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Some Remarks on Integral Operators in Banach Function Spaces and Representation Theorems in Banach-Sobolev Spaces

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Abstract. In this paper, we consider convolution operators, integral operators with weak singularity, Riesz potentials, in particular, those with kernels $K_i(x,y) = \frac{x_i - y_i}{|x-y|^n}$ acting in special classes of Banach function spaces $X(\Omega)$ and their subspaces $X_s(\Omega)$), and we prove some representation theorems for the functions from Banach-Sobolev spaces. We also prove the boundedness of Riesz potential in additive-invariant spaces.

Key Words and Phrases: Banach function space, rearrangement-invariant space, additive-invariant, Sobolev space, integral operator, weak singularity, Riesz potential.

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1. Introduction

In recent years there has been increasing interest towards non-standard function spaces. The emergence of new function spaces such as Morrey space, grand-Lebesgue space, etc. naturally requires the development of corresponding theory. That's why various problems in such spaces and corresponding Sobolev spaces generated by these spaces began to be intensively studied (see [1, 2, 3, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]). In general, the Banach function spaces are not separable. Therefore, using classical methods for establishing classical facts in these spaces requires the essential modification of classical methods and a lot of preparation, concerning correctness of substitution operator, problems related to the extension operator in such spaces, etc. To this aim, based on the additive shift operator $(T_\delta f)(x) = f(x+\delta)$, corresponding separable subspaces $X_s(\Omega)$ of these spaces have been introduced, in which the set of compactly supported infinitely differentiable functions is dense

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([3, 6, 7, 9, 11, 12, 13, 14, 15, 16]). Corresponding subspaces of grand Lebesgue, Marcinkiewich, weak type L_p^w, Morrey spaces are described, for example, in ([8, 9]). In rearrangement-invariant case these subspaces coincide with the set of absolutely continuous functions, that makes it easier to describe such subspaces.

In the classical case, Riesz potential and Sobolev integral identity (see [4, 5, 22]) play an exceptional role in establishing many properties of functions from Sobolev class $W_p^k(\Omega)$ and proving embedding theorems. In [3], under some conditions it is proved that the corresponding embedding operators act compactly from grand-Lebesgue space to $C(\overline{\Omega})$ or to classical Lebesgue spaces defined on manifolds. In [6], some generalizations for the convolution operator acting in rearrangement-invariant Banach function space are established.

In this paper we study convolution operators, integral operators with weak singularity, Riesz potential, in particular, those with kernels $K_i(x,y) = \frac{x_i - y_i}{|x-y|^n}$ acting in special classes of Banach function spaces and their subspaces $X_s(\Omega)$, and representation theorems for functions from Banach-Sobolev spaces. In particular, boundedness of Riesz potential in additive-invariant spaces is proved.

2. Needful information

In this section, we give notations, concepts and results to be used in the sequel. We will use the following standard notations: Z_+ will denote the set of non-negative integers, $|x| = \sqrt{x_1^2 + ... + x_n^2}$ will be the norm of $x = (x_1, ..., x_n)$, $m = mes(M) = |M|$ will stand for the Lebesgue measure of the set $M \subset R^n$, $\partial\Omega$ will denote the boundary of the domain Ω, $\overline{\Omega} = \Omega \bigcup \partial\Omega$ will be the closure of Ω.

$$B_r(x_0) = \{x : |x - x_0| < r\}, \; B_r = B_r(0), \Omega - \delta = \{x : x + \delta \in \Omega\} \; (\forall \delta \in R^n),$$

$$\Omega_\varepsilon = \{x : dist(x, \Omega) < \varepsilon\}, \; (\forall \varepsilon > 0).$$

By $[X, Y]$ ($[X]$ if $X = Y$) we will denote the space of bounded operators acting from Banach function space X to Y, $\|T\|_{[X,Y]}$ will be the norm of the operator T in $[X.Y]$. $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ will stand for the multiindex with the coordinates $\alpha_k \in Z_+$, $|\alpha| = \alpha_1 + ... + \alpha_n$. For every $\xi = (\xi_1, \xi_2, ..., \xi_n)$ we assume $\xi^\alpha = (\xi_1^{\alpha_1}, \xi_2^{\alpha_2}, ..., \xi_n^{\alpha_n})$, $\alpha_k \in Z_+$, $\forall k = \overline{1, n}$. By $\partial_i = \frac{\partial}{\partial x_i}$ we denote the differentiation operator and $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} ... \partial_n^{\alpha_n}$. $f|_\Omega$ will be the restriction of the function f to the set Ω; χ_E will denote the characteristic function of the set E. By $C_0^\infty(\Omega)$ we will denote the set of all infinitely differentiable functions, whose supports are compact subsets of Ω, and by $C^{(m)}$ - the class of all domains with m-th order smooth boundary.

$X(\Omega)$ will denote a Banach function space on domain $\Omega \subset R^n$ with Lebesgue measure, and $X'(\Omega)$ will be its associated space. $\|f\|_X$ will be the norm of

$f \in X(\Omega)$, and the corresponding associated norm will be denoted by $\|.\|_{X'}$. Unit balls in Banach function space $X(\Omega)$ and its associate space will be denoted by S and S', respectively. $X_b(\Omega)$ will be the closure of the set of all simple functions in $X(\Omega)$. The set of all functions from $X(\Omega)$ with absolutely continuous norm will be denoted by $X_a(\Omega)$.

Many monographs have been dedicated to the theory of Banach function spaces. We refer the reader, for example, to [1], or [17].

2.1. Convention

Throughout this paper we assume that $K = \left\{(x_1,...,x_n) : |x_i| < \frac{d}{2}\right\} \subset R^n$ is any cube or $K = R^n$, $X(K)$ is a Banach function space defined on K with Lebesgue measure and the norm $\|f\|_{X(K)} = \rho(f)$. For a domain $\Omega \subset K : \overline{\Omega} \subset K$, by $X(\Omega)$ we mean the space of restrictions of all functions from $X(K)$ to Ω with corresponding norm, i.e.

$$X(\Omega) = \left\{ f \in X(K) : \|f\|_{X(\Omega)} = \|f\chi_\Omega\|_{X(K)} < \infty \right\}.$$

We will mainly consider only the bounded domain case. Depending on circumstances, we assume that $f \in X(\Omega)$ is extended by zero to K, or to the whole of R^n, or periodically on R^n considered as a function from $X(K)$, i.e. for a function f defined on $\Omega \subset K$ we define the new function f_d on R^n in a following way: firstly we continue f by zero to the whole of K, and then periodically to the whole of R^n with

$$\|f_d(. + kd)\|_{X(K)} = \|f_d(.)\|_{X(K)} = \|f\|_{X(K)}.$$

If $X(K)$ is a rearrangement-invariant space, it follows that $f_d(.)$ and $f_d(. + y)$, $(\forall y \in R^n)$ are equimeasurable functions, consequently we have

$$\|f_d(.+)\|_{X(\Omega)} = \|f_d\|_{X(\Omega)} = \|f\|_{X(\Omega)}.$$

The following theorem holds ([1, 17]).

Theorem 1. *a) The inclusions* $X_a(\Omega) \subset X_b(\Omega) \subset X(\Omega)$ *are true.*

b) $|\Omega| < \infty \Rightarrow L_\infty(\Omega) \subset X(\Omega) \subset L_1(\Omega)$.

c) Subspaces $X_a(\Omega)$ *and* $X_b(\Omega)$ *coincide if and only if for every finite measure set* E *the characteristic function* χ_E *has an absolutely continuous norm.*

Throughout this paper, we will use the Fatou's following lemma.

Lemma 1 (Fatou, [1] Lemma 1.5)**.** *Let* X *be a Banach function space and* $f_n \in X$, $(n = 1, 2, ...)$. *If* $f_n \to f$ $\mu - a.e.$ *and* $\liminf_{n\to\infty} \|f_n\|_X < \infty$, *then* $f \in X$ *and*

$$\|f\|_X \le \lim \inf_{n\to\infty} \|f_n\|_X.$$

$\Omega+\delta=\{t+\delta : t\in\Omega\}$ means that $\Omega+\delta\subset K$. For arbitrary function $f\in X(\Omega)$ and for arbitrarily small $\delta\in R^n : \Omega-\delta\subset K$, by $T_\delta f$ we denote the additive shift operator defined as

$$(T_\delta f)(x)=\begin{cases} f(x+\delta), & x+\delta\in\Omega, \\ 0, & x+\delta\notin\Omega. \end{cases}$$

By $X_s(\Omega)$ we will denote the subspace of all functions from $X(\Omega)$ with the following property:

Property α**).** $\|T_\delta(f)-f\|_{X(K)}\to 0,\ \delta\to 0$, *where* $\delta\in R^n$ *is a shift vector.*

Let us consider the following property:

$$\forall\Omega:\overline{\Omega}\subset K,\ \forall\delta:\ \Omega-\delta\subset K,\ \forall f\in X(\Omega)\Rightarrow T_\delta f\in X(K),\ \|f\|_{X(K)}=\|T_\delta f\|_{X(K)}.$$

In the sequel, the spaces with this property will be called the spaces with additive-invariant norm or the additive-invariant Banach function spaces.

For example, rearrangement-invariant Banach function spaces and Morrey spaces have this property.

Property β**).** $\forall E_n\to\emptyset\Rightarrow\|\chi_{E_n}\|_{X(K)}\to 0$.

The Propositions 1-2 below have been proved in [6, 7, 11].

Proposition 1. *Let* $X(K)$ *be an additive-invariant Banach function space and* $\Omega:\overline{\Omega}\subset K$ *be any domain. If Property* β*) holds, then* $X_s(\Omega)=X_a(\Omega)=X_b(\Omega)=\overline{C_0^\infty(\Omega)}$ *(the closure is taken in topology of* $X(\Omega)$*).*

Proposition 2. *Let* $X(K)$ *be an additive-invariant Banach function space and* $\Omega:\overline{\Omega}\subset K$ *be any domain. If Property* β*) holds, then* $\forall\varphi\in L_\infty(\Omega)$*. It follows that* $\varphi f\in X_s(\Omega)$*.*

By α_X and β_X we will denote the lower and upper Boyd indices of the space X, respectively. For more information about Boyd indices we refer the reader to [1, 2, 17].

Theorem 2. *([2]) Let* X *be a rearrangement-invariant space. For arbitrary* p *and* q *with* $1\le q<\frac{1}{\beta_X}\le\frac{1}{\alpha_x}<p\le\infty$*, the following embedding holds:* $L_p\subset X\subset L_q$*.*

Theorem 3. *Let* X *be a rearrangement-invariant Banach function space with Boyd indices* $\alpha_X,\beta_X: 0<\alpha_X\le\beta_X<1$*. Then singular operator* K *is bounded in* X*:* $K\in[X]$*.*

Theorem 2 has the following

Corollary 1. *Let* $X(K)$ *be a rearrangement-invariant Banach function space with Boyd indices* $\alpha_X,\beta_X: 0<\alpha_X\le\beta_X<1$*, and* $\Omega:\overline{\Omega}\subset K$ *be any bounded domain. Then Property* β*) holds.*

Proof. Under conditions of this corollary, from Theorem 2 it follows that

$$0 < \alpha_X \leq \beta_X < 1 \Rightarrow \exists p, q \in [1; \infty) : \ L_p \subset X \subset L_q.$$

Let $E \to \emptyset$. Taking into account that $\forall E \subset \Omega \Rightarrow \chi_E \in L_p$, we have

$$\|\chi_E\|_X \leq const \, \|\chi_E\|_{L_p} \to 0.$$

The corollary is proved. ◄

We will denote the following spaces of functions by $W_X^m(\Omega)$, $WX_s^m(\Omega)$ and $\overset{0}{W}{}_{X_s}^m(\Omega)$, respectively:

$$W_X^m(\Omega) = \{f \in X : \ \partial^p f \in X, \forall p \in Z_+ : \ |p| \leq m\},$$

$$WX_s^m(\Omega) = \left\{f \in WX^m : \ \|T_\delta f - f\|_{WX^m(\Omega)} \to 0, \, \delta \to 0\right\},$$

$\overset{0}{W}{}_{X_s}^m(\Omega) = \overline{C_0^\infty}(\Omega)$ (closure taken in the space $W_X^m(\Omega)$), with the corresponding norm

$$\|f\|_{W_X^m(\Omega)} = \sum_{|p| \leq m} \|\partial^p f\|_{X(\Omega)}. \tag{1}$$

2.2. Convolution operator

By the convolution of the functions $f, \ h$ defined on $\Omega \subset K$, $f \in L_1(\Omega)$, $h \in X(\Omega)$, we will mean the following relation:

$$(f * h)(x) = \int_{R^n} f_d(x-y) \, h_d(y) \, dy, \tag{2}$$

denoted as $f * g$. In the sequel, we suppose that $\Omega \pm \Omega = \{x \pm y : \ x, y \in \Omega\} \subset K$. In this case, for $x \in \Omega$ the convolution can be defined as follows:

$$(f * h)(x) = \int_\Omega f(x-y) \, h(y) \, dy.$$

Let's state the following well-known classical facts (see, e.g., [4, pp.38-39]).

Let $C_0(R^n)$ be the space of functions $f : \lim\limits_{|x| \to \infty} f(x) = 0$ with the supremum norm and $B(R^n) = (C_0(R^n))^*$ which can be identified with Banach space of finite measure with the norm $\|d\mu\| = \int_{R^n} |d\mu|$, and the space $L^1(R^n)$ can be identified with some subspace of $B(R^n)$ by the map $f(x) \to f(x) \, dx$.

It is well known that $\forall f, g \in L_1(R^n)$ the convolution $(f * g)(.)$ is defined a.e. on R^n and belongs to $L^1(R^n)$. Moreover, the following statement is true:

Let $f \in L^p(R^n)$, $1 \le p \le \infty$. *Then* $g = f * \mu = \int_{R^n} f(x-y)\, d\mu(y) \in L^p(R^n)$ *and*

$$\|g\|_p \le \|f\|_p \|d\mu\|,$$

i.e. the convolution acts continuously in $L^p(R^n)$ ([4]).

What can we say if one of the given functions belongs to a Banach function space? Will the result also belong to the considered Banach space? If the answer to the latter question is positive, will be the convolution operator bounded?

For $f \in X$, the theorem and lemma below have been proved in [6].

Theorem 4. *[6, Th.2.2] Let* X *be a rearrangement-invariant space, and* X' *be an associate space. Then*

$$\|f * g\|_\infty \le \|f\|_X \|g\|_{X'}, \ \ f \in X,\, g \in X'.$$

Moreover, if $f \in X_s$, *or* $g \in X_s'$, *then the convolution operator is continuous in* $L_\infty(K)$.

Lemma 2. *[6, Lemma 2.1] Let* $X(K)$ *be a rearrangement-invariant Banach function space on* K *and* $\Omega : \overline{\Omega} \subset K$ *be some domain. Then for arbitrary pair* $f, g \in X(\Omega)$ *the convolution* $f * g$ *belongs to* X *and the estimate*

$$\|f * g\|_{X(\Omega)} \le \|f\|_{X(\Omega)} \|g\|_{L_1(\Omega)}$$

holds.

It should be noted that the assertions of the above theorem and lemma are true for an additive-invariant spaces. Their proofs are the same as those of Theorem 2.2 and Lemma 2.1 in [6], respectively.

For example, let's prove the lemma in additive-invariant case. Let S' be the unit ball of the associate space X'. Then we have

$$\|f * g\|_{X(\Omega)} = \sup_{v \in S'} \left| \int_\Omega (f * g)(x)\, v(x)\, dx \right| = \sup_{v \in S'} \left| \int_K \int_K f(x-y)\, g(y)\, v(x)\, dy dx \right| =$$
$$\text{(by Fubini's theorem)} = \sup_{v \in S'} \left| \int_K \int_K f(x-y)\, v(x)\, dx g(y)\, dy \right| \le$$
$$\le \int_K \sup_{v \in S'} \left| \int_K f(x-y)\, v(x)\, dx \right| |g(y)|\, dy = \int_K \|f(.-y)\|_X |g(y)|\, dy =$$
$$= \|f\|_{X(\Omega)} \int_\Omega |g(y)|\, dy = \|f\|_{X(\Omega)} \|g\|_{L_1(\Omega)}.$$

Indeed, we only used the relation $\|f(.-y)\|_K = \|f(.)\|_{X(\Omega)}$, i.e. an additive-invariance property of the space.

Therefore, the assertions of the above-mentioned theorem and lemma can be restated as follows.

Theorem 5. *Let $X(K)$ be an additive-invariant Banach function space on K, $\Omega : \overline{\Omega} \subset K$ be some domain and $X'(\Omega)$ be an associate space. Then*

$$\|f * g\|_\infty \leq \|f\|_{X(\Omega)} \|g\|_{X'(\Omega)}, \quad f \in X(\Omega), \, g \in X'(\Omega).$$

If $f \in X_s(\Omega)$, or $g \in X_s'(\Omega)$, then the convolution operator is continuous in $L_\infty(K)$.

Lemma 3. *Let $X(K)$ be an additive-invariant Banach function space on K and $\Omega : \overline{\Omega} \subset K$ be some domain. Then for arbitrary pair $f, g \in X(\Omega)$ the convolution $f * g$ belongs to $X(\Omega)$ and the estimate*

$$\|f * g\|_{X(\Omega)} \leq \|f\|_{X(\Omega)} \|g\|_{L_1(\Omega)}$$

holds.

Lemma 3 has the following corollary.

Corollary 2. *Let $X(K)$ be an additive-invariant Banach function space on K and $\Omega : \overline{\Omega} \subset K$ be some domain. Then for arbitrary functions $f \in L_1(\Omega), g \in X(\Omega)$ the convolution $f * g$ belongs to $X(\Omega)$ and the estimate*

$$\|f * g\|_{X(\Omega)} \leq \|f\|_{L_1(\Omega)} \|g\|_{X(\Omega)}$$

holds.

Proof. It is clear that $\overline{X(\Omega)}_{L_1(\Omega)} = L_1(\Omega)$. Let $f \in L_1(\Omega)$, $\{f_n\} \subset X$: $\lim f_n = f$ in $L_1(\Omega)$ and $\|f_n\|_{L_1} \leq C$. Moreover, let $f_n \to f$ a.e. on Ω. Then, by Lemma 3, we have

$$\|f_n * g\|_{X(\Omega)} \leq \|f_n\|_{L_1(\Omega)} \|g\|_X \leq C \|g\|_X .$$

Taking into account the continuity of the convolution operator in $L_1(\Omega)$, without loss of generality we can assume that $f_n * g \to f * g$ a.e. on Ω. Then from Fatou's lemma it follows that $f * g \in X$ and

$$\|f * g\|_{X(\Omega)} \leq \lim \|f_n * g\|_{X(\Omega)} \leq C \|g\|_{X(\Omega)} .$$

The corollary is proved. ◄

3. Integral operators

3.1. Riesz potential

Riesz potential and Sobolev integral identity play an exceptional role in the study of the properties of functions from $W_p^m(\Omega)$. Recall that by Sobolev identity we mean the following relation ([5, 10]):

$$u(x) = \sum_{|\alpha|=0}^{m-1} x^\alpha \int_\Omega b_\alpha(y)\, u(y)\, dy + \sum_{|\alpha|=m} \int_\Omega \frac{A_\alpha(x,y)}{|x-y|^{n-m}} \partial^\alpha u(y)\, dy \ , \forall u \in C^m(\Omega),$$

where $b_\alpha \in C\left(\overline{\Omega}\right)$, $A_\alpha \in L_\infty(\Omega\times\Omega)$ (in general, $x \neq y : A_\alpha(x,y)$ is infinitely differentiable).

Let $\Omega \subset R^n$ be a bounded domain, $0 \le \alpha < n$, $A(x,y) \in L_\infty(\Omega\times\Omega)$. The operator defined as

$$(R_{A,\alpha} f)(x) = \int_\Omega \frac{A(x,y)}{|x-y|^\alpha} f(y)\, dy \tag{3}$$

is called a Riesz potential.

Let $k_\alpha(x) = \frac{1}{|x|^\alpha}$, $0 \le \alpha < n$. Consider the integral operator K_α with the kernel $k_\alpha(x-y) = \frac{1}{|x-y|^\alpha}$, i.e.

$$(K_\alpha u)(x) = (k_\alpha * u)(x) = \int_\Omega \frac{u(y)}{|x-y|^\alpha} dy. \tag{4}$$

It is clear that the boundedness of the operator K_α implies the boundedness of the operator $R_{A,\alpha}$. In the sequel, we will use the following well-known equality:

$$\int_{|x-y|<r} \frac{dx}{|x-y|^\alpha} = \frac{|B_1|}{n-\alpha} r^{n-\alpha}, \forall y \in R^n, \tag{5}$$

where B_1 is a unit ball (see, e.g., [5, p.19]). Hence it follows that $k_\alpha(.) \in L_1(\Omega)$, $\alpha \in (0, n)$. So we can apply Corollary 2 in this case to get

Corollary 3. *Let $X(K)$ be an additive-invariant Banach function space on K, $\Omega : \overline{\Omega} \subset K$ be any domain and $\alpha \in [0,n)$. Then the integral operator K_α is bounded in $X(\Omega)$ and the estimate*

$$\|K_\alpha g\|_{X(\Omega)} = \|f * g\|_X \le \|k_\alpha(.)\|_{L_1(\Omega)} \|g\|_{X(\Omega)} \le C\|g\|_{X(\Omega)}, \forall g \in X(\Omega) \tag{6}$$

holds. Consequently, Riesz potential is bounded in $X(\Omega)$.

Remark 1. *Let $R > 0$ be some fixed positive number and $\Omega \subset K$: $\Omega \subset B_R(0)$. Then the constant C can be chosen independent of $\Omega \subset K$. In particular, if K is a cube, then we can take $C = \|k_\alpha\|_{L_1(K)}$.*

In the sequel, we are going to study some properties, in particular, compactness of the integral operators generated by the kernels $\psi_{i_1,\dots,i_k}(x) = \frac{x_{i_1} \dots x_{i_k}}{|x|^n}$, i.e. the operators defined by

$$v(x) = (\Psi_{i_1,\dots,i_k} u)(x) =$$

$$= \frac{1}{(k-1)!\sigma_n} \sum_{i_1,\dots,i_k=1}^{n} \int_\Omega \frac{(x_{i_1} - y_{i_1}) \dots (x_{i_k} - y_{i_k})}{|x-y|^n} u(y)\, dy,\ k \in N,$$

on the b.f.s $X(\Omega)$, or on its subspace $X_s(\Omega)$. We will also prove some representation theorems for the functions from $\overset{0}{W}{}^m_{X_s}(\Omega)$, which play an exceptional role in getting embedding theorems. We will denote the corresponding integral operators by $\psi_{i_1\dots i_k}$. It is clear that the estimate

$$|\psi_{i_1\dots i_k}(x)| = \frac{|x_{i_1} \dots x_{i_k}|}{|x|^n} \le \frac{|x|^k}{|x|^n} = \frac{1}{|x|^{n-k}} = k_{n-k}(x),$$

holds. Hence, by Corollary 3 it follows that the considered integral operators are bounded in $X(\Omega)$. Moreover

$$|(\Psi_{i_1\dots i_k} u)(x)| \le (K_{n-k}|u|)(x) \ \ a.e. \Rightarrow \|(\Psi_{i_1\dots i_k} u)\|_{X(\Omega)} \le \|(K_{n-k}|u|)\|_{X(\Omega)},$$

which implies

i) if $U \subset X(\Omega) \Rightarrow K_{n-k}(|U|)$ is relatively compact in $X(\Omega)$, then $\Psi_{i_1\dots i_k}(U)$ is also relatively compact in $X(\Omega)$, where $|U| = \{|u| : u \in U\}$. Consequently, compactness of the operator K_{n-k} in $X(\Omega)$ implies compactness of the operator $\Psi_{i_1\dots i_k}$ in $X(\Omega)$.

ii) $K_{n-k}u \in X_a \Rightarrow \Psi_{i_1\dots i_k} u \in X_a$.

First let's prove that the operator K_α, $0 \le \alpha < n$, acts boundedly in $X_s(\Omega)$, i.e. $X_s(\Omega)$ is an invariant subspace of the operator K_α. It should be noted that we assume that the function from $X(\Omega)$ is extended by zero outside of Ω and the given function is identified with this extended function.

Lemma 4. *Let $X(K)$ be an additive-invariant space with Property β) and Ω : $\overline{\Omega} \subset$ K be any bounded domain in R^n. Then $K_\alpha \in [X_s(\Omega)]$, $0 \le \alpha < n$.*

Proof. Let $u \in X_s(\Omega)$, $v = K_\alpha u \in X(\Omega)$ and $\delta < \delta_0 : \overline{\Omega_{\delta_0}} \subset K$. $\forall x \in \Omega$ we have

$$\begin{aligned}
&|v(x+\delta) - v(x)| = \left|\int_\Omega (k_\alpha(x+\delta-y) - k_\alpha(x-y))\, u(y)\, dy\right| = \\
&= \left|\int_{(\Omega-\delta)} \tfrac{1}{|x-z|^\alpha} u(z+\delta)\, dz - \int_\Omega \tfrac{1}{|x-y|^\alpha} u(y)\, dy\right| = \\
&= \left|\int_{\Omega-\delta \bigcup \Omega} \tfrac{1}{|x-y|^\alpha} (u(y+\delta)) - u(y)\, dy\right| = |(K_\alpha(T_\delta |u(.)| - |u(.)|)(x)| \Rightarrow \\
&\Rightarrow \|v(.+\delta) - v(.)\|_{X(K)} \le const \, \|K_\alpha\|_{[X(K)]} \, \|T_\delta |u| - |u|\|_{X(K)} \underset{\delta \to 0}{\longrightarrow} 0 \Rightarrow \\
&\Rightarrow v(.) \in X_s(\Omega).
\end{aligned}$$

The lemma is proved. ◄

Now let's prove that the operator K_α, $0 \le \alpha < n$, is compact in $X_s(\Omega)$. For this, we will prove that the operator K_α can be uniformly approximated by compact operators.

Consider the general case. Let $\Omega \subset K \subset R^n$, $\Omega' \subset R^m$, $X(\Omega)$ be a Banach function space, and $k(x,y)$ be a function defined on $\Omega' \times \Omega$. Consider the integral operator K defined as

$$v(x) = Ku = \int_\Omega k(x,y)\, u(y)\, dy.$$

Let the kernel $k(x,y)$ have the following properties:

i) $k(x,y)$ is a bounded function, i.e., $\exists b > 0 : \sup\limits_{x \in K, y \in \Omega} |k(x,y)| \le b < \infty$;

ii) $k(x,y)$ is uniformly continuous with respect to the first variable:

$$\sup_{\substack{x, z \in \Omega' \\ |x-z| \le r}} \sup_{y \in \Omega} |k(x,y) - k(z,y)| \le \omega(r) \underset{r \to 0}{\longrightarrow} 0.$$

Lemma 5. *Let $\Omega \subset K$ be any domain, $\Omega' \subset R^m$ be an arbitrary bounded measurable subset, and $k(x,y)$ have the properties i)-ii). Then the operator K acts compactly from $X(\Omega)$ to $C\left(\overline{\Omega'}\right)$.*

Proof. Let $U \subset X(\Omega)$ be a bounded subset. For an arbitrary function $u \in U$, using Holder's inequality, we have

$$\begin{aligned}
&|v(x)| = \left|\int_\Omega k(x,y)\, u(y)\, dy\right| \le \int_\Omega |k(x,y)\, u(y)|\, dy \le b \int_\Omega 1 \cdot |u|\, dy \le \\
&\le b \, \|1\|_{X'(\Omega)} \, \|u\|_{X(\Omega)}, \ \forall x \in \overline{\Omega'}.
\end{aligned}$$

Thus, the range of a bounded set of $X(\Omega)$ is a bounded set in $C\left(\overline{\Omega'}\right)$.

If $|x-z| \le r$, using *ii)* we have

$$|v(x) - v(z)| \le \int_\Omega |k(x,y) - k(z,y)|\,|u(y)|\,dy \le$$

$$\le \omega(r) \int_\Omega |u(y)|\,dy \le \omega(r) \cdot \|1\|_{X'(\Omega)} \|u\|_{X(\Omega)} \underset{r\to 0}{\longrightarrow} 0.$$

Hence, the operator K transforms a bounded set of $X(\Omega)$ to an equicontinuous family in $C\left(\overline{\Omega'}\right)$. From the Arzela-Ascoli theorem it follows that $K(U)$ is a relatively compact set in $C\left(\overline{\Omega'}\right)$. Consequently, K is a compact operator acting from $X(\Omega)$ into $C\left(\overline{\Omega'}\right)$.

The lemma is proved. ◄

Corollary 4. *Let* $\Omega = \Omega' \subset K$ and $k(x,y)$ *have the properties i)-ii). Then the operator* K *acts compactly from* $X(\Omega)$ *into* $X_s(\Omega)$*.*

Proof. As is known, the continuous embedding $C\left(\overline{\Omega}\right) \subset X(\Omega)$ holds, i.e. if a sequence converges in $C\left(\overline{\Omega}\right)$, then it also converges in $X_s(\Omega)$.

The corollary is proved. ◄

Corollary 5. *Let* $\Omega = \Omega' \subset K$ and $k(x,y)$ *have the properties i)-ii). Then the operator* K *acts compactly from* $X_s(\Omega)$ *into* $X_s(\Omega)$*.*

Consider the operator K_α, $0 \le \alpha < n$. Define the following kernels and corresponding integral operators. Let $\varphi(r)$ be a smooth monotone function equal to 1 if $r \ge 1$, and to 0 if $r \le \frac{1}{2}$. Let

$$\varphi_h(r) = \varphi\left(\frac{r}{h}\right),\ k_{\alpha,h}(x,y) = \frac{1}{|x-y|^\alpha}\varphi_h(|x-y|).$$

It is obvious that the kernel $k_{\alpha,h}(x,y)$ is a bounded and continuous function of x and y. Consequently, it has the properties *i)-ii)*. Hence it follows that the integral operators $K_{\alpha,h}$ act compactly from $X(\Omega)$, and also from $X_s(\Omega)$, into $X_s(\Omega)$.

Corollary 6. *Let* $X(K)$ *be an additive-invariant space with Property* β*). Let* $\Omega : \overline{\Omega} \subset K$ *be any bounded domain in* R^n*. Then the estimate*

$$\|K_\alpha - K_{\alpha h}\|_{[X(\Omega)]} \underset{h\to 0}{\longrightarrow} 0$$

holds. Consequently, K_α *is a compact operator acting in* $X(\Omega)$*.*

Proof. Let $u \in X_s(\Omega)$. Taking into account the relation

$$k_\alpha(x-y) - k_{\alpha h}(x-y) =$$

$$= \begin{cases} 0, & |x-y| > h, \\ |k_\alpha(x-y)(1-\varphi_h(|x-y|))| \le k_\alpha(x-y), & \frac{h}{2} < |x-y| \le h, \\ k_\alpha(x-y), & |x-y| \le \frac{h}{2}, \end{cases}$$

by Corollary 4 we have

$$\begin{aligned} &\|((K_\alpha - K_{\alpha h})u)(x)\|_{X(\Omega)} = \left\| \int_\Omega (k_\alpha(x-y) - k_{\alpha h}(x-y))u(y)\,dy \right\|_{X(\Omega)} \le \\ &\le \|k_\alpha(.) - k_{\alpha,h}(.)\|_{L_1(\Omega)} \|u\|_{X(\Omega)} = \|k_\alpha(.) - k_{\alpha,h}(.)\|_{L_1(B_h(0))} \|u\|_{X(\Omega)} \le \\ &\le \|k_\alpha(.) - k_{\alpha,h}(.)\|_{L_1(B_h(0))} \|u\|_{X(\Omega)}. \end{aligned}$$

Hence, by the relation (5) it follows that

$$\|K_\alpha - K_{\alpha h}\|_{[X(\Omega)]} \underset{h\to 0}{\longrightarrow} 0,$$

i.e. the operator K_α can be approximated by compact operators. Consequently, it is a compact operator.

The corollary is proved. ◀

Theorem 6. *Let $X(K)$ be an additive-invariant space with Property β) and $\Omega : \overline{\Omega} \subset K$ be any bounded domain in R^n. Then, for an arbitrary $u \in \overset{0}{W}{}^1_{X_s}(\Omega)$ the following representation is true:*

$$u(x) = \frac{1}{\sigma_n} \sum_{i=1}^{n} \int_\Omega \frac{x_i - y_i}{|x-y|^n} \frac{\partial u}{\partial y_i} dy,$$

where σ_n is an area of the unit sphere in R^n, i.e. $\sigma_n = 2\pi^{\frac{n}{2}} \left(\Gamma\left(\frac{n}{2}\right)\right)^{-1}$.

Proof. The proof is a modification of the classical one. In case $u \in C_0^\infty(\Omega)$, all argumentations of classical case hold. In general case of $\forall u \in \overset{0}{W}{}^1_{X_s}(\Omega)$, we use the fact that $\exists \{u_k\} \subset C_0^\infty(\Omega) : \lim u_k = u$ in $W^1_{X_s}(\Omega)$. Taking into account the continuity of the operator K_i in $X(\Omega)$ and differentiation operator from $W^1_X(\Omega)$ into $X(\Omega)$, we have

$$\frac{\partial u_k}{\partial x_j} \to \frac{\partial u}{\partial x_j},\ k \to \infty,\ j = \overline{1,n},$$

which ends the proof. ◀

Corollary 7. *The space* $\overset{0}{W}{}^{1}_{X_s}(\Omega)$ *can be compactly embedded into* $X_s(\Omega)$*.*

Proof. This is a direct consequence of Theorem 6 and Corollary 6. ◀

Corollary 8. *Let* $X(K)$ *be an additive-invariant Banach function space and the domain* $\Omega : \overline{\Omega} \subset K$ *admit extension of the functions from* $X(\Omega)$*. Then the assertion of Corollary 6 is true for the space* $W^1_{X_s}(\Omega)$*.*

Proof. Let $\Omega_1 : \overline{\Omega} \subset \Omega_1, \overline{\Omega_1} \subset K$ and $\theta : W^1_{X_s}(\Omega) \to \overset{0}{W}{}^{1}_{X_s}(\Omega_1)$ be an extension operator. Let $U \subset W^1_{X_s}(\Omega)$ be any bounded subset. It is clear that $\theta(U) \subset \overset{0}{W}{}^{1}_{X_s}(\Omega_1)$ will be a bounded subset of $\overset{0}{W}{}^{1}_{X_s}(\Omega_1)$. From Corollary 6 it follows that $\theta(U)$ is compact in $X_s(\Omega_1)$, which implies that $U = \theta U|_{\Omega}$ is compact in $X_s(\Omega)$.

The corollary is proved. ◀

To generalize the assertion of Theorem 6, we need the following statement (see [10, p.174]):

Let $v \in C_0^{\infty}(R^n)$ and $k \in N$ *be a given number. Then the following representation formula is true:*

$$v(x) = \frac{1}{(k-1)!\sigma_n} \sum_{|i|=k} \int_{R^n} \frac{(x_{i_1} - y_{i_1})^{l_i} \ldots (x_{i_n} - y_{i_n})^{i_n}}{|x-y|^n} \frac{\partial^k v(y)}{\partial y^i} dy, \tag{7}$$

where $i = (i_1, \ldots, i_n)$*.*

Indeed, for fixed $i : |i| = k$ the following is true:

$$\sum_{|j|=1}^{n} \frac{\partial}{\partial x_j} \frac{(x_{i_1})^{l_i} \ldots (x_{i_n})^{i_n}}{|x|^n} = k \frac{(x_{i_1})^{l_i} \ldots (x_{i_n})^{i_n}}{|x|^n}.$$

From this it follows that the right-hand side in (7) is equal to

$$\sum_{|i|=k} \int_{R^n} \frac{(x_{i_1} - y_{i_1})^{l_i} \ldots (x_{i_n} - y_{i_n})^{i_n}}{|x-y|^n} \frac{\partial^k v(y)}{\partial y^i} dy =$$
$$= (k-1)! \sum_{|i|=k} \int_{R^n} \frac{(x_{i_1} - y_{i_1})^{l_i} \ldots (x_{i_n} - y_{i_n})^{i_n}}{|x-y|^n} \frac{\partial v(y)}{\partial y_j} dy \overset{by\, th.3.1}{=} (k-1)!\sigma_n v(x).$$

Corollary 9. *Let* $X(K)$ *be an additive-invariant Banach function space with Property* β) *and* $\Omega : \overline{\Omega} \subset K$ *be any bounded domain. For arbitrary* $u \in \overset{0}{W}{}^{m}_{X_s}(\Omega)$*, the representation formula*

$$v(x) = \frac{1}{(m-1)!\sigma_n} \sum_{|i|=m} \int_{R^n} \frac{(x_{i_1} - y_{i_1})^{l_i} \ldots (x_{i_n} - y_{i_n})^{i_n}}{|x-y|^n} \frac{\partial^m v(y)}{\partial y^i} dy$$

is true.

Proof. For $k = 1$, the assertion has been proved in Theorem 6. In general case, it is a direct consequence of the relation $\overset{0}{W}{}^{m}_{X_s}(\Omega) = \overline{C_0^\infty(\Omega)}$, boundedness of the potential-type integral operators and representation formula (7). ◄

In particular, from this corollary we obtain the following embedding-type statement.

Corollary 10. *Let $X(K)$ be an additive-invariant Banach function space. Then*

a) the space $\overset{0}{W}{}^{m}_{X_s}(\Omega)$ can be compactly embedded into $X_s(\Omega)$;

b) if the domain $\Omega : \overline{\Omega} \subset K$ admits extension of the functions from $W^m_{X_s}(\Omega)$, then $W^m_{X_s}(\Omega)$ can be compactly embedded into $X_s(\Omega)$.

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References

[1] C. Bennett, R. Sharpley, *Interpolation of Operators*, Academic Press, 1988, 469 p.

[2] D.W. Boyd, *Spaces between a pair of reflexive Lebesgue spaces*, Proc. Amer. Math. Soc., **18(2)**, 1967, 215-219.

[3] B.T. Bilalov, S.R. Sadigova, *On the Fredholmness of the Dirichlet problem for a second-order elliptic equation in grand-Sobolev spaces*, Ricerche mat., **73**, 2024, 283–322.

[4] E.M. Stein, *Singular operator and differentiability properties of functions*, Moscow, Mir, 1973 (translation into Russian)

[5] S.G. Mikhlin, *Linear partial differential equations*, Moscow, Visshaya skola, 1977 (in Russian).

[6] B.T. Bilalov, S.R. Sadigova, *On local solvability of higher order elliptic equations in rearrangement invariant spaces*, Siberian Mathematical Journal, **63(3)**, 2022, 516-530.

[7] E.M. Mamedov, *On substitution and extension operators in Banach-Sobolev function spaces*, Proceedings of the Institute of Math. and Mech., Nat. Ac. of Sciences of Azer., **48(1)**, 2022, 88-103.

[8] R.E. Castillo, H. Rafeiro, *An introductory course in Lebesgue spaces*, Springer, 2016.

[9] E.M. Mamedov, N.A. Ismailov, *On structural theorems in Banach function spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics, **43(4)**, 2023, 114-127.

[10] V.A. Solonnikov, N.N. Uraltseva, *Sobolev spaces*, in: Selected Topics of Higher Algebra and Analysis, Leningrad, Leningrad Gos. Univ., 1981, 129-199 (in Russian).

[11] B.T. Bilalov, E.M. Mamedov, Y. Sezer, N.P. Nasibova, Compactness in Banach function spaces. Poincare and Friedrichs inequalities. (submitted)

[12] B.T. Bilalov, S.R. Sadigova, *On solvability in the small of higher order elliptic equations in grand-Sobolev spaces*, Complex Variables and Elliptic Equations, **66(12)**, 2021, 2117-2130.

[13] B.T. Bilalov, S.R. Sadigova, *Interior Schauder-type estimates for higher-order elliptic operators in grand-Sobolev spaces*, Sahand Communications in Mathematical Analysis, **1(2)**, 2021, 129-148.

[14] E.M. Mamedov, S. Cetin, *Interior Schauder-type estimates for m-th order elliptic operators in rearrangement-invariant Sobolev spaces*, Turk J Math., **48(4)**, 2024, 793-816.

[15] B.T. Bilalov, T.M. Ahmadov, Y. Zeren, S.R. Sadigova, *Solution in the Small and Interior Shauder-type Estimate for the m-th Order Elliptic Operator in Morrey-Sobolev Spaces*, Azerb. J. Math, **12(2)**, 2022, 190-219.

[16] B.T. Bilalov, S.R. Sadigova, S. Cetin, *The concept of a trace and boundedness of the trace operator in Banach-Sobolev function spaces*, Numerical Functional Analysis and Optimization, **43(9)**, 2022, 1069-1094.

[17] B.T. Bilalov, *Hardy's Banach functional classes and methods of boundary value problems in questions of bases*, Baku, Elm, 2022, 272 p. (in Russian).

[18] S.S. Byun, D.K. Palagachev, L.G. Softova, *Survey on gradient estimates for nonlinear elliptic equations in various function spaces*, St. Petersburg Math. J., **31(3)**, 2020, 401-419 and Algebra Anal., **31(3)**, 2019, 10-35.

[19] D.K. Palagachev, L.G. Softova, *Elliptic Systems in Generalized Morrey Spaces*, Azerb. J. Math., **11(2)**, 2021, 153-162.

[20] L. Caso, R. D'Ambrosio, L. Softova, *Generalized Morrey Spaces over Unbounded Domains*, Azerb. J. Math., **10(1)**, 2020, 193-208.

[21] V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral Operators in Non-Standard Function Spaces*, Volume 1: Variable Exponent Lebesgue and Amalgam Spaces, Springer, 2016.

[22] V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral Operators in Non-Standard Function Spaces*, Volume 2: Variable Exponent Hölder, Morrey–Campanato and Grand Spaces, Springer, 2016.

[23] D. Cruz-Uribe, O.M. Guzman, H. Rafeiro, *Weighted Riesz Bounded Variation Spaces and the Nemytskii operator*, Azerb J. Math., **10(2)**, 2020, 125-139.

[24] R.E. Castillo, H. Rafeiro, E.M. Rojas, *Unique Continuation of the Quasilinear Elliptic Equation on Lebesgue Spaces L_p*, Azerb J. Math., **11(1)**, 2021, 136-153.

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