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Matrix Theory on Lorentz-Minkowski Scalar Product

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Abstract. In this paper, we investigate Lorentz-Minkowski matrix multiplication introduced in [3] by O. Keçilioğlu and H. Gündoğan. Unlike [3], we will define some transformations to get a connection between Lorentz-Minkowski matrix multiplication and Euclidean matrix multiplication which allows for a more practical use instead of direct approach.

Key Words and Phrases: pseudo matrix multiplication, isomorphism.

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1. Introduction

Matrix theory is an essential tool to get results while dealing with any kind of geometry. Determining the notions of matrix theory, semi-Riemannian geometry is an attractive area for researchers. Some basics of matrix theory in semi-Riemannian geometry was studied by B. O'Neill ([1]) who obtained his results with the help of sign matrix. After that, it was realized that a more suitable matrix multiplication can be used to eliminate the need of sign matrix. That matrix multiplication was named Lorentz matrix multiplication. Later, Lorentz matrix multiplication was generalized to pseudo matrix multiplication and named Lorentz-Minkowski matrix multiplication. There is a great deal of studies in that area, [2, 3, 4]. In these studies, the calculations had been conducted by a direct approach independent of Euclidean matrix multiplication.

In this paper, we define some transformations to get a connection between Lorentz-Minkowski matrix multiplication and Euclidean matrix multiplication which enables to obtain the fundamentals of our approach. We define Lorentz-Minkowski scalar product and then introduce a transformation that allows us

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to write Lorentz-Minkowski scalar product in terms of Euclidean inner product. Then we define Lorentz-Minkowski matrix multiplication and introduce two more very alike transformations. These two transformations will allow us to represent Lorentz-Minkowski matrix multiplication in terms of Euclidean matrix multiplication. The crucial example of this approach is that it is very easy to find the unit matrix of Lorentz-Minkowski matrix multiplication without using sign matrix that O'Neill used in [1]. It is possible to find it using these transformations and Euclidean unit matrix. Finally, we define matrix inverses considering Lorentz-Minkowski product and find those inverses using our transformations.

2. Lorentz-Minkowski Scalar Product and a Transformation

In this section, we will define Lorentz-Minkowski scalar product. Then we define a transformation that will allow us to connect it to Euclidean inner product.

Throughout this paper, unless stated otherwise, the vector space $V = \mathbb{R}^n$ over the field $F = \mathbb{R}$ is considered.

Definition 1. *Let $n \in \mathbb{N}$. For any vectors $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ the Euclidean inner product is given as follows ([5]):*

$$< X, Y >_e = \sum_{i=1}^{n} x_i y_i .$$

Let us denote the set of non-negative integers $\mathbb{N}_0$.

Definition 2. *Let $n, \upsilon \in \mathbb{N}_0$ with $n \geq \upsilon$. For any vectors $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, the Lorentz-Minkowski scalar product is defined as follows ([5]):*

$$< X, Y >_\upsilon = -\sum_{i=1}^{\upsilon} x_i y_i + \sum_{i=\upsilon+1}^{n} x_i y_i .$$

From now on, unless stated otherwise, we will assume that $n \in \mathbb{N}$ and $n \geq \upsilon \in \mathbb{N}_0$.

Lemma 1. *Let us define the transformation*

$$\begin{aligned} &\psi : \mathbb{R}^n \to \mathbb{R}^n \\ &(x_1, x_2, \ldots, x_n) \mapsto \psi(x_1, x_2, \ldots, x_n) = (-x_1, -x_2, \ldots, -x_\upsilon, x_{\upsilon+1}, \ldots, x_n). \end{aligned}$$

Then, ψ has the following properties:

1. *ψ is a linear transformation.*

2. ψ *is an isomorphism.*

3. $\psi = \psi^{-1}$.

4. ψ *is self-adjoint in Euclidean inner product.*

Proof. Let $X = (x_i) \in \mathbb{R}^n$ and $Y = (y_i) \in \mathbb{R}^n$ be two arbitrary vectors and $\lambda \in \mathbb{R}$ be an arbitrary scalar.

1. It can be seen that

$$\begin{aligned}\psi(X+Y) &= \psi((x_i+y_i)) \\ &= (-x_1-y_1, -x_2-y_2, \dots, -x_\upsilon - y_\nu, x_{\nu+1}+y_{\nu+1}, \dots, x_n+y_n) \\ &= (-x_1, -x_2, \dots, -x_\nu, x_{\nu+1}, \dots, x_n) \\ &\ + (-y_1, -y_2, \dots, -x_\nu, x_{\nu+1}, \dots, y_n) \\ &= \psi(X) + \psi(Y).\end{aligned}$$

For any $\lambda \in \mathbb{R}$, we can easily obtain $\psi(\lambda X) = \lambda\psi(X)$. Hence ψ is a linear transformation.

2. Let us suppose that $\psi(X) = \psi(Y)$. Then we have:

$$(-x_1, -x_2, \dots, -x_\nu, x_{\nu+1}, \dots, x_n) = (-y_1, -y_2, \dots, -x_\nu, x_{\nu+1}, \dots, y_n)$$

which implies $X = Y$.

For surjectivity, whenever an arbitrary vector $X = (x_1, x_2, \dots, x_n)$ is given, there is a $(-x_1, -x_2, \dots, -x_\nu, x_{\nu+1}, \dots, x_n) \in \mathbb{R}^n$ satisfying $\psi\left((-x_1, -x_2, \dots, -x_\nu, x_{\nu+1}, \dots, x_n)\right) = X$.

3. It is easily seen that $\psi(\psi(X)) = X$.

4. Calculations yield:

$$< \psi(X), Y >_e = < (-x_1, -x_2, \dots, -x_\nu, x_{\nu+1}, \dots, x_n), (y_i) >_e$$

$$= -x_1.y_1 + (-x_2.y_2) + \ldots + (-x_\nu.y_\nu) + x_{\nu+1}.y_{\nu+1} + \ldots + x_n.y_n$$

$$= -y_1.x_1 + (-y_2.x_2) + \ldots + (-y_\nu.x_\nu) + y_{\nu+1}.x_{\nu+1} + \ldots + y_n.x_n = < X, \psi(Y) >_e .$$

◄

Now, we will give our theorem that establishes the connection between Lorentz-Minkowski scalar product and Euclidean inner product.

Theorem 1. *For any* $X = (x_i) \in \mathbb{R}^n$ *and* $Y = (y_i) \in \mathbb{R}^n$, *the equation* $< X, Y >_\nu = < \psi(X), Y >_e = < X, \psi(Y) >_e$ *is satisfied.*

Proof. Let $X = (x_i) \in \mathbb{R}^n$ and $Y = (y_i) \in \mathbb{R}^n$ be two arbitrary vectors. From Definition 2, we obtain:

$$\begin{aligned} < X, Y >_\nu &= -\sum_{i=1}^{\nu} x_i y_i + \sum_{i=\nu+1}^{n} x_i y_i \\ &= < \psi(X), Y >_e \\ &= < X, \psi(Y) >_e . \end{aligned}$$

◀

3. Lorentz-Minkowski Matrix Multiplication

Let us recall Lorentz-Minkowski matrix multiplication.

Definition 3. *Let* $A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \in \mathbb{R}^m_n$ *and* $B = \begin{bmatrix} B_1 & B_2 & \dots & B_p \end{bmatrix} \in \mathbb{R}^n_p$ *be arbitrary matrices, where* $A_i \in \mathbb{R}^n$ *and* $(B_j)^T \in \mathbb{R}^n$, $(i = 1, 2 \dots, m\ j = 1, 2, \dots, p)$. *The Lorentz-Minkowski multiplication of* A *and* B *is denoted by* $A \bullet_\nu B$ *and defined as ([5])*

$$A \bullet_\nu B = \begin{bmatrix} < A_1, B_1^T >_\nu & < A_1, B_2^T >_\nu & \dots & < A_1, B_p^T >_\nu \\ < A_2, B_1^T >_\upsilon & < A_2, B_2^T >_\upsilon & \dots & < A_2, B_p^T >_\nu \\ \vdots & & & \vdots \\ < A_n, B_1^T >_\nu & < A_n, B_2^T >_\nu & \dots & < A_n, B_p^T >_\nu \end{bmatrix}.$$

Lemma 2. *The transformation*

$$\theta : \mathbb{R}^m_n \to \mathbb{R}^m_n$$

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \quad \mapsto \quad \theta(A) = \begin{bmatrix} \psi(A_1) \\ \psi(A_2) \\ \vdots \\ \psi(A_m) \end{bmatrix}$$

is a linear isomorphism.

Proof. Let $A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \in \mathbb{R}_n^m$ and $B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix} \in \mathbb{R}_n^m$ be two matrices. Then we can write:

$$\theta(A+B) = \theta\left(\begin{bmatrix} A_1 + B_1 \\ A_2 + B_2 \\ \vdots \\ A_m + B_m \end{bmatrix}\right) = \begin{bmatrix} \psi\left(A_1 + B_1\right) \\ \psi\left(A_2 + B_2\right) \\ \vdots \\ \psi(A_m) + \psi(B_m) \end{bmatrix}.$$

Since ψ is a linear isomorphism (see Lemma 1), we have:

$$\begin{aligned} \theta\left(A+B\right) &= \begin{bmatrix} \psi\left(A_1 + B_1\right) \\ \psi\left(A_2 + B_2\right) \\ \vdots \\ \psi\left(A_m + B_m\right) \end{bmatrix} \\ &= \begin{bmatrix} \psi\left(A_1\right) + \psi\left(B_1\right) \\ \psi\left(A_2\right) + \psi\left(B_2\right) \\ \vdots \\ \psi\left(A_m\right) + \psi\left(B_m\right) \end{bmatrix} \\ &= \begin{bmatrix} \psi\left(A_1\right) \\ \psi\left(A_2\right) \\ \vdots \\ \psi\left(A_m\right) \end{bmatrix} + \begin{bmatrix} \psi\left(B_1\right) \\ \psi\left(B_2\right) \\ \vdots \\ \psi\left(B_m\right) \end{bmatrix} \\ &= \theta\left(A\right) + \ \theta\left(B\right) \end{aligned}$$

as desired. The scalar multiplication is very similar, hence θ is a linear transformation.

To show that this transformation is 1-1, suppose that $A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \in \mathbb{R}_n^m$ and

$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix} \in \mathbb{R}_n^m$ are arbitrary matrices such that $\theta(A) = \theta(B)$. Hence we get

$$\begin{bmatrix} \psi(A_1) \\ \psi(A_2) \\ \vdots \\ \psi(A_m) \end{bmatrix} = \begin{bmatrix} \psi(B_1) \\ \psi(B_2) \\ \vdots \\ \psi(B_m) \end{bmatrix}$$

implying for all $i = 1, 2, \ldots, m$, $\psi(A_i) = \psi(B_i)$. Since ψ is an isomorphism, this means that $A_i = B_i$ for all $i = 1, 2, \ldots, m$ and that is $A = B$. For any arbitrary matrix X it is easily seen that $\theta(\theta(X)) = X$, proving surjectivity. ◄

Now we define another transformation which is very alike to the previous one. Since the proof is very similar, we can omit it.

Lemma 3. *The transformation defined as*

$$\phi : \mathbb{R}_n^m \to \mathbb{R}_n^m$$

$$A = \begin{bmatrix} A_1 & A_2 & \ldots & A_n \end{bmatrix} \mapsto \phi(A) = \left[\left[\psi\left(A_1^T\right)\right]^T \quad \left[\psi\left(A_2^T\right)\right]^T \quad \ldots \quad \left[\psi\left(A_n^T\right)\right]^T \right]$$

is a linear isomorphism.

Lemma 4. *Let A be a square matrix. Then*

1. $\theta(A) = \left[\phi\left(A^T\right)\right]^T$,
2. $\phi(A) = \left[\theta\left(A^T\right)\right]^T$.

Proof.

1. Let $A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} \in \mathbb{R}_n^n$. Then $A^T = \begin{bmatrix} A_1^T & A_2^T & \ldots & A_n^T \end{bmatrix}$. It follows that

$$\begin{aligned} \phi\left(A^T\right) &= \left[\left[\psi\left(A_1^T\right)^T\right]^T \quad \left[\psi\left(A_2^T\right)^T\right]^T \quad \ldots \quad \left[\psi\left(A_n^T\right)^T\right]^T \right] \\ &= \left[[\psi(A_1)]^T \quad [\psi(A_2)]^T \quad \ldots \quad [\psi(A_n)]^T \right], \end{aligned}$$

and

$$\left[\phi\left(A^T\right)\right]^T = \begin{bmatrix} \left([\psi(A_1)]^T\right)^T \\ \left([\psi(A_2)]^T\right)^T \\ \vdots \\ \left([\psi(A_n)]^T\right)^T \end{bmatrix} = \begin{bmatrix} \psi(A_1) \\ \psi(A_2) \\ \vdots \\ \psi(A_n) \end{bmatrix} = \theta(A).$$

2. Considering part (1), it is quickly proven by picking $A = X^T$.

◀

Lemma 5. *Let θ and ϕ be the transformations given in Lemma 2 and Lemma 3, respectively. Then $\theta = \theta^{-1}$ and $\phi = \phi^{-1}$.*

4. Building Matrix Theory over Lorentz-Minkowski Matrix Multiplication

In this section, we will define the fundamentals of Lorentz-Minkowski (ν) matrix multiplication based on Euclidean matrix multiplication, unlike the direct approach of [3]. It will be conducted by using the transformations defined in Lemma 2 and Lemma 3 in the previous section.

Theorem 2. *Let $A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \in \mathbb{R}^m_n$ and $B = \begin{bmatrix} B_1 & B_2 & \dots & B_p \end{bmatrix} \in \mathbb{R}^n_p$. Then $A \bullet_\nu B = \theta(A) \bullet B = A \bullet \phi(B)$.*

Proof. Let us begin with showing that $A \bullet_\nu B = \theta(A) \bullet B$. From Definition 3, we have:

$$A \bullet_\nu B = \begin{bmatrix} < A_1, B_1^T >_\nu & < A_1, B_2^T >_\nu & \dots & < A_1, B_p^T >_\nu \\ < A_2, B_1^T >_\nu & < A_2, B_2^T >_\nu & \dots & < A_2, B_p^T >_\nu \\ \vdots & & & \vdots \\ < A_m, B_1^T >_\nu & < A_m, B_2^T >_\nu & \dots & < A_m, B_p^T >_\nu \end{bmatrix}.$$

Here we can transform the Lorentz-Minkowski scalar product into Euclidean inner product by using Theorem 1. Hence, we have:

$$
\begin{aligned}
A \bullet_\nu B &= \begin{bmatrix} <\psi(A_1), B_1^T> & <\psi(A_1), B_2^T> & \dots & <\psi(A_1), B_p^T> \\ <\psi(A_2), B_1^T> & <\psi(A_2), B_2^T> & \dots & <\psi(A_2), B_p^T> \\ \vdots & & & \vdots \\ <\psi(A_m), B_1^T> & <\psi(A_m), B_2^T> & \dots & <\psi(A_m), B_p^T> \end{bmatrix} \\
&= \begin{bmatrix} \psi(A_1) \\ \psi(A_2) \\ \vdots \\ \psi(A_m) \end{bmatrix} \bullet B \\
&= \theta(A) \bullet B.
\end{aligned}
$$

Now, let us show that $A \bullet_\nu B = A \bullet \phi(B)$. From Definition 3 and again using Theorem 1, we can write:

$$
\begin{aligned}
A \bullet_\nu B &= \begin{bmatrix} <A_1, \psi\left(B_1^T\right)^T> & <A_1, \psi\left(B_2^T\right)^T> & \dots & <A_1, \psi\left(B_p^T\right)^T> \\ <A_2, \psi\left(B_1^T\right)^T> & <A_2, \psi\left(B_2^T\right)^T> & \dots & <A_2, \psi\left(B_p^T\right)^T> \\ \vdots & & & \vdots \\ <A_m, \psi\left(B_1^T\right)^T> & <A_m, \psi\left(B_2^T\right)^T> & \dots & <A_m, \psi\left(B_p^T\right)^T> \end{bmatrix} \\
&= A \bullet \phi(B).
\end{aligned}
$$

◄

Definition 4. *The matrix $I^{n,\nu} \in \mathbb{R}_n^n$ is called the $\nu-$unit matrix if it satisfies $I^{n,\nu} \bullet_\nu A = A \bullet_\nu I^{n,\nu}$ for every $A \in \mathbb{R}_n^n$.*

Theorem 3. *Let I_n be the Euclidean unit matrix. Then $\theta(I_n) = \phi(I_n)$ is the unit matrix in Lorentz-Minkowski matrix multiplication.*

Proof. We shall begin with proving that $\theta(I_n) = \phi(I_n)$. Notice that in the unit matrix the 1's are equally distant from the row and columns. Hence the transformations θ and ϕ will result in the same matrix.
Now, we show that $\theta(I_n) \bullet_\nu A$. If we use Lemma 5, we can write:

$$
\begin{aligned}
\theta(I_n) \bullet_\nu A &= \theta(\theta(I_n)) \bullet A \\
&= I_n \bullet A \\
&= A.
\end{aligned}
$$

For the right-hand side multiplication, we similarly get:

$$\begin{aligned} A \bullet_\nu \theta\left(I_n\right) &= A \bullet_\nu \phi\left(I_n\right) \\ &= A \bullet_\nu \phi\left(\phi\left(I_n\right)\right) \\ &= A \bullet I_n \\ &= A. \end{aligned}$$

◀

Now, we can express the $\nu-$unitary matrix using our transformations (θ and ϕ). We will do it the other way around to express our transformations using the $\nu-$unitary matrix.

Theorem 4. *For any $(m \times n)$ matrix A, $\theta\left(A\right) = A \bullet I^{n,\nu}$ and $\phi\left(A\right) = I^{n,\nu} \bullet A$, where $I^{n,\nu}$ denotes the $(n \times n)$ $\nu -$ unit matrix.*

Proof.

$$\begin{aligned} A \bullet I^{n,\nu} &= \ \theta^{-1}\left(A\right) \bullet_\nu I^{n,\nu} \\ &= \theta^{-1}\left(A\right) \end{aligned}$$

and since $\theta = \theta^{-1}$ (see Lemma 5), we have $A \bullet I^{n,\nu} = \theta(A)$. The proof for ϕ is very similar, so we omit it. ◀

This theorem will help us to have a better understanding of the transformations as shown in the following theorem.

Lemma 6. *Let $A \in \mathbb{R}^n_m$ and $B \in \mathbb{R}^p_n$ be two arbitrary matrices. The following equalities are satisfied:*

1. $\theta(A \bullet B) = A \bullet \theta(B)$,
2. $\phi(A \bullet B) = \phi(A) \bullet B$.

Proof.

1. By Theorem 4, we can write $\theta(A \bullet B) = (A \bullet B) \bullet I^{n,\nu}$. Since Euclidean matrix multiplication is associative, we have

$$\begin{aligned} \theta(A \bullet B) &= (A \bullet B) \bullet I^{n,\nu} \\ &= A \bullet (B \bullet I^{n,\nu}). \end{aligned}$$

By Theorem 4, this means that $B \bullet I^{n,\nu} = \theta(B)$ and we obtain:

$$\begin{aligned} \theta(A \bullet B) &= (A \bullet B) \bullet I^{n,\nu} \\ &= A \bullet (B \bullet I^{n,\nu}) \\ &= A \bullet \theta(B). \end{aligned}$$

2. The proof is very similar to that of 1.

◀

Now we express Lorentz-Minkowski matrix multiplication by only using its unitary matrix and Euclidean matrix multiplication.

Theorem 5. *Let $A \in \mathbb{R}_m^n$ and $B \in \mathbb{R}_n^p$ be two arbitrary matrices. Then the following equation holds:*

$$A \bullet_\nu B = (A \bullet I^{n,\nu}) \bullet B = A \bullet (I^{n,\nu} \bullet B).$$

Proof. This can immediately be seen by using Theorem 2 and Theorem 4 with the associativity of Euclidean matrix multiplication. ◀

Now we give a lemma about the matrix $I^{n,\nu}$.

Lemma 7. *The matrix $I^{n,\nu}$ is Euclidean invertible and its inverse is $I^{n,\nu}$.*

Proof. Using Lemma 5 and Theorem 2,

$$\begin{aligned} I^{n,\nu} \bullet I^{n,\nu} &= \theta\left(I^{n,\nu}\right) \bullet_\nu I^{n,\nu} \\ &= I_n \bullet_\nu I^{n,\nu} \\ &= I_n \end{aligned}$$

and

$$\begin{aligned} I^{n,\nu} \bullet I^{n,\nu} &= I^{n,\nu} \bullet_\nu \phi\left(I^{n,\nu}\right) \\ &= I^{n,\nu} \bullet_\nu I_n \\ &= I_n, \end{aligned}$$

as desired. ◀

Theorem 6. *Let A be a square matrix with the size $n \times n$. Then A is Euclidean invertible if and only if $\theta(A)$ is Euclidean invertible.*

Proof. ($\Rightarrow$) Suppose that A is Euclidean invertible. By Theorem 4, we have $\theta(A) = A \bullet I^{n,\nu}$. Besides, by Lemma 7, $I^{n,\nu}$ is Euclidean invertible. Thus $\theta(A)$ is a multiplication of two Euclidean invertible matrices which implies $\theta(A)$ is also Euclidean invertible.

($\Leftarrow$) Suppose that $\theta(A)$ is Euclidean invertible. By Theorem 4, we have $\theta(A) = A \bullet I^{n,\nu}$ and when the equation is multiplied by $I^{n,\nu}$ from the right we get $A = \theta(A) \bullet I^{n,\nu}$ (see Lemma 7). Hence $\theta(A)$ is Euclidean invertible and, by Lemma 7, $I^{n,\nu}$ is also Euclidean invertible. Hence A is the result of the multiplication of two Euclidean invertible matrices, implying A is Euclidean invertible.◄

Now we will give a theorem to connect our transformations when A is an invertible matrix.

Theorem 7. *Let $A \in \mathbb{R}_n^n$ be an invertible matrix. Then the followings hold:*

1. $\phi(A^{-1}) = (\theta(A))^{-1}$,
2. $\phi(A) = \left(\theta(A^{-1})\right)^{-1}$.

Proof.

1. By Theorem 4, we have $\theta(A) = A \bullet I^{n,\nu}$. Since we consider Euclidean inverses, we can write:

$$(\theta(A))^{-1} = (I^{n,\nu})^{-1} \bullet A^{-1}.$$

By Lemma 7, we can substitute $(I^{n,\nu})^{-1} = I^{n,\nu}$ to get

$$(\theta(A))^{-1} = I^{n,\nu} \bullet A^{-1},$$

which allows us to write $I^{n,\nu} \bullet A^{-1} = \phi(A^{-1})$ to get

$$(\theta(A))^{-1} = \phi(A^{-1}).$$

2. Let X be an Euclidean invertible matrix. By picking $A = X^{-1}$ in part 1, we will obtain the desired result immediately.

◄

Now we will define invertibility in Lorentz-Minkowski matrix multiplication.

Definition 5. *Let A be a $n \times n$ square matrix. If there exists a matrix $A^{-1,\nu}$ of the same dimension satisfying*

$$A \bullet_\nu A^{-1,\nu} = I^{n,\nu} = A^{-1,\nu} \bullet_\nu A,$$

then the matrix A is said to be $\nu-$invertible and $A^{-1,\nu}$ is called the $\nu-$inverse of A.

Now we will give a theorem to have a criterion for $\nu-$invertibility using our transformations given in Lemma 2 and Lemma 3.

Theorem 8. *Let A be a square matrix, then A is $\nu-$invertible if and only if $\theta\,(A)$ is Euclidean invertible.*

Proof. Let A be a $n \times n$ matrix.
($\Rightarrow$) Suppose that A is $\nu-$invertible. Then there exists $A^{-1,\nu}$ such that

$$A \bullet_\nu A^{-1,\nu} = I^{n,\nu}.$$

We know that $A \bullet_\nu A^{-1,\nu} = \theta(A) \bullet A^{-1,\nu}$. Thus, we have

$$\theta(A) \bullet A^{-1,\nu} = I^{n,\nu}.$$

Euclidean multiplying this equation from the right by $I^{n,\nu}$ results in

$$\left(\theta(A) \bullet A^{-1,\nu}\right) \bullet I^{n,\nu} = I^{n,\nu} \bullet I^{n,\nu} = I_n.$$

This implies that $\theta(A)$ is Euclidean invertible and that its inverse is $A^{-1,\nu} \bullet I^{n,\nu}$.

($\Leftarrow$) Suppose that $\theta(A)$ is Euclidean invertible. Then, there is a matrix $[\theta(A)]^{-1}$ such that $\theta(A) \bullet [\theta(A)]^{-1} = I_n$. Now denote $[\theta(A)]^{-1} \bullet I^{n,\nu} = X$ and notice that

$$\begin{aligned}
A \bullet_\nu X &= \theta(A) \bullet X \\
&= \theta(A) \bullet \left([\theta(A)]^{-1} \bullet I^{n,\nu}\right) \\
&= \left(\theta(A) \bullet [\theta(A)]^{-1}\right) \bullet I^{n,\nu} \\
&= I_n \bullet I^{n,\nu} \\
&= I^{n,\nu}.
\end{aligned}$$

On the other hand, we obtain:

$$\begin{aligned}
\left([\theta(A)]^{-1} \bullet I^{n,\nu}\right) \bullet_\nu A &= \left([\theta(A)]^{-1} \bullet I^{n,\nu}\right) \bullet \phi(A) \\
&= [\theta(A)]^{-1} \bullet \left(I^{n,\nu} \bullet \phi(A)\right) \\
&= [\theta(A)]^{-1} \bullet \phi(\phi(A)) \\
&= [\theta(A)]^{-1} \bullet A \\
&= \left([\theta(A)]^{-1} \bullet A\right) \bullet I_n \\
&= \left([\theta(A)]^{-1} \bullet A\right) \bullet \left(I^{n,\nu} \bullet I^{n,\nu}\right) \\
&= [\theta(A)]^{-1} \bullet \left(A \bullet I^{n,\nu}\right) \bullet I^{n,\nu} \\
&= \left([\theta(A)]^{-1} \bullet \theta(A)\right) \bullet I^{n,\nu} \\
&= I_n \bullet I^{n,\nu} \\
&= I^{n,\nu}.
\end{aligned}$$

◄

While proving this theorem, we have also found the $\nu-$inverse of the given matrix. Let us state that as a corollary.

Corollary 1. *Let A be an Euclidean invertible matrix. Then $\nu-$inverse of A can be given as follows:*

$$A^{-1,\nu} = \theta\left(\theta(A)^{-1}\right).$$

Corollary 2. *The $\nu-$inverse of a matrix is unique and left and right inverses are the same.*

In fact we can make the $\nu-$invertibility criterion even simpler. The following theorem does that.

Theorem 9. *A square matrix A is $\nu-$invertible if and only if it is Euclidean invertible.*

Proof. Immediately seen when Theorem 6 and Theorem 8 are used. ◄

Lemma 8. *Let A be an Euclidean invertible matrix. Then the $\bullet_\nu$ inverse of A can be found from the following equations:*

1. $A^{-1,\nu} = \theta\left(\phi(A^{-1})\right)$,
2. $A^{-1,\nu} = \left[\phi\left(\theta\left(A\right)\right)\right]^{-1}$.

Proof.

1. Let A be Euclidean invertible. Then by Corollary 1, we have $A^{-1,\nu} = \theta\left([\theta(A)]^{-1}\right)$. Here by Theorem 7, $[\theta(A)]^{-1}$ can be written as $\phi(A^{-1})$, from which we can get $A^{-1,\nu} = \theta\left(\phi(A^{-1})\right)$ as desired.

2. By Corollary 1, we have $A^{-1,\nu} = \theta\left([\theta(A)]^{-1}\right)$. Then, in part 1 of Theorem 7 if we pick A as $[\theta(A)]^{-1}$ we get

$$\phi(\left([\theta(A)]^{-1}\right)^{-1}) = \left(\theta([\theta(A)]^{-1})\right)^{-1},$$

which can be written as $\phi[(\theta(A))] = \left(\theta([\theta(A)]^{-1})\right)^{-1}$. Here taking Euclidean inverses of both sides results in

$$(\phi[(\theta(A)])^{-1} = \theta\left((\theta(A))^{-1}\right) = A^{-1,\nu}$$

as desired.

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