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Super and Strong γ $\mathcal{H}$-Lindelöfness in Hereditary m-Spaces

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Abstract. Let $(X, m, \mathcal{H})$ be a hereditary m-space and $\gamma : m \to P(X)$ be a γ-operation on m. A subset A of X is said to be $\gamma\mathcal{H}$-Lindelöf relative to X [1] if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of A by m-open sets of X, there exists a countable subset Δ_0 of Δ such that $A \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$. In this paper, we define and investigate two kinds of strong forms of "γ $\mathcal{H}$-Lindelöf relative to X".

Key Words and Phrases: hereditary m-space, $\gamma\mathcal{H}$-Lindelöfness, strong $\gamma\mathcal{H}$-Lindelöfness, super $\gamma\mathcal{H}$-Lindelöfness.

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1. Introduction

In 1991, Ogata [2] introduced the notions of γ-operations and γ-open sets in a topological space and investigated the associated topology τ_γ and weak separation axioms γ-T_i $(i = 0, 1/2, 1, 2)$. In [3], a minimal structure and a minimal space (X, m) are introduced and investigated. In 2011, Noiri [4] defined an $m\gamma$-operation on an m-structure with property $\mathcal{B}$ (the generalized topology in the sense of Lugojan [5]). Császár [6] introduced the notion of hereditary classes as a generalization of ideals. Let $(X, m, \mathcal{H})$ be a hereditary m-space and $\gamma : m \to P(X)$ be an operation on m. A subset A of X is said to be $\gamma\mathcal{H}$-Lindelöf relative to to X (resp. γ-Lindelöf relative to X) [1] if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of A by m-open sets of X, there exists a countable subset Δ_0 of Δ such that $A \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$ (resp. $A \subseteq \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$).

In this paper, we define a subset A of a hereditary m-space $(X, m, \mathcal{H})$ to be *super* $\gamma\mathcal{H}$-Lindelöf relative to X if for every family $\{U_\alpha : \alpha \in \Delta\}$ of m-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$, there exists a countable subset Δ_0

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of Δ such that $A \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$. Similarly, we define a subset called *strongly* $\gamma\mathcal{H}$-Lindelöf relative to X and investigate their properties. Also, papers [7, 8, 9] have introduced some properties related to minimal spaces with hereditary classes.

2. Preliminaries

Definition 1. A subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [3] on X if m satisfies the following conditions:

(1) $\emptyset \in m$ and $X \in m$,

(2) The union of any family of subsets belonging to m belongs to m.

A set X with an m-structure m on X is denoted by (X, m) and is called an *m-space*. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*. In this paper, the m-structure [3] having property $\mathcal{B}$ is briefly called the *m-structure*.

Definition 2. Let (X, m) be an m-space and A be a subset of X. The *m-closure* $\mathrm{mCl}(A)$ and the *m-interior* $\mathrm{mInt}(A)$ of A [10] are defined as follows:

(1) $\mathrm{mCl}(A) = \cap\{F \subset X : A \subset F, X \setminus F \in m\}$,

(2) $\mathrm{mInt}(A) = \cup\{U \subset X : U \subset A, U \in m\}$.

Lemma 1. [3]. *Let (X, m) be an m-space and A be a subset of X.*

(1) $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m(x)$.

(2) A is m-closed if and only if $\mathrm{mCl}(A) = A$.

Definition 3. A nonempty subfamily $\mathcal{H}$ of $\mathcal{P}(X)$ is called a *hereditary class* on X [6] if it satisfies the following properties: $A \in \mathcal{H}$ and $B \subset A$ implies $B \in \mathcal{H}$. A hereditary class $\mathcal{H}$ is called an *ideal* [11], [12] if it satisfies the additional condition: $A \in \mathcal{H}$ and $B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$.

A minimal space (X, m) with a hereditary class $\mathcal{H}$ on X is called a *hereditary minimal space* (briefly *hereditary m-space*) and is denoted by $(X, m, \mathcal{H})$. The notion of ideals has been introduced in [11] and [12] and further investigated in [13].

Definition 4. Let (X, m) be an m-space. Let $m\gamma : m \to P(X)$ be a function from m into $P(X)$ such that $U \subset m\gamma(U)$ for each $U \in m$. The function $m\gamma$ is called an *$m\gamma$-operation* on m [4] and the image $m\gamma(U)$ is simply denoted by $\gamma(U)$. In this paper, an $m\gamma$-operation is simply called a γ-operation.

Definition 5. Let (X, m) be an m-space and $\gamma : m \to P(X)$ be a γ-operation. A subset A of X is said to be *γ-open* [4] if for each $x \in A$ there exists $U \in m$ such that $x \in U \subset \gamma(U) \subset A$. The complement of a γ-open set is said to be *γ-closed*. The family of all γ-open sets of (X, m) is denoted by $\gamma(X)$. The *γ-closure* of A, $\gamma\mathrm{Cl}(A)$, is defined as follows: $\gamma\mathrm{Cl}(A) = \cap\{F \subset X : A \subset F, X \setminus F \in \gamma(X)\}$.

Definition 6. Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. A subset A of X is said to be *$\gamma\mathcal{H}$-Lindelöf relative to X* [1] (resp. *γ-Lindelöf relative to X*) if for each cover $\{U_\alpha : \alpha \in \Delta\}$ of A by m-open sets of X, there exists a countable subset Δ_0 of Δ such that $A \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$ (resp. $A \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$.

Definition 7. Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. The space $(X, m, \mathcal{H})$ is said to be *$\gamma\mathcal{H}$-Lindelöf* [1] (resp. *γ-Lindelöf*) if X is *$\gamma\mathcal{H}$-Lindelöf relative to X* (resp. *γ-Lindelöf relative to X*).

3. Super $\gamma\mathcal{H}$-Lindelöf spaces

Definition 8. Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m.

(1) A subset A of X is said to be *super $\gamma\mathcal{H}$-Lindelöf relative to X* if for every family $\{U_\alpha : \alpha \in \Delta\}$ of m-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$, there exists a countable subset Δ_0 of Δ such that $A \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$.

(2) $(X, m\ \mathcal{H})$ is called a *super $\gamma\mathcal{H}$-Lindelöf space* if X is super $\gamma\mathcal{H}$-Lindelöf relative to X.

Remark 1. Let $(X, m, \mathcal{H})$ be a hereditary m-space. If $\mathcal{H} = \{\emptyset\}$, then "super $\gamma\mathcal{H}$-Lindelöf relative to X" coincides with "γ-Lindelöf relative to X".

Theorem 1. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. For a subset A of X, the following properties are equivalent:*

(1) A is super $\gamma\mathcal{H}$-Lindelöf relative to X;

(2) for every family $\{F_\alpha : \alpha \in \Delta\}$ of m-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$, there exists a countable subset Δ_0 of Δ such that $A \cap (\cap\{[X \setminus \gamma(X \setminus F_\alpha)] : \alpha \in \Delta_0\}) = \emptyset$.

Proof. (1) $\Rightarrow$ (2): Let $\{F_\alpha : \alpha \in \Delta\}$ be any family of m-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$. Then, we have $A \setminus (\cup\{X \setminus F_\alpha : \alpha \in \Delta\}) = A \setminus (X \setminus \cap\{F_\alpha : \alpha \in \Delta\}) = A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$. Since $X \setminus F_\alpha$ is m-open for each $\alpha \in \Delta$, by (1) there exists a countable subset Δ_0 of Δ such that $A \subseteq \cup\{\gamma(X \setminus F_\alpha) : \alpha \in \Delta_0\}$. Therefore, we have

$$A \cap [X \setminus (\cup\{\gamma(X \setminus F_\alpha) : \alpha \in \Delta_0\})] = A \cap (\cap\{[X \setminus \gamma(X \setminus F_\alpha)] : \alpha \in \Delta_0\}) = \emptyset.$$

(2) $\Rightarrow$ (1): Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then $\{X \setminus U_\alpha : \alpha \in \Delta\}$ is a family of m-closed sets such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} = A \cap (X \setminus \cup\{U_\alpha : \alpha \in \Delta\}) = A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta\})$ and hence $A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$. By (2), there exists a countable subset Δ_0 of Δ such that $A \cap (\cap\{[X \setminus \gamma(X \setminus (X \setminus U_\alpha))] : \alpha \in \Delta_0\}) = A \cap (\cap\{[X \setminus \gamma(U_\alpha)] : \alpha \in \Delta_0\}) = \emptyset$. Therefore, $A \cap (X \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}) = \emptyset$ and hence $A \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$. This shows that A is super $\gamma\mathcal{H}$-Lindelöf relative to X. ◄

Corollary 1. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. Then, the following properties are equivalent:*

(1) $(X, m, \mathcal{H})$ is super $\gamma\mathcal{H}$-Lindelöf;

(2) for every family $\{F_\alpha : \alpha \in \Delta\}$ of m-closed sets of X such that $\cap\{F_\alpha : \alpha \in \Delta\} \in \mathcal{H}$, there exists a countable subset Δ_0 of Δ such that $\cap\{[X \setminus \gamma(X \setminus F_\alpha)] : \alpha \in \Delta_0\} = \emptyset$.

Definition 9. Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. A subset A of X is said to be *$\mathcal{H}\gamma g$-closed* if $\gamma\mathrm{Cl}(A) \subset U$ whenever $A \setminus U \in \mathcal{H}$ and U is m-open.

Theorem 2. *Let $(X, m, \mathcal{H})$ be a hereditary m-space, γ be a γ-operation on m and A, B be subsets of X such that $A \subset B \subset \gamma\mathrm{Cl}(A)$ and A is $\mathcal{H}\gamma g$-closed. Then the following properties hold:*

(1) if $\gamma\mathrm{Cl}(A)$ is γ-Lindelöf relative to X, then B is super $\gamma\mathcal{H}$-Lindelöf relative to X,

(2) if B is γ-Lindelöf relative to X, then A is super $\gamma\mathcal{H}$-Lindelöf relative to X.

Proof. (1): Suppose that $\gamma\mathrm{Cl}(A)$ is γ-Lindelöf relative to X. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $B \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since A is $\mathcal{H}\gamma g$-closed, $\gamma\mathrm{Cl}(A) \subset \cup\{U_\alpha : \alpha \in \Delta\}$. Since $\gamma\mathrm{Cl}(A)$ is γ-Lindelöf relative to X, there exists a countable subset Δ_0 of Δ such that $\gamma\mathrm{Cl}(A) \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$. Since $B \subset \gamma\mathrm{Cl}(A)$, we have $B \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$. Therefore, B is super $\gamma\mathcal{H}$-Lindelöf relative to X.

(2): Suppose that B is γ-Lindelöf relative to X. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets in X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since A is $\mathcal{H}\gamma g$-closed, $\gamma\mathrm{Cl}(A) \subset \cup\{U_\alpha : \alpha \in \Delta\}$. Hence, we have $B \subset \gamma\mathrm{Cl}(A) \subset \cup\{U_\alpha : \alpha \in \Delta\}$. Since B is γ-Lindelöf relative to X, there exists a countable subset Δ_0 of Δ such that $B \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$. Since $A \subset B, A \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$. Therefore, A is super $\gamma\mathcal{H}$-Lindelöf relative to X. ◄

Theorem 3. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. If subsets A and B of X are super $\gamma\mathcal{H}$-Lindelöf relative to X, then $A \cup B$ is super $\gamma\mathcal{H}$-Lindelöf relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $(A \cup B) \setminus \cup\{U_\alpha \in \Delta\} \in \mathcal{H}$. Then, we have $A \setminus \cup\{U_\alpha \in \Delta\} \in \mathcal{H}$ and $B \setminus \cup\{U_\alpha \in \Delta\} \in \mathcal{H}$. Since A and B are super $\gamma\mathcal{H}$-Lindelöf relative to X, there exist countable subsets Δ_A and Δ_B of Δ such that $A \subset \cup\{\gamma\mathrm{Cl}(U_\alpha) : \alpha \in \Delta_A\}$ and $B \subset \cup\{\gamma\mathrm{Cl}(U_\alpha) : \alpha \in \Delta_B\}$. Hence we have $A \cup B \subset \cup\{\gamma\mathrm{Cl}(U_\alpha) : \alpha \in \Delta_A \cup \Delta_B\}$. $\Delta_A \cup \Delta_B$ is a countable subset of Δ. Therefore, $A \cup B$ is super $\gamma\mathcal{H}$-Lindelöf relative to X. ◀

Theorem 4. *Let $(X, m, \mathcal{H})$ be a hereditary m-space, γ be a γ-operation on m and A, B be subsets of X. If A is super $\gamma\mathcal{H}$-Lindelöf relative to X and B is γ-closed, then $A \cap B$ is super $\gamma\mathcal{H}$-Lindelöf relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of m-open sets of X such that $(A \cap B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since B is γ-closed, $X \setminus B$ is γ-open and for each $x \in X \setminus B$, there exists $V_x \in m$ such that $x \in V_x \subset \gamma(V_x) \subset X \setminus B$. Hence $\{U_\alpha : \alpha \in \Delta\} \cup [\cup\{V_x : x \in X \setminus B\}]$ is a family of m-open sets of X. $(A \cap B) \setminus \cup\{U_\alpha : \alpha \in \Delta\}$ $= A \setminus [(X \setminus B) \cup (\cup\{U_\alpha : \alpha \in \Delta\})] = A \setminus [(\cup\{V_x : x \in X \setminus B\}) \cup (\cup\{U_\alpha : \alpha \in \Delta\})] \in \mathcal{H}$. Since A is super $\gamma\mathcal{H}$-Lindelöf relative to X, there exist countable subset Δ_0 of Δ and countable points $x_1, x_2, ..., x_n, ...$ in $X \setminus B$ such that $A \subset [(\cup\{\gamma(V_{x_i}) : i = 1, 2, ..., n, ...\}) \cup (\cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\})]$. Since $B \cap \gamma(V_{x_i}) = \emptyset$ for each x_i $(i = 1, 2, ..., n, ...)$, $A \cap B \subset [\cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}] \cap B \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$. Therefore, $A \cap B$ is super $\gamma\mathcal{H}$-Lindelöf relative to X. ◀

Corollary 2. *If a hereditary m-space $(X, m, \mathcal{H})$ is super $\gamma\mathcal{H}$-Lindelöf and B is γ-closed, then B is super $\gamma\mathcal{H}$-Lindelöf relative to X.*

Definition 10. *A function $f : (X, m) \to (Y, n)$ is said to be* (γ, δ)-closed *if for each $y \in Y$ and $U \in m$ containing $f^{-1}(y)$, there exists $V \in n$ containing y such that $f^{-1}(\delta(V)) \subseteq \gamma(U)$.*

Definition 11. Let $(X, m, \mathcal{H})$ be a hereditary m-space.

(1) A subset A of X is said to be super $\mathcal{H}$*-Lindelöf relative to* X if for every family $\{U_\alpha : \alpha \in \Delta\}$ of m-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$, there exists a countable subset Δ_0 of Δ such that $A \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$.

(2) $(X, m\ \mathcal{H})$ is called a super $\mathcal{H}$*-Lindelöf space* if X is super $\mathcal{H}$- Lindelöf relative to X.

An m-structure m is said to have countable additive property for an operation $\gamma : m \to \mathcal{P}(X)$ if $\gamma(\cup\{V_\alpha : \alpha \in \Delta\}) = \cup\{\gamma(V_\alpha) : \alpha \in \Delta\}$ for $V_\alpha \in m$ and a countable set Δ.

Theorem 5. *Let $f : (X, m) \to (Y, n, \mathcal{H})$ be a (γ, δ)-closed surjective function such that m has countable additive property. If $f^{-1}(y)$ is super $\mathcal{H}$-Lindelöf relative to X for each $y \in Y$ and B is δ-Lindelöf relative to Y, then $f^{-1}(B)$ is super $\gamma f^{-1}(\mathcal{H})$-Lindelöf relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $f^{-1}(B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in f^{-1}(\mathcal{H})$. Then for each $y \in B$, since $f^{-1}(y)$ is super $\mathcal{H}$-Lindelöf relative to X, there exists a countable subset $\Delta(y)$ of Δ such that $f^{-1}(y) \subseteq \cup\{U_\alpha : \alpha \in \Delta(y)\} = U_y$. Since U_y is an m-open set of X containing $f^{-1}(y)$ and f is (γ, δ)-closed, there exists an n-open set V_y containing y such that $f^{-1}(\delta(V_y)) \subseteq \gamma(U_y)$. Since $\{V_y : y \in B\}$ is an n-open cover of B and B is δ-Lindelöf relative to Y, there exists a countable subset B_0 of B such that $B \subseteq \cup\{\delta(V_y) : y \in B_0\}$. Hence we have

$$\begin{aligned} f^{-1}(B) &\subseteq \cup\,\{f^{-1}(\delta(V_y)) : y \in B_0\} \subseteq \cup\{\gamma(U_y) : y \in B_0\} \\ &\subseteq \cup\,\{\gamma(U_\alpha) : \alpha \in \Delta(y), y \in B_0\}. \end{aligned}$$

We obtain $f^{-1}(B) \subseteq \cup\{\gamma(U_\alpha) : \alpha \in \Delta(y), y \in B_0\}$. This shows that $f^{-1}(B)$ is super $\gamma f^{-1}(\mathcal{H})$-Lindelöf relative to Y. ◄

Corollary 3. *Let $f : (X, m) \to (Y, n, \mathcal{H})$ be a (γ, δ)-closed surjective function such that m has a countable additive property. If $f^{-1}(y)$ is super $\mathcal{H}$-Lindelöf relative to X for each $y \in Y$ and B is super $\delta\mathcal{H}$-Lindelöf relative to Y, then $f^{-1}(B)$ is super $\gamma f^{-1}(\mathcal{H})$-Lindelöf relative to X.*

Corollary 4. *Let $f : (X, m) \to (Y, n, \mathcal{H})$ be a (γ, δ)-closed surjective function such that m has a countable additive property. If $f^{-1}(y)$ is super $\mathcal{H}$-Lindelöf relative to X for each $y \in Y$ and Y is δ-Lindelöf, then X is super $\gamma f^{-1}(\mathcal{H})$-Lindelöf.*

4. Strongly $\gamma\mathcal{H}$-Lindelöf spaces

Definition 12. Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m.

(1) A subset A of X is said to be *strongly $\gamma\mathcal{H}$-Lindelöf relative to X* if for every family $\{U_\alpha : \alpha \in \Delta\}$ of m-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$, there exists a countable subset Δ_0 of Δ such that $A \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in$

$\mathcal{H}$.

(2) $(X, m, \mathcal{H})$ is said to be *strongly $\gamma\mathcal{H}$-Lindelöf* if X is *strongly $\gamma\mathcal{H}$-Lindelöf relative to X.*

Theorem 6. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. For a subset A of X, the following properties are equivalent:*

(1) A is strongly $\gamma\mathcal{H}$-Lindelöf relative to X;

(2) for every family $\{F_\alpha : \alpha \in \Delta\}$ of m-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$, there exists a countable subset Δ_0 of Δ such that $A \cap (\cap\{[X \setminus \gamma(X \setminus F_\alpha)] : \alpha \in \Delta_0\}) \in \mathcal{H}$.

Proof. (1) $\Rightarrow$ (2): Let $\{F_\alpha : \alpha \in \Delta\}$ be any family of m-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$. Then $A \setminus \cup\{X \setminus F_\alpha : \alpha \in \Delta\}) = A \setminus (X \setminus \cap\{F_\alpha : \alpha \in \Delta\}) = A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$. Since $X \setminus F_\alpha$ is m-open for each $\alpha \in \Delta$ and A is strongly $\gamma\mathcal{H}$-Lindelöf relative to X, by (1) there exists a countable subset Δ_0 of Δ such that $A \setminus \cup\{\gamma(X \setminus F_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$. This implies that $A \cap (\cap\{[X \setminus \gamma(X \setminus F_\alpha)] : \alpha \in \Delta_0\}) = A \setminus (X \setminus (\cap\{[X \setminus \gamma(X \setminus F_\alpha)] : \alpha \in \Delta_0\})) = A \setminus \cup\{\gamma(X \setminus F_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$.

(2) $\Rightarrow$ (1): Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of m-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then $\{X \setminus U_\alpha : \alpha \in \Delta\}$ is a family of m-closed sets of X and also $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} = A \cap (X \setminus \cup\{U_\alpha : \alpha \in \Delta\}) = A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$. Thus, by (2) there exists a countable subset Δ_0 of Δ such that $A \cap (\cap\{X \setminus \gamma(U_\alpha) : \alpha \in \Delta_0\}) \in \mathcal{H}$. Therefore, we have $A \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} = A \cap (X \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}) = A \cap (\cap\{X \setminus \gamma(U_\alpha) : \alpha \in \Delta_0\}) \in \mathcal{H}$. This shows that A is strongly $\gamma\mathcal{H}$-Lindelöf relative to X. ◄

Corollary 5. *For a hereditary m-space $(X, m, \mathcal{H})$, the following properties are equivalent, where γ is a γ-operation on m:*

(1) $(X, m, \mathcal{H})$ is strongly $\gamma\mathcal{H}$-Lindelöf;

(2) for every family $\{F_\alpha : \alpha \in \Delta\}$ of m-closed sets of X such that $\cap\{F_\alpha : \alpha \in \Delta\} \in \mathcal{H}$, there exists a countable subset Δ_0 of Δ such that $\cap\{[X \setminus \gamma(X \setminus F_\alpha)] : \alpha \in \Delta_0\} \in \mathcal{H}$.

Theorem 7. *Let $(X, m, \mathcal{H})$ be a hereditary m-space, γ be a γ-operation on m and A, B be subsets of X such that A is $\mathcal{H}\gamma g$-closed and $A \subset B \subset \gamma\mathrm{Cl}(A)$. Then the following properties hold:*

(1) if $\gamma\mathrm{Cl}(A)$ is $\gamma\mathcal{H}$-Lindelöf relative to X, then B is strongly $\gamma\mathcal{H}$-Lindelöf relative to X,

(2) if B is $\gamma\mathcal{H}$-Lindelöf relative to X, then A is strongly $\gamma\mathcal{H}$-Lindelöf relative to X.

Proof. (1): Suppose that $\gamma\text{Cl}(A)$ is $\gamma\mathcal{H}$-Lindelöf relative to X. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $B \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$ and $\cup\{U_\alpha : \alpha \in \Delta\} \in m$. Since A is $\mathcal{H}mg$-closed, $\gamma\text{Cl}(A) \subset \cup\{U_\alpha : \alpha \in \Delta\}$. Since $\gamma\text{Cl}(A)$ is $\gamma\mathcal{H}$-Lindelöf relative to X, there exists a countable subset Δ_0 of Δ such that $\gamma\text{Cl}(A) \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$ and hence $B \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$. Therefore, B is strongly $\gamma\mathcal{H}$-Lindelöf relative to X.

(2): Suppose that B is $\gamma\mathcal{H}$-Lindelöf relative to X. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since A is $\mathcal{H}mg$-closed, we have $B \subset \gamma\text{Cl}(A) \subset \cup\{U_\alpha : \alpha \in \Delta\}$. Since B is $\gamma\mathcal{H}$-Lindelöf relative to X, there exists a countable subset Δ_0 of Δ such that $B \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$. Since $A \subset B, A \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$. Hence, A is strongly $\gamma\mathcal{H}$-Lindelöf relative to X. ◄

Theorem 8. *Let $(X, m, \mathcal{H})$ be an ideal m-space and γ be a γ-operation on m. If the subsets A and B of X are strongly $\gamma\mathcal{H}$-Lindelöf relative to X, then $A \cup B$ is strongly $\gamma\mathcal{H}$-Lindelöf relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $(A \cup B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$ and $B \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since A and B are strongly $\gamma\mathcal{H}$-Lindelöf relative to X, there exist countable subsets Δ_A and Δ_B of Δ and subsets H_A and H_B of $\mathcal{H}$ such that $A \subset \cup\{U_\alpha : \alpha \in \Delta_A\} \cup H_A$ and $B \subset \cup\{U_\alpha : \alpha \in \Delta_B\} \cup H_B$. Hence we have $(A \cup B) \subset \cup\{U_\alpha : \alpha \in \Delta_A \cup \Delta_B\} \cup (H_A \cup H_B)$. Since $\mathcal{H}$ is an ideal, we have $(A \cup B) \setminus \cup\{U_\alpha : \alpha \in \Delta_A \cup \Delta_B\} \in \mathcal{H}$. This shows that $A \cup B$ is strongly $\gamma\mathcal{H}$-Lindelöf relative to X. ◄

Theorem 9. *Let $(X, m, \mathcal{H})$ be a hereditary m-space, γ be a γ-operation on m and A, B be subsets of X. If A is strongly $\gamma\mathcal{H}$-Lindelöf relative to X and B is γ-closed, then $A \cap B$ is strongly $\gamma\mathcal{H}$-Lindelöf relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $(A \cap B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since B is γ-closed, $X \setminus B$ is γ-open and for each $x \in X \setminus B$, there exists $V_x \in m$ such that $x \in V_x \subset \gamma(V_x) \subset X \setminus B$. Hence $\{U_\alpha : \alpha \in \Delta\} \cup [\cup\{V_x : x \in X \setminus B\}]$ is a family of m-open sets of X. $(A \cap B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} = A \setminus [(X \setminus B) \cup (\cup\{U_\alpha : \alpha \in \Delta\})] = A \setminus [\cup\{V_x : x \in X \setminus B\} \cup (\cup\{U_\alpha : \alpha \in \Delta\})] \in \mathcal{H}$. Since A is strongly $\gamma\mathcal{H}$-Lindelöf relative to X, there exist countable subset Δ_0 of Δ and countable points $x_1, x_2, ..., x_n, ...$ in $X \setminus B$ such that $A \setminus [\cup\{\gamma(V_{x_i}) : i = 1, 2, ..., n, ...\} \cup (\cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\})] \in \mathcal{H}$. Since $B \cap \gamma(V_{x_i}) = \emptyset$ for each x_i $(i = 1, 2, ..., n)$, $A \cap B \setminus [\cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}] \in \mathcal{H}$. Therefore, $A \cap B$ is strongly $\gamma\mathcal{H}$-Lindelöf relative to X. ◄

Corollary 6. *If a hereditary m-space $(X, m, \mathcal{H})$ is strongly $\gamma\mathcal{H}$-Lindelöf and B is γ-closed, then B is strongly $\gamma\mathcal{H}$-Lindelöf relative to X.*

Theorem 10. *Let $f : (X, m) \to (Y, n, \mathcal{H})$ be a (γ, δ)-closed surjective function such that m has a countable additive property. If $f^{-1}(y)$ is super $\mathcal{H}$-Lindelöf relative to X for each $y \in Y$ and B is $\delta\mathcal{H}$-Lindelöf relative to Y, then $f^{-1}(B)$ is strongly $\gamma f^{-1}(\mathcal{H})$-Lindelöf relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $f^{-1}(B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in f^{-1}(\mathcal{H})$. Then for each $y \in B$, since $f^{-1}(y)$ is super $\mathcal{H}$-Lindelöf relative to X, there exists a countable subset $\Delta(y)$ of Δ such that $f^{-1}(y) \subseteq \cup\{U_\alpha : \alpha \in \Delta(y)\} = U_y$. Since U_y is an m-open set of X containing $f^{-1}(y)$ and f is (γ, δ)-closed, there exists an n-open set V_y containing y such that $f^{-1}(\delta(V_y)) \subseteq \gamma(U_y)$. Since $\{V_y : y \in B\}$ is an n-open cover of B and B is $\delta\mathcal{H}$-Lindelöf relative to Y, there exists a countable subset B_0 of B such that $B \setminus \cup\{\delta(V_y) : y \in B_0\} \in \mathcal{H}$. Therefore, $B \subseteq \cup\{\delta(V_y) : y \in B_0\} \cup H_0$, where $H_0 \in \mathcal{H}$. Hence we have

$$\begin{aligned} f^{-1}(B) &\subseteq \cup \{f^{-1}(\delta(V_y)) : y \in B_0\} \cup f^{-1}(H_0) \\ &\subseteq \cup \{\gamma(U_y) : y \in B_0\} \cup f^{-1}(H_0) \\ &\subseteq \cup \{\gamma(U_\alpha) : \alpha \in \Delta(y), y \in B_0\} \cup f^{-1}(H_0). \end{aligned}$$

We obtain $f^{-1}(B) \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta(y), y \in B_0\} \in f^{-1}(\mathcal{H})$. This shows that $f^{-1}(B)$ is strongly $\gamma f^{-1}(\mathcal{H})$-Lindelöf relative to Y. ◄

Corollary 7. *Let $f : (X, m) \to (Y, n, \mathcal{H})$ be a (γ, δ)-closed surjective function such that m has a countable additive property. If $f^{-1}(y)$ is super $\mathcal{H}$-Lindelöf relative to X for each $y \in Y$ and B is strongly $\delta\mathcal{H}$-Lindelöf relative to Y, then $f^{-1}(B)$ is strongly $\gamma f^{-1}(\mathcal{H})$-Lindelöf relative to X.*

Corollary 8. *Let $f : (X, m) \to (Y, n, \mathcal{H})$ be a (γ, δ)-closed surjective function such that m has a countable additive property. If $f^{-1}(y)$ is super $\mathcal{H}$-Lindelöf relative to X for each $y \in Y$ and Y is $\delta\mathcal{H}$-Lindelöf, then X is strongly $\gamma f^{-1}(\mathcal{H})$-Lindelöf.*

Remark 2. We have the following relationships:

$$\begin{array}{ccc} \text{super } \gamma\mathcal{H}\text{-Lindelöf relative to } X & \Rightarrow & \text{strongly } \gamma\mathcal{H}\text{-Lindelöf relative to } X \\ \Downarrow & & \Downarrow \\ \gamma\text{-Lindelöf relative to } X & \Rightarrow & \gamma\mathcal{H}\text{-Lindelöf relative to } X. \end{array}$$

Remark 3. The following examples show that "γ-Lindelöf relative to X" and "strongly $\gamma\mathcal{H}$-Lindelöf relative to X" are independent of each other. Therefore, the converse of the above four implications is not necessarily true.

Example 1. Let $X = [0, \infty)$, $m = \{X, (a, \infty) : a \geq 0\} \cup \{\emptyset\}$ be an m-structure, $\mathcal{H} = \mathcal{H}_f$ the hereditary classes of all finite subsets of X and γ be a γ-operation on m such that $\gamma(U) = id(U) = U$ for each $U \in m$. Then

(1) $(X, m, \mathcal{H})$ is γ-Lindelöf relative to X. To prove this, let $\{V_\alpha : \alpha \in \Delta\}$ be any m-open cover of X. Then there exists $\alpha_0 \in \Delta$ with $V_{\alpha_0} = X$, and so there exists a countable subset Δ_0 of Δ such that $X \subseteq \cup\{\gamma(V_\alpha) : \alpha \in \Delta_0\}$.

(2) $(X, m, \mathcal{H})$ is not strongly γ-Lindelöf relative to X, because $X \setminus \cup\{(a, \infty) : a > 0\} = \{0\} \in \mathcal{H}_f$. But if we consider the increasing sequence $\{a_i : a_1 > 0, i \in \mathbb{Z}^+\}$, then $X \setminus \cup\{\gamma(a_i, \infty) : i \in \mathbb{Z}^+\} = X \setminus \cup\{(a_i, \infty) : i \in \mathbb{Z}^+\} = X \setminus (a_1, \infty) = [0, a_1] \notin \mathcal{H}_f$.

Example 2. *Let $X = \mathbb{R} \times \mathbb{R}^+$. For $(x, y) \in X$ and $r > 0$, let*

$$N_r(x, y) = \begin{cases} B_r(x, y) & \text{if } r \leq y; \\ B_r(x, r) \cup \{(x, 0)\} \cup B_r(0, r), & \text{if } y = 0. \end{cases}$$

We take $\{N_r(x, y)\}$ as a basis for the topology on X which is an m-structure and let $\mathcal{H} = \mathcal{P}(X)$ be the hereditary classes and γ be a γ-operation on m such that $\gamma(U) = id(U) = U$ for each $U \in m$, then

(1) $(X, m, \mathcal{H})$ is not γ-Lindelöf relative to X, because $\{N_1(x, 0)\} \cup \{N_1(x, y) : y \geq 1\}$ is an m-open cover of X. Since $(z, 0) \notin \{N_1(x, y) : y \geq 1\}$ and $(z, 0) \in \{N_1(x, 0)\}$ if and only if $x = z$, the above m-open cover has no countable subcover. Thus, X is not γ-Lindelöf.

(2) $(X, m, \mathcal{H})$ is strongly γ-Lindelöf relative to X, since $\mathcal{H} = \mathcal{P}(X)$.

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