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Carleman's Integral Formula in Cartesian Product of Matrix Upper Half-Plane

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Abstract. Carleman's formulas solve the problem of recovery of a function from such a class of its values on the set of uniqueness $M \subset \partial D$ for this class that does not contain the Shilov boundary ∂D. In this paper, by using the matrix upper half-plane and the biholomorphic equivalence of the matrix unit disc, the Carleman formula for the Cartesian product of matrix upper half-planes is proved;

Key Words and Phrases: matrix unit disc, automorphism, holomorphic function, Cauchy-Szegő's kernel, Carleman's formulas, Cartesian product.

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1. Introduction, preliminaries and problem statement

The problem of finding the Carleman formulas has long attracted the attention of many specialists in complex analysis and mathematical physics. These formulas are useful for solving many problems of mathematical physics and analysis. For example, to restore the values of holomorphic or harmonic functions in a domain from their values on sets of uniqueness. The first result in this direction was obtained by T. Carleman in 1926 (see [15]), therefore, all formulas of this type are called Carleman formulas. This research was continued by G.M. Goluzin and V.I. Krylov (see [2]), who gave a general method for obtaining Carleman's formulas in one-dimensional complex analysis. Their method did not work in the multidimensional case. In 1956, M.M. Lavrent'ev (see [5]) suggested his own method based on the approximation of kernels of integral representations. Many examples of the construction of Carleman's formulas in various problems of mathematical physics can be found in the monograph [6].

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Further development of this theory in multidimensional complex analysis can be found in the book by L.A. Aizenberg (see [1]), who constructed the Carleman formula on the basis of the Bochner-Martinelli integral representation in convex (or linearly convex) bounded domains. In recent years, A. Tumanov, A. Vidras and G. Chailos obtained important results concerning Carleman-type integral formulas. Such integral formulas are useful for studying dependency of the values of holomorphic functions inside domain on its values on the boundary or on a part of the boundary of the domain. They are also useful in holomorphic continuation problems (see, for example, [21, 22, 23, 24, 25]). In complex analysis it is important to use the holomorphic continuation of a function defined on the skeleton or on the part of the skeleton of the matrix ball associated with the classical domain into the matrix ball (see [9, 11, 17, 18, 19, 20]). In homogeneous domains in $\mathbb{C}^n$, automorphism of groups can be used for finding such formulas (see [1]). In [4], the case of Siegel domains, i.e., unbounded realizations of homogeneous domains has been considered and Carleman formulas have been stated that restore the values of holomorphic functions on the skeleton of the Siegel domain.

1.1. Matrix unit circle and matrix upper half-plane

Consider the space $\mathbb{C}^{m^2}$, the space of complex m^2 variables. In some problems, it is convenient to represent the point Z of this space as $Z = (z_{ij})_{i,j=1}^m$, i.e., as square $[m \times m]$-matrices. With such a representation for this point, the space $\mathbb{C}^{m^2}$ will be denoted by $\mathbb{C}[m \times m]$. Denote by $\mathbb{C}^n[m \times m]$ the direct product

$$\underbrace{\mathbb{C}[m \times m] \times \cdots \times \mathbb{C}[m \times m]}_{n}$$

of n instances of $[m \times m]$-matrix spaces.

Let $Z = (Z_1, ..., Z_n)$ be a vector composed by square matrices Z_j of order m, considered over the field of complex numbers $\mathbb{C}$. Let us write the elements of the vector $Z = (Z_1, ..., Z_n)$ as points z of the space $\mathbb{C}^{nm^2}$:

$$z = (z_{11}^{(1)}, ..., z_{1m}^{(1)}, ..., z_{m1}^{(1)}, ..., z_{mm}^{(1)}, ..., z_{11}^{(n)}, ..., z_{1m}^{(n)}, ..., z_{m1}^{(n)}, ..., z_{mm}^{(n)}) \in \mathbb{C}^{nm^2}. \quad (1)$$

Hence, we can assume that Z is an element of the space $\mathbb{C}^n\,[m \times m]$, i.e., we came to the isomorphism $\mathbb{C}^n\,[m \times m] \cong \mathbb{C}^{nm^2}$.

The unit matrix disk is defined as the set τ (see [10, 11]):

$$\tau = \{Z \in \mathbb{C}\,[m \times m] : ZZ^* < I\},$$

where $Z^* = \bar{Z}'$ is the conjugate transpose of the matrix Z, the notation $ZZ^* < I$ (I is the unit $[m \times m]$-matrix) means that the Hermitian matrix $I - ZZ^*$ is positive definite, thus, all its eigenvalues are positive.

The skeleton (Shilov's boundary) of a matrix disk is the set

$$S(\tau) = \{Z \in \mathbb{C}[m \times m] : ZZ^* = I\} \ .$$

Let $Z = (Z_1,\, Z_2,\, ...,\, Z_n) \in \mathbb{C}^n\,[m \times m]$ be a vector composed of square matrices of order m, considered over the field of complex numbers $\mathbb{C}$.

The matrix unit polydisc T^n in the space $\mathbb{C}^n\,[m \times m]$ is defined as the direct product of n matrix discs (see [13]):

$$T^n = \underbrace{\tau \times \tau \times ... \times \tau}_{n\ ta} = \left\{Z = (Z_1,\, Z_2,\, ...,\, Z_n)\ : Z_j(Z_j)^* < I,\, j = \overline{1,n}\right\}.$$

The skeleton of the matrix unit polydisc is the set

$$S\left(T^n\right) = \underbrace{S\left(\tau\right) \times S\left(\tau\right) \times ... \times S\left(\tau\right)}_{n\ ta} = \left\{U \in \mathbb{C}^n\left[m \times m\right]\ : U_j(U_j)^* = I,\, j = \overline{1,n}\right\}.$$

Consider an automorphism of the matrix unit polydisc T^n, permuting the points $A \in T^n$ and 0. Such an automorphism has the form (see [8])

$$\Phi_A\left(Z\right) = \left(\Phi_A^1\left(Z^1\right),\, ...\, ,\, \Phi_A^n\left(Z^n\right)\right),$$

where

$$\Phi_A^j\left(Z_j\right) = Q_j\left(Z_j - A_j\right)\left(I - (A_j)^* Z_j\right)^{-1}(R_j)^{-1}, \ \ j = \overline{1,n},$$

$[m \times m]$ matrices Q_j and R_j satisfy the conditions

$$\overline{Q}_j\left(I - \bar{A}_j(A_j)'\right)(Q_j)' = I,$$

$$\overline{R}_j\left(I - (A_j)'\bar{A}_j\right)(R_j)' = I,$$

$$Q_j A_j + A_j R_j = 0,$$

In particular, for $A = 0$ we get

$$\Phi_0\left(Z\right) = \left(\Phi_0^1\left(Z_1\right),\, ...\, ,\, \Phi_0^n\left(Z_n\right)\right)$$

and

$$\Phi_0^j\left(Z_j\right) = Q_j Z_j (R_j)^{-1}.$$

Let $M = (M_1,\, ...\, ,\, M_n) \subset S\left(T^n\right)$ and $\mu\left(M\right) > 0$. Denote $'U = (U_2, ..., U_n)$, where $S\left(T^{n-1}\right)$ is a projection of the skeleton $S\left(T^n\right)$ on the space $\mathbb{C}^{n-1}\left[m \times m\right]$.

Let's define the sets

$$M_{0,U} = \left\{Z :\ Z \in M,\ \Phi_0^1\left(Z_1\right) = \lambda,\ \Phi_0^j\left(Z_j\right) = \lambda\Phi_0^j\left(U_j\right),\ j = \overline{2.n},\, \lambda \in S\left(\tau\right)\right\},$$

$$\tilde{M}_0 = \{Z \in M \,:\, \mu_1 M_{0,U} > 0\},$$

The sets $M_{0,U}$ and $\tilde{M}_0$ are subsets with positive measure of the set M and their Cartesian product coincides with M, i.e. $M = M_{0,U} \times \tilde{M}_0$.

We introduce an auxiliary function $\varphi_0 = \exp\,\psi_0$, where

$$\psi_0\,(U) = \frac{1}{2\pi i} \int\limits_{M^1_{0,U^1}} \frac{n+\lambda}{n-\lambda} \cdot \frac{d\eta}{\eta},\ M^1_{0,U^1} \times \tilde{M}_{0,U} = M_{0,U}.$$

We define the class $H^p(D)$ $(p > 0)$ as the class of all functions f, holomorphic in D for which

$$\sup_{0<r<1} \int\limits_{S(D)} |f(rU)|^p d\mu < +\infty,$$

where $rU = (ru_{11}, ru_{12}, ..., ru_{mm})$, and $d\mu$ is a normalized Lebesgue measure on the manifold $S(D)$, which is invariant under rotation. The following statement holds (see [14]).

Lemma 1. *Let $f \in H^1\,(T^n)\,,\ M \subset S\,(T^n)$ and $\mu\,(M) > 0$. Then the following formula is valid:*

$$f\,(0) = \frac{m}{\int\limits_{\tilde{M}_0} d\mu_{n-1} \int\limits_{\tilde{M}_{0,U}} d\mu_0} \lim_{l\to\infty} \int\limits_M f\,(Z) \left[\frac{\varphi_0\,(U)}{\varphi_0\,(0)}\right]^l d\mu. \tag{2}$$

The formula (2) restores the value of the function f at the point 0 from its values on M. Consider the following set:

$$M_{A,U} = \left\{Z :\ Z \in M,\ \Phi^1_A\,(Z_1) = \lambda,\ \Phi^j_A\,(Z_j) = \lambda \Phi^j_A\,(U_j)\,,\ j = \overline{2.n},\ \lambda \in S\,(\tau)\right\}.$$

This set is measurable with respect to the measure μ_1 for almost all A and $'U$. The set $\{'U\ :\ 'U \in S\,(T_{n-1})\,,\ \mu_1 M_{A,U} > 0\}$ is denoted by $\tilde{M}_A$. It is obvious that $M_{A,U} \times \tilde{M}_A = M$. It follows from Fubini's theorem that the Lebesgue $(mn - m)$ dimensional measure of this set is positive.

Similarly to the previous case, we introduce an auxiliary function

$$\varphi_A = \exp\psi_A,\ \psi_A\,(U) = \frac{1}{2\pi i} \int\limits_{M^1_{A^1,K^1}} \frac{n+\lambda}{n-\lambda} \frac{d\eta}{\eta},$$

where

$$M_{A^1,K^1} = \left\{U^1 \in M^1,\ U = \left(\Phi^1_A\right)^{-1} \left(\lambda \left(\Phi^1_A\right)^{-1}(K)\right),\ |\lambda| = 1\right\},\ K \in \Phi^1_A\,(SU\,(m))\,.$$

The following theorem holds (see [14]).

Theorem 1. *Let $f \in H^1(T^n)$, $M \subset S(T^n)$, $\mu(M) > 0$. Then any point $A \in T^n$ satisfies Carleman's formula*

$$f(A) = \frac{m}{\mu_{mn-m}\left(\Phi_A^{-1}\left(\tilde{M}_A\right)\right)\mu\left(\Phi_A^{-1}\left(\tilde{M}_{A,U}\right)\right)} \times$$

$$\times \lim_{l\to\infty}\int\limits_M f(Z)\left[\frac{\varphi_A(Z)}{\varphi_A(A)}\right]^l \prod_{j=1}^n H\left(A_j, \bar{Z}_j\right) d\mu(Z),$$

where

$$H\left(A_j, \bar{Z}_j\right) = \frac{1}{\det\left(I - A_j Z_j\right)^m}$$

is a Cauchy-Szegő kernel for the matrix unit circle.

Consider the following unbounded domain (see [13]):

$$D = D_1 \times D_2 \times ... \times D_n = \left\{W = (W_1, ..., W_n) \in \mathbb{C}^n[m \times m] \ : \operatorname{Im} W_j > 0,\ j = \overline{1, n}\right\}$$

where $W_j \in \mathbb{C}[m \times m]$.

The skeleton of this domain is denoted by Γ :

$$\Gamma = \Gamma_1 \times \Gamma_2 \times ... \times \Gamma_n = \left\{V = (V_1,\, V_2,\, ...\,,\, V_n) \in \mathbb{C}^n[m \times m] : \operatorname{Im} V_j = 0,\, j = \overline{1, n}\right\}.$$

Let $\Phi = \left(\Phi^1, ..., \Phi^n\right)$ denote the Cayley transform

$$W_j = \Phi^j(Z) = i\left(I + Z_j\right)\left(I - Z_j\right)^{-1},\ j = \overline{1, n}, \tag{3}$$

which biholomorphically maps T^n onto D, while $S(T^n)$ goes onto Γ (see [12, 13]). Let $\dot{U}$ be a volume element in $S(T^n)$, and $\dot{V}$ be a volume element in Γ. It is known that

$$\dot{U} = \prod_{j=1}^n \dot{U}_j,\ \ \dot{V} = \prod_{j=1}^n \dot{V}_j,$$

where $\dot{U}_j \in \{U_j(U_j)^* = I\}$, $\dot{V}_j \in \{\operatorname{Im} V_j = 0\}$. In [8], the following relation between $\dot{U}_j$ and $\dot{V}_j$ under the mapping Φ was proved:

$$\dot{U}_j = 2^{m^2}|\det\left(U_j + iI\right)|^{-2m}\dot{V}_j.$$

The lemma below is true (see [13]):

Lemma 2. *The following relation is true:*

$$\dot{U} = \dot{U}_1 \wedge ... \wedge \dot{U}_n = 2^{nm^2}\prod_{j=1}^n |\det\left(V_j - iI\right)|^{-2m}\dot{V}.$$

Let $f \in \mathcal{O}(D)$. Note that the formula

$$f\left(i\,(I+Z_j)\,(I-Z_j)^{-1}\right) \in H^1(T^n),\ j=\overline{1,n}$$

is true if and only if

$$\frac{f\,(W_j)}{\det^2(W_j+iI)} \in H^1\,(D_n)\,,\ \ j=\overline{1,n}. \tag{4}$$

(see [3, 16, 20])

2. Main result

Theorem 2. *If the function $f \in \mathcal{O}(D)$ satisfies the condition* (4) *and the set $\tilde{M} = \tilde{M}_1 \times ... \times \tilde{M}_n \subset \Gamma$ has a positive Lebesgue measure, then the following Carleman formula is true:*

$$f\,(W) = \prod_{j=1}^{n} \frac{\det{}^{m}\,(W_j+iI)}{i^{nm^2}} \cdot \lim_{l\to\infty} \int\limits_{\tilde{M}} f\,(V) \left[\frac{\tilde{\varphi}\,(V)}{\tilde{\varphi}\,(W)}\right]^l \times$$

$$\times \prod_{j=1}^{n} \det{}^{-m}\,((V_j)^* - W_j)\det{}^{-m}\,(V_j+iI)\,d\mu_{V_j}, \tag{5}$$

where the convergence is uniform on compact sets of the skeleton, with $V_j \in \tilde{M}_j,\ V \in \tilde{M}$.

Proof. Let

$$F(Z_j) = f\left(i\,(I+Z_j)\,(I-Z_j)^{-1}\right).$$

Then $F(Z) \in H^1(T^n)$ and by Theorem 1 it satisfies the Carleman formula

$$F\,(Z) = \lim_{l\to\infty} \int\limits_{M} F\,(U) \left[\frac{\varphi\,(U)}{\varphi\,(Z)}\right]^l \prod_{j=1}^{n} \frac{d\mu_U}{\det^m(I - Z_j U_j^*)},$$

where $M = M_1 \times M_2 \times ... \times M_n \subset S(T)$ is an image of $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2 \times ... \times \tilde{M}_n \subset \Gamma$ by the mapping $Z_j = (W_j + iI)(W_j - iI)^{-1}$ of the matrix upper half-plane to the matrix unit circle.

Next, consider the inverse mapping to (3)

$$Z_j = (W_j+iI)^{-1}(W_j-iI),\ \ U_j = (V_j+iI)^{-1}(V_j-iI),$$

and do the following calculations:

$$I-Z_jU_j^*=I-(W_j+iI)^{-1}(W_j-iI)(V_j^*+iI)(V_j^*-iI)^{-1}=$$

$$=(W_j+iI)^{-1}\left[(W_j+iI)(V_j^*-iI)-(W_j-iI)(V_j^*+iI)\right](V_j^*-iI)^{-1}=$$

$$=(W_j+iI)^{-1}\left[W_jV_j^*-iW_j+iV_j^*+I-W_jV_j^*-iW_j-iV_j^*-I\right](V_j^*-iI)^{-1}=$$

$$=2i(W_j+iI)^{-1}\left[V_j^*-W_j\right](V_j^*-iI)^{-1}.$$

Then the condition of Lemma 2 will be satisfied:

$$d\mu_U=2^{nm^2}\prod_{j=1}^{n}|\det(V_j+iI)|^{-2m}d\mu_V.$$

Calculations show that

$$\prod_{j=1}^{n}\frac{d\mu_U}{\det^m(I-Z_jU_j^*)}=\frac{1}{(2i)^{m^2}\prod\limits_{j=1}^{n}\det^{-m}(W_j+iI)\det^m(V_j^*-W_j)\det^{-m}(V_j^*-iI)}\times$$

$$\times 2^{nm^2}\prod_{j=1}^{n}\det|(V_j+iI)|^{-2m}d\mu_V=\frac{\prod\limits_{j=1}^{n}\det^m(W_j+iI)\prod\limits_{j=1}^{n}\det^m(V_j^*-iI)}{(2i)^{nm^2}\prod\limits_{j=1}^{n}\det^m(V_j^*-W_j)}\times$$

$$\times\frac{2^{nm^2}d\mu_V}{\prod\limits_{j=1}^{n}\overline{\det^m(V_j+iI)}\det^m(V_j+iI)}=\prod_{j=1}^{n}\frac{\det^m(W_j+iI)}{i^{nm^2}\det^m(V_j^*-W_j)\det^m(V_j+iI)}d\mu_V.$$

Further, φ plays the role of $\tilde{\varphi}$ for the set $M=M_1\times M_2\times ...\times M_n\subset S(T^n)$. By M.A. Lavrent'ev theorem (see [7]), $M=M_1\times M_2\times ...\times M_n\subset S(T^n)$ is also the set of positive Lebesgue measures such that the harmonic measure $M=M_1\times M_2\times ...\times M_n\subset S(T^n)$ maps into harmonic measure $\tilde{M}=\tilde{M}_1\times\tilde{M}_2\times...\times\tilde{M}_n\subset\Gamma$, therefore, φ maps to $\tilde{\varphi}$, and we get the formula (4).

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The results obtained in this article are the general cases of the results of L. Aizenberg, G. Khudaiberganov, and their results coincide with our results in particular cases. If $n=1$ in Theorem 2, then formula (5) overlaps with the Carleman formula for the upper half-plane of the matrix (see [16]), particularly, when $m=n=1$ it gives Carleman formula for the upper half-plane (see [1]).

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