in X)$$

is the projection onto $\ker(T - I)$. The projection P_T will be called *mean ergodic projection* associated with T.

An operator $T \in B(X)$ is said to be *power bounded* if

$$C_T := \sup_{n \geq 0} \|T^n\| < \infty.$$

A power bounded operator T on a Banach space X is mean ergodic if and only if

$$X = \ker(T - I) \oplus \overline{\operatorname{ran}(T - I)}. \tag{1}$$

Recall also that a power bounded operator on a reflexive Banach space is mean ergodic [6, Chapter 2].

The following result is an immediate consequence of the identity (1).

Proposition 1. *Let T be a power bounded operator on a Banach space X and assume that*

$$\lim_{n\to\infty} \left\|T^{n+1}x - T^n x\right\| = 0 \quad \textit{for all } x \in X.$$

If T is mean ergodic (so if X is reflexive), then $T^n \to P_T$ in the strong operator topology, where P_T is the mean ergodic projection associated with T.

As usual, by $\sigma(T)$ we denote the spectrum of $T \in B(X)$. If T is a power bounded operator, then, clearly, $\sigma(T) \subseteq \overline{\mathbb{D}}$, where $\mathbb{D}$ is an open unit disc in the complex plane.

There is an operator T with $\sigma(T) \subseteq \overline{\mathbb{D}}$, which is not power bounded. To see this, let $R \in B(X)$ be such that $R \neq 0$ and $R^2 = 0$. If $T = I + R$, then $\sigma(T) = \{1\}$ and as $T^n = I + nR$, we have $\lim_{n\to\infty} \|T^n\| = \infty$.

If T is a mean ergodic operator, then by the Principle of Uniform Boundedness, T is Cesàro bounded, that is,

$$\sup_{n\in\mathbb{N}} \left\| \frac{1}{n} \sum_{i=1}^{n} T^i \right\| < \infty.$$

It follows from the spectral mapping theorem that if T is mean ergodic, then $r(T) \leq 1$, where $r(T)$ is a spectral radius of T.

Recall that the Assani matrix

$$T = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

is Cesàro bounded, but not power bounded.

An operator $T \in B(X)$ is called *uniformly mean ergodic* if there exists $Q \in B(X)$ such that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} T^i = Q \quad \text{in norm operator topology.}$$

A power bounded operator T on a Banach space is uniformly mean ergodic if and only if $\operatorname{ran}(T - I)$ is closed [8].

2. Normal operators

Let N be a normal operator on a complex Hilbert space H with the spectral measure E. If N is mean ergodic, then N is a contraction, that is, $\|N\| = r(N) \leq 1$. If N is a contraction (normal operator is power bounded if and only if it is a contraction), then by the mean ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} N^i x = E\{1\} x \quad \text{in norm for all } x \in H.$$

Recall that ran$(N - I)$ is closed if and only if 1 is an isolated point of $\sigma(N)$ [3, Chapter IX]. Hence, a normal operator N is uniformly mean ergodic if and only if $\|N\| \leq 1$ and 1 is an isolated point of $\sigma(N)$. Under these conditions,

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} N^i = E\{1\} \quad \text{in norm operator topology.}$$

For every bounded Borel function ϕ on $\sigma(N)$, we can define $\phi(N) \in B(H)$ by

$$\langle \phi(N) x, y\rangle = \int_{\sigma(N)} \phi(z)\, d\langle E(z) x, y\rangle \quad (x, y \in H). \tag{2}$$

As $\phi(N)^* = \overline{\phi}(N)$, $\phi(N)$ is a normal operator and

$$\|\phi(N)\| \leq \|\phi\|_\infty .$$

The spectral measure E_ϕ of $\phi(N)$ is defined by

$$E_\phi(B) = E\left(\phi^{-1}\{B\}\right),$$

for every Borel subset B of complex plane. It follows that if $\|\phi\|_\infty \leq 1$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(N)^i x = E\left(\phi^{-1}\{1\}\right) x \quad \text{in norm for all } x \in H. \tag{3}$$

Let $C(\sigma(N))$ be the space of all complex continuous functions on $\sigma(N)$. For an arbitrary $\phi \in C(\sigma(N))$, we put

$$\mathcal{F}_N^\phi := \{z \in \sigma(N) : \phi(z) = 1\} \text{ and } \mathcal{E}_N^\phi := \{z \in \sigma(N) : |\phi(z)| = 1\}.$$

Both $\mathcal{F}_N^\phi$ and $\mathcal{E}_N^\phi$ are closed subsets of $\sigma(N)$ and $\mathcal{F}_N^\phi \subseteq \mathcal{E}_N^\phi$.

Proposition 2. *Let N be a normal operator on a Hilbert space H with the spectral measure E and let ϕ be a continuous function on $\sigma(N)$ with $\|\phi\|_\infty \leq 1$. The sequence $\left\{\phi(N)^k x\right\}_{k\in\mathbb{N}}$ converges in norm for every $x \in H$ if and only if*

$$E\left(\mathcal{E}_N^\phi \diagdown \mathcal{F}_N^\phi\right) = 0.$$

In this case,

$$\phi(N)^k x \to E\left(\mathcal{F}_N^\phi\right) x \quad \text{in norm for all } x \in H.$$

Proof. By Proposition 1, the sequence $\left\{\phi(N)^k x\right\}_{k\in\mathbb{N}}$ converges in norm for every $x \in H$ if and only if

$$\lim_{k\to\infty} \left\|\phi(N)^{k+1} x - \phi(N)^k x\right\| = 0.$$

By the identity (2), we can write

$$\begin{aligned}
&\lim_{k\to\infty} \left\|\phi(N)^{k+1} x - \phi(N)^k x\right\|^2 \\
=\ &\lim_{k\to\infty} \int_{\sigma(N)} \left|\phi(z)^{k+1} - \phi(z)^k\right|^2 d\langle E(z)x, x\rangle \\
=\ &\lim_{k\to\infty} \int_{\sigma(N)\diagdown\mathcal{E}_N^\phi} |\phi(z)|^{2k} |\phi(z) - 1|^2 d\langle E(z)x, x\rangle \\
&+ \lim_{k\to\infty} \int_{\mathcal{E}_N^\phi} |\phi(z)|^{2k} |\phi(z) - 1|^2 d\langle E(z)x, x\rangle \\
=\ &\int_{\mathcal{E}_N^\phi} |\phi(z) - 1|^2 d\langle E(z)x, x\rangle \\
=\ &\int_{\mathcal{E}_N^\phi\diagdown\mathcal{F}_N^\phi} |\phi(z) - 1|^2 d\langle E(z)x, x\rangle.
\end{aligned}$$

It follows that

$$\lim_{k\to\infty} \left\|\phi(N)^{k+1} x - \phi(N)^k x\right\| = 0 \text{ for all } x \in H$$

if and only if $E\left(\mathcal{E}_N^\phi \diagdown \mathcal{F}_N^\phi\right) = 0$.

By (3), we get

$$\phi(N)^k x \to E\left(\mathcal{F}_N^\phi\right) x \quad \text{in norm for every } x \in H,$$

where $E\left(\mathcal{F}_N^{\phi}\right)$ is the mean ergodic projection associated with $\phi(N)$.

For an arbitrary $x \in H$, let μ_x be the measure defined on the Borel subsets of $\sigma(N)$ by

$$\mu_x(B) = \langle E(B)x, x\rangle = \|E(B)x\|^2. \tag{4}$$

As $\sigma(N) = suppE$ [3, Chapter IX], we have

$$\sigma(N) = \cup_{x\in H} supp\mu_x.$$

The proof of the following proposition being very easy is omitted.

Lemma 1. *Let N be a normal operator on a Hilbert space H with the spectral measure E and let $\phi \in C(\sigma(N))$. For an arbitrary $x, y \in H$, the following assertions hold:*

(a) $supp\mu_{x+y} \subseteq supp\mu_x \cup supp\mu_y$.

(b) $supp\mu_{\phi(N)x} \subseteq supp\phi \cap supp\mu_x$.

(c) For a closed subset S of $\mathbb{C}$, we have

$$\{x \in H : supp\mu_x \subseteq S\} = E(S)H.$$

Next, we have the following

Theorem 1. *Let N be a normal operator on a Hilbert space H and let $\phi \in C(\sigma(N))$ be such that $\|\phi\|_\infty \leq 1$. If $S := \mathcal{E}_N^{\phi} = \mathcal{F}_N^{\phi}$, then there is a subspace F (not necessarily closed) of H with the following properties:*

(i) $H = \overline{F} \oplus F^{\perp}$.

(ii)

$$\sum_{n=1}^{\infty} \|\phi(N)^n x\|^2 < \infty \text{ for all } x \in F \text{ and } \phi(N)y = y \text{ for all } y \in F^{\perp}.$$

(iii) If S is a relatively open subset of $\sigma(N)$, then F is closed.

Proof. For a given $x \in H$, let μ_x be the measure defined by (4) and let

$$F := \{x \in H : supp\mu_x \cap S = \emptyset\}.$$

By Lemma 1 (a), F is a linear subspace of H. Let us show that $\phi(N)y = y$ for all $y \in F^{\perp}$. To see this, let $y \in F^{\perp}$ and assume that the function $h \in C(\sigma(N))$ vanishes in a neighborhood of S. Then, as $supph \cap S = \emptyset$, by Lemma 1 (b) we have

$$supp\mu_{h(N)x} \cap S = \emptyset.$$

Consequently, $h(N)x \in F$ and therefore $\langle h(N)x, y\rangle = 0$ or $\langle x, h(N)^* y\rangle = 0$ for all $x \in H$. Hence, $h(N)^* y = 0$. Since $h(N)$ is a normal operator, we have $h(N)y = 0$. Now, assume that $h \in C(\sigma(N))$ vanishes on S. Then there is a sequence $\{h_n\}$ in $C(\sigma(N))$ such that each h_n vanishes on a neighborhood of S and

$$\lim_{n\to\infty} \|h_n - h\|_\infty = 0.$$

In other words, every closed subset of $\sigma(N)$ is a set of synthesis for $C(\sigma(N))$ (see, for instance [7, Section 8.3]). This implies

$$\lim_{n\to\infty} \|h_n(N) - h(N)\| = 0.$$

Since $h_n(N)y = 0$ for all $n \in \mathbb{N}$, we have $h(N)y = 0$. Since the function $h(z) := \phi(z) - 1$ vanishes on S, we have $\phi(N)y = y$.

If $x \in F$, then as $supp\mu_x \cap S = \emptyset$, we get

$$\sup_{z\in supp\mu_x} |\phi(z)| = \delta < 1,$$

so that

$$\|\phi(N)^n x\|^2 = \int_{supp\mu_x} |\phi(z)|^{2n} d\mu_x \le \delta^{2n} \|x\|^2 .$$

Consequently, we have

$$\sum_{n=1}^{\infty} \|\phi(N)^n x\|^2 < \infty \quad \text{for all } x \in E.$$

If S is an open set, then $\sigma(N) \setminus S$ is closed and

$$F = \{x \in H : supp\mu_x \subseteq \sigma(N) \setminus S\} .$$

By Lemma 1 (c), $F = E[\sigma(N) \setminus S]$ and therefore F is closed. ◄

Let (Ω, Σ, m) be a measure space with m being an σ-finite positive measure and let $L^2(\Omega) := L^2(\Omega, \Sigma, m)$ be the usual Lebesgue space.

Corollary 1. *Let N be a normal operator on $L^2(\Omega)$ and let $\phi \in C(\sigma(N))$ be such that $\|\phi\|_\infty \le 1$. If $S := \mathcal{E}_N^\phi = \mathcal{F}_N^\phi$. Then:*

(a) The limit

$$\lim_{n\to\infty} [\phi(N)^n f](\omega) \quad \textit{exists a.e.,}$$

for every f in a dense subspace of $L^2(\Omega)$.

(b) If S is an open set, then the limit

$$\lim_{n\to\infty} [\phi(N)^n f](\omega) \quad \textit{exists a.e.,}$$

for every f in $L^2(\Omega)$.

Proof. (a) By Theorem 1, there is a subspace F of $L^2(\Omega)$ with the following three properties:

$$L^2 = \overline{F} \oplus F^{\perp},$$

$$\sum_{n=1}^{\infty} \|\phi(N)^n f\|_2^2 < \infty \text{ for all } f \in F,$$

and

$$\phi(N) h = h \text{ for all } h \in F^{\perp}.$$

Now, it suffices to show that

$$\lim_{n\to\infty} [\phi(N)^n f](\omega) = 0 \text{ a.e. for all } f \in F.$$

Indeed, if $f \in F$, then as

$$\sum_{n=1}^{\infty} \int_{\Omega} |[\phi(N)^n f](\omega)|^2 \, dm(\omega) < \infty,$$

by the Beppo-Levi theorem, the series

$$\sum_{n=1}^{\infty} |[\phi(N)^n f](\omega)|^2$$

converges almost everywhere. It follows that

$$\lim_{n\to\infty} [\phi(N)^n f](\omega) = 0 \text{ a.e.}$$

(b) follows from (a) since the subspace F is closed by Theorem 1. ◄

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Received 13 December 2023
Accepted 03 February 2024