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Approximation of the Hilbert Transform in Hölder Spaces

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Abstract. The Hilbert transform plays an important role in the theory and practice of signal processing operations in continuous system theory because of its relevance to such problems as envelope detection and demodulation, as well as its use in relating the real and imaginary components, and the magnitude and phase components of spectra. The Hilbert transform is a multiplier operator and is widely used in the theory of Fourier transforms. It is also the main part of the theory of singular integral equations on the real line. Therefore, approximations of Hilbert transform are of great interest. Many papers have dealt with the numerical approximation of singular integrals in case of bounded intervals. On the other hand, the literature concerning the numerical integration on unbounded intervals is much sparser than the one on bounded intervals. There is very little literature concerning the case of Hilbert transform. This article is dedicated to the approximation of Hilbert transform in Hölder spaces by the operators introduced by V.R.Kress and E.Mortensen to approximate the Hilbert transform of analytic functions in a strip.
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1. Introduction

Let the function u be defined on the real axis and $\alpha \in (0, 1]$. If there exists a number $M > 0$ such that for any $x, y \in R$

$$|u(x) - u(y)| \leq M \cdot |x - y|^{\alpha} \tag{1}$$

and for any $x, y \in R \backslash \{0\}$

$$|u(x) - u(y)| \leq M \cdot \left| \frac{1}{x} - \frac{1}{y} \right|^{\alpha}, \tag{2}$$

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then the function u is said to be Hölder continuous with exponent α on the real axis (see [10, 18]). The class of Hölder continuous functions with exponent α on the real axis with norm

$$\|u\|_\alpha = \|u\|_\infty + \sup_{x,y\in R, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} + \sup_{x,y\in R\setminus\{0\}, x\neq y} \frac{|u(x)-u(y)|}{|1/x-1/y|^\alpha}$$

forms a Banach space and is denoted by $\mathcal{H}_\alpha(R)$, where $\|u\|_\infty = \max_{x\in R}|u(x)|$.

It follows from (1) and (2) that for any $u \in \mathcal{H}_\alpha(R)$ there exist $u(\infty) = \lim_{x\to\pm\infty} u(x)$ and for any $x \neq 0$

$$|u(x)-u(\infty)| \leq \frac{\|u\|_\alpha}{|x|^\alpha}.$$

Denote

$$\mathcal{H}^0_\alpha(R) = \{u \in \mathcal{H}_\alpha(R) : u(\infty) = 0\} \subset \mathcal{H}_\alpha(R).$$

The Hilbert transform of the function $u \in \mathcal{H}^0_\alpha(R)$, $\alpha \in (0,1]$ is defined as the Cauchy principle value integral

$$(Hu)(t) = \frac{1}{\pi}\int_R \frac{u(\tau)}{t-\tau}d\tau \equiv \frac{1}{\pi}\lim_{\varepsilon\to 0+}\int_{R\setminus(t-\varepsilon,t+\varepsilon)} \frac{u(\tau)}{t-\tau}d\tau, \quad t\in R.$$

It is well known (see [12, 21]) that the Hilbert transform of the function $u \in \mathcal{H}^0_\alpha(R)$, $\alpha \in (0,1]$ exists for any $t \in R$. In case $\alpha \in (0,1)$, the Hilbert transform is a bounded map in the space $\mathcal{H}^0_\alpha(R)$.

The Hilbert transform plays an important role in the theory and practice of signal processing operations in continuous system theory because of its relevance to such problems as envelope detection and demodulation, as well as its use in relating the real and imaginary components, and the magnitude and phase components of spectra. The Hilbert transform is a multiplier operator and is widely used in the theory of Fourier transforms. It is also the main part of the theory of singular integral equations on the real line (see [24]). Therefore, approximations of Hilbert transform are of great interest.

Many papers have dealt with the numerical approximation of Hilbert transform in case of bounded intervals and the reader can refer to [1, 3, 4, 5, 7, 9, 13, 14, 17, 18, 20, 24, 25, 26, 27, 31] and the references therein. On the other hand, the literature concerning the numerical integration on unbounded intervals is much sparser than the one on bounded intervals. There is very little literature concerning the case of Hilbert transform on the real axis and the reader may refer to [2, 6, 8, 10, 11, 15, 16, 19, 22, 23, 28, 29, 30].

In particular, in [15] the authors assume that the function u is analytic in the strip $\{z \in C : |\Im z| < d\}$ and show that the series $\frac{2}{\pi}\sum_{k\in Z, k\neq even}\frac{u(t+k\delta)}{-k}$ uniformly converges to $(Hu)(t)$ as $\delta \to 0$. In [6], the above series is replaced by the following one:

$$(H_\delta u)(t) = \frac{1}{\pi} \sum_{k \in Z} \frac{u(t + (k + 1/2)\delta)}{-k - 1/2}, \quad \delta > 0.$$

In [2], it was proved that the operators H_δ are bounded maps in the space $L_p(R)$, $1 < p < \infty$, satisfy the equality

$$H_\delta^2 = -I$$

in $L_p(R)$, and for any $\delta > 0$ the sequence of operators $\{H_{\delta/n}\}_{n \in N}$ strongly converges to the operator H in $L_p(R)$. This article is dedicated to the approximation of the Hilbert transform of functions from the space $\mathcal{H}_\alpha^0(R)$ by the operators H_δ, $\delta > 0$. It is proved that if the function u belongs to the space $\mathcal{H}_\alpha^0(R)$, $\alpha \in (0,1)$, then a family of functions $\{H_\delta u(t)\}$ uniformly converges to the function $(Hu)(t)$ as $\delta \to 0$.

2. Main result

The main result of the paper is the following theorem.

Theorem 1. *If the function u belongs to the space $\mathcal{H}_\alpha^0(R)$, $\alpha \in (0,1]$, then a family of functions $\{H_\delta u(t)\}$ uniformly converges to the function $(Hu)(t)$ as $\delta \to 0$ and the following inequality holds:*

$$\|Hu - H_\delta u\|_\infty \le \|u\|_\alpha \left[\frac{6}{\alpha} + \ln(e + 1/\delta) \right] \delta^\alpha, \quad \delta > 0. \tag{3}$$

Proof. It is obvious that the uniform convergence of the family of functions $\{H_\delta u(t)$ to the function $(Hu)(t)$ as $\delta \to 0$ follows from the inequality (3). Therefore, it suffices to prove the inequality (3).

We write the difference $(Hu)(t) - (H_\delta u)(t)$, $t \in R$ in the form

$$\pi[(Hu)(t) - (H_\delta u)(t)] = \int_R \frac{u(\tau)}{t - \tau} d\tau - \sum_{k=-\infty}^{\infty} \frac{u(t + (k + 1/2)\delta)}{-k - 1/2}$$

$$= \int_{t-3\delta}^{t+3\delta} \frac{u(\tau)}{t - \tau} d\tau + \sum_{k=-3}^{2} \frac{u(t + (k + 1/2)\delta)}{k + 1/2}$$

$$+ \left[\int_{R \setminus (t-3\delta, t+3\delta)} \frac{u(\tau)}{t - \tau} d\tau + \sum_{k=3}^{\infty} \frac{u(t + (k + 1/2)\delta)}{k + 1/2} + \sum_{k=-\infty}^{-4} \frac{u(t + (k + 1/2)\delta)}{k + 1/2} \right]$$

$$= J_1(t) + J_2(t) + J_3(t).$$

Let's estimate $J_k(t)$, $k = 1, 2, 3$. For any $t \in R$

$$|J_1(t)| = \left| \int_{t-3\delta}^{t+3\delta} \frac{u(\tau)}{t-\tau} d\tau \right| = \left| \int_0^{3\delta} \frac{u(t+\tau) - u(t-\tau)}{\tau} d\tau \right|$$

$$\leq \int_0^{3\delta} \frac{|u(t+\tau) - u(t-\tau)|}{\tau} d\tau \leq \int_0^{3\delta} \frac{\|u\|_\alpha \cdot (2\tau)^\alpha}{\tau} d\tau \leq \frac{6\|u\|_\alpha}{\alpha} \cdot \delta^\alpha, \tag{4}$$

$$|J_2(t)| \leq \frac{|u(t+\delta/2) - u(t-\delta/2)|}{1/2} + \frac{|u(t+3\delta/2) - u(t-3\delta/2)|}{3/2}$$

$$+ \frac{|u(t+5\delta/2) - u(t-5\delta/2)|}{5/2}$$

$$\leq (2\delta^\alpha + 2(3\delta)^\alpha/3 + 2(5\delta)^\alpha/5) \cdot \|u\|_\alpha \leq 6\|u\|_\alpha \cdot \delta^\alpha. \tag{5}$$

Let's write $J_3(t)$ in the form

$$J_3(t) = -\int_{3\delta}^{\infty} \frac{u(t+\tau) - u(t-\tau)}{\tau} d\tau + \sum_{k=3}^{\infty} \frac{u(t+(k+1/2)\delta) - u(t-(k+1/2)\delta)}{k+1/2}$$

$$= \sum_{k=3}^{\infty} \left(\frac{u(t+(k+1/2)\delta) - u(t-(k+1/2)\delta)}{k+1/2} - \int_{k\delta}^{(k+1)\delta} \frac{u(t+\tau) - u(t-\tau)}{\tau} d\tau \right)$$

$$= \sum_{k=3}^{\infty} [u(t+(k+1/2)\delta) - u(t-(k+1/2)\delta)] \cdot \left(\frac{1}{k+1/2} - \int_{k\delta}^{(k+1)\delta} \frac{d\tau}{\tau} \right)$$

$$+ \sum_{k=3}^{\infty} \int_{k\delta}^{(k+1)\delta} \left[\frac{u(t+(k+1/2)\delta) - u(t-(k+1/2)\delta)}{\tau} - \frac{u(t+\tau) - u(t-\tau)}{\tau} \right] d\tau$$

$$= J_3^{(1)}(t) + J_3^{(2)}(t). \tag{6}$$

It follows from inequality

$$\left| \frac{1}{k+1/2} - \int_{k\delta}^{(k+1)\delta} \frac{d\tau}{\tau} \right| = \ln(1+1/k) - \frac{1}{k+1/2} \leq \frac{1}{12k^3}, \quad k \in N$$

that

$$|J_3^{(1)}(t)| \leq \sum_{k=3}^{\infty} \|u\|_\alpha \cdot ((2k+1)\delta)^\alpha \cdot \frac{1}{12k^3} \leq \|u\|_\alpha \cdot \frac{\delta^\alpha}{12} \cdot \sum_{k=3}^{\infty} \frac{2k+1}{k^3}$$

$$\leq \|u\|_\alpha \cdot \frac{\delta^\alpha}{12} \cdot \int_2^\infty \frac{2x+1}{x^3} dx \leq \frac{\|u\|_\alpha}{6} \delta^\alpha. \tag{7}$$

For $J_3^{(2)}(t)$ we have

$$|J_3^{(2)}(t)|$$

$$\leq \sum_{k=3}^\infty \int_{k\delta}^{(k+1)\delta} \frac{|u(t+(k+1/2)\delta) - u(t+\tau)| + |u(t-(k+1/2)\delta) - u(t-\tau)|}{\tau} d\tau$$

$$\leq \sum_{k=3}^\infty \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} |u(t+(k+1/2)\delta) - u(t+\tau)| d\tau$$

$$+ \sum_{k=3}^\infty \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} |u(t-(k+1/2)\delta) - u(t-\tau)| d\tau$$

$$= J_3^{(2,1)}(t) + J_3^{(2,2)}(t). \tag{8}$$

Consider the case $t \geq 0$ (the case $t < 0$ is treated similarly).

If $\delta \geq 1/2$, then

$$J_3^{(2,1)}(t) \leq \sum_{k=3}^\infty \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} \left| \frac{1}{t+(k+1/2)\delta} - \frac{1}{t+\tau} \right|^\alpha \cdot \|u\|_\alpha d\tau$$

$$= \sum_{k=3}^\infty \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} \frac{|\tau - (k+1/2)\delta|^\alpha}{(t+(k+1/2)\delta)^\alpha (t+\tau)^\alpha} \cdot \|u\|_\alpha d\tau$$

$$\leq \sum_{k=3}^\infty \frac{1}{k\delta} \cdot \frac{(\delta/2)^\alpha}{(k\delta)^{2\alpha}} \cdot \delta \cdot \|u\|_\alpha \leq (\delta/2)^\alpha \cdot \|u\|_\alpha \cdot \sum_{k=3}^\infty \frac{1}{k^{1+2\alpha}\delta^{2\alpha}}$$

$$\leq (\delta/2)^\alpha \cdot \|u\|_\alpha \cdot \sum_{k=3}^\infty \frac{2^{2\alpha}}{k^{1+2\alpha}} \leq (2\delta)^\alpha \cdot \|u\|_\alpha \cdot \int_2^\infty \frac{dx}{x^{1+2\alpha}} dx \leq \frac{\|u\|_\alpha}{2\alpha} \cdot \delta^\alpha, \tag{9}$$

and if $\delta < 1/2$, then

$$J_3^{(2,1)}(t) \leq \sum_{k=3}^{[1/\delta]+1} \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} |(k+1/2)\delta - \tau|^\alpha \cdot \|u\|_\alpha d\tau$$

$$+ \sum_{k=[1/\delta]+2}^\infty \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} \left| \frac{1}{t+(k+1/2)\delta} - \frac{1}{t+\tau} \right|^\alpha \cdot \|u\|_\alpha d\tau$$

$$\leq \sum_{k=3}^{[1/\delta]+1} \frac{1}{k\delta}\left(\frac{\delta}{2}\right)^{\alpha} \cdot \|u\|_{\alpha} \cdot \delta + \sum_{k=[1/\delta]+2}^{\infty} \frac{1}{k\delta} \cdot \frac{(\delta/2)^{\alpha}}{(k\delta)^{2\alpha}} \cdot \delta \cdot \|u\|_{\alpha}$$

$$\leq \|u\|_{\alpha} \cdot \delta^{\alpha} \ln \frac{1}{\delta} + \|u\|_{\alpha} \cdot \delta^{\alpha} \int_{1/\delta}^{\infty} \frac{dx}{x \cdot (\delta x)^{2\alpha}} \leq \|u\|_{\alpha} \cdot \delta^{\alpha}\left[\ln \frac{1}{\delta} + \frac{1}{2\alpha}\right], \tag{10}$$

where $[1/\delta]$ is an integer part of the number $1/\delta$. It follows from (9) and (10) that for any $\delta > 0$

$$J_3^{(2,1)}(t) \leq \|u\|_{\alpha} \cdot \delta^{\alpha}\left[\ln(e + 1/\delta) + \frac{1}{2\alpha}\right]. \tag{11}$$

Let's estimate $J_3^{(2,2)}(t)$.

Case 1. $t < 20\delta$, $\delta > 1/2$. In this case,

$$J_3^{(2,2)}(t) \leq \sum_{k=3}^{[2t/\delta]+3} \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} |(k+1/2)\delta - \tau|^{\alpha} \cdot \|u\|_{\alpha} d\tau$$

$$+ \sum_{k=[2t/\delta]+4}^{\infty} \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} \left|\frac{1}{t-(k+1/2)\delta} - \frac{1}{t-\tau}\right|^{\alpha} \cdot \|u\|_{\alpha} d\tau$$

$$\leq \sum_{k=3}^{[2t/\delta]+3} \frac{1}{k\delta}\left(\frac{\delta}{2}\right)^{\alpha} \cdot \|u\|_{\alpha} \cdot \delta + \sum_{k=[2t/\delta]+4}^{\infty} \frac{1}{k\delta} \cdot \frac{(\delta/2)^{\alpha}}{(k\delta - t)^{2\alpha}} \cdot \delta \cdot \|u\|_{\alpha}$$

$$\leq \|u\|_{\alpha} \cdot \delta^{\alpha}\left[\sum_{k=3}^{43} \frac{1}{k} + \sum_{k=[2t/\delta]+4}^{\infty} \frac{1}{k(k\delta - t)^{2\alpha}}\right] \leq \|u\|_{\alpha} \cdot \delta^{\alpha}\left[3 + \int_{2+2t/\delta}^{\infty} \frac{dx}{x(x\delta - t)^{2\alpha}}\right]$$

$$= \|u\|_{\alpha} \cdot \delta^{\alpha}\left[3 + \int_{2\delta+t}^{\infty} \frac{dy}{(y+t)y^{2\alpha}}\right] \leq \|u\|_{\alpha} \cdot \delta^{\alpha}\left[3 + \int_{1}^{\infty} \frac{dy}{y^{1+2\alpha}}\right]$$

$$= \|u\|_{\alpha} \cdot \delta^{\alpha}\left[3 + \frac{1}{2\alpha}\right]. \tag{12}$$

Case 2. $t < 20\delta$, $\delta \leq 1/2$. In this case,

$$J_3^{(2,2)}(t) \leq \sum_{k=3}^{[20/\delta]+1} \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} |(k+1/2)\delta - \tau|^{\alpha} \cdot \|u\|_{\alpha} d\tau$$

$$+ \sum_{k=[20/\delta]+2}^{\infty} \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} \left|\frac{1}{t-(k+1/2)\delta} - \frac{1}{t-\tau}\right|^{\alpha} \cdot \|u\|_{\alpha} d\tau$$

$$\leq \sum_{k=3}^{[20/\delta]+1} \frac{1}{k\delta} \left(\frac{\delta}{2}\right)^{\alpha} \cdot \|u\|_{\alpha} \cdot \delta + \sum_{k=[20/\delta]+2}^{\infty} \frac{1}{k\delta} \cdot \frac{(\delta/2)^{\alpha}}{(k\delta - t)^{2\alpha}} \cdot \delta \cdot \|u\|_{\alpha}$$

$$\leq \|u\|_{\alpha} \cdot \delta^{\alpha} \sum_{k=3}^{[20/\delta]+1} \frac{1}{k} + \|u\|_{\alpha} \sum_{k=[20/\delta]+2}^{\infty} \frac{(\delta/2)^{\alpha}}{k(k\delta/2)^{2\alpha}}$$

$$\leq \|u\|_{\alpha} \cdot \delta^{\alpha} \ln \frac{20}{\delta} + \|u\|_{\alpha} \cdot (2\delta)^{\alpha} \cdot \int_{20/\delta}^{\infty} \frac{dx}{x \cdot (\delta x)^{2\alpha}}$$

$$\leq \|u\|_{\alpha} \cdot \delta^{\alpha} \ln \frac{20}{\delta} + \frac{\|u\|_{\alpha}}{2\alpha} \cdot \delta^{\alpha}. \tag{13}$$

Case 3. $t \geq 20\delta$, $\delta \geq 1/4$. In this case,

$$J_3^{(2,2)}(t) \leq \sum_{k=3}^{[t/2\delta]-2} \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} \left| \frac{1}{t-(k+1/2)\delta} - \frac{1}{t-\tau} \right|^{\alpha} \cdot \|u\|_{\alpha} d\tau$$

$$+ \sum_{k=[t/2\delta]-1}^{[2t/\delta]} \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} |(k+1/2)\delta - \tau|^{\alpha} \cdot \|u\|_{\alpha} d\tau$$

$$+ \sum_{k=[2t/\delta]+1}^{\infty} \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} \left| \frac{1}{t-(k+1/2)\delta} - \frac{1}{t-\tau} \right|^{\alpha} \cdot \|u\|_{\alpha} d\tau$$

$$\leq \sum_{k=3}^{[t/2\delta]-2} \frac{1}{k\delta} \cdot \frac{(\delta/2)^{\alpha}}{(t-(k+1)\delta)^{2\alpha}} \cdot \delta \cdot \|u\|_{\alpha} + \sum_{k=[t/2\delta]-1}^{[2t/\delta]} \frac{1}{k\delta} \left(\frac{\delta}{2}\right)^{\alpha} \delta \cdot \|u\|_{\alpha}$$

$$+ \sum_{[2t/\delta]+1}^{\infty} \frac{1}{k\delta} \cdot \frac{(\delta/2)^{\alpha}}{(k\delta - t)^{2\alpha}} \cdot \delta \cdot \|u\|_{\alpha}$$

$$\leq \delta^{\alpha} \cdot \|u\|_{\alpha} \left[\int_{2}^{t/2\delta-2} \frac{dx}{x(t-\delta(x+2))^{2\alpha}} + \int_{t/2\delta-2}^{2t/\delta} \frac{dx}{x} + \int_{2t/\delta}^{\infty} \frac{dx}{x(\delta x - t)^{2\alpha}} \right]$$

$$\leq \delta^{\alpha} \cdot \|u\|_{\alpha} \left[\left(\frac{2}{t}\right)^{2\alpha} \int_{2}^{t/2\delta-2} \frac{dx}{x} + \ln 5 + \int_{2t}^{\infty} \frac{dy}{y(y-t)^{2\alpha}} \right]$$

$$\leq \delta^{\alpha} \cdot \|u\|_{\alpha} \left[\left(\frac{2}{t}\right)^{2\alpha} \ln \frac{t}{4\delta} + \ln 5 + \int_{t}^{\infty} \frac{dy}{y^{1+2\alpha}} \right]$$

$$\leq \delta^{\alpha} \cdot \|u\|_{\alpha} \left[\frac{2}{e \cdot \alpha} + \ln 5 + \frac{1}{2\alpha} \right]. \tag{14}$$

Case 4. $t \in [20\delta, 5]$, $\delta < 1/4$. In this case,

$$J_3^{(2,2)}(t) \le \sum_{k=3}^{[10/\delta]+1} \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} |(k+1/2)\delta - \tau|^\alpha \cdot \|u\|_\alpha d\tau$$

$$+ \sum_{k=[10/\delta]+2}^{\infty} \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} \left| \frac{1}{t-(k+1/2)\delta} - \frac{1}{t-\tau} \right|^\alpha \cdot \|u\|_\alpha d\tau$$

$$\le \sum_{k=3}^{[10/\delta]+1} \frac{1}{k\delta} \left(\frac{\delta}{2}\right)^\alpha \delta \cdot \|u\|_\alpha + \sum_{k=[10/\delta]+2}^{\infty} \frac{1}{k\delta} \cdot \frac{(\delta/2)^\alpha}{(k\delta - t)^{2\alpha}} \cdot \delta \cdot \|u\|_\alpha$$

$$\le \delta^\alpha \cdot \|u\|_\alpha \left[\sum_{k=3}^{[10/\delta]+1} \frac{1}{k} + \int_{10/\delta}^{\infty} \frac{dx}{x(\delta x - t)^{2\alpha}} \right]$$

$$\le \delta^\alpha \cdot \|u\|_\alpha \left[\ln \frac{10}{\delta} + \int_{10t/\delta}^{\infty} \frac{dx}{x(\delta x - 5)^{2\alpha}} \right]$$

$$\le \delta^\alpha \cdot \|u\|_\alpha \left[\ln \frac{10}{\delta} + \frac{1}{2\alpha} \right]. \tag{15}$$

Case 5. $t > 5$, $\delta < 1/4$. In this case, similar to Case 3, we have

$$J_3^{(2,2)}(t) \le \sum_{k=3}^{[t/2\delta]-2} \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} \left| \frac{1}{t-(k+1/2)\delta} - \frac{1}{t-\tau} \right|^\alpha \cdot \|u\|_\alpha d\tau$$

$$+ \sum_{k=[t/2\delta]-1}^{[2t/\delta]} \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} |(k+1/2)\delta - \tau|^\alpha \cdot \|u\|_\alpha d\tau$$

$$+ \sum_{k=[2t/\delta]+1}^{\infty} \frac{1}{k\delta} \int_{k\delta}^{(k+1)\delta} \left| \frac{1}{t-(k+1/2)\delta} - \frac{1}{t-\tau} \right|^\alpha \cdot \|u\|_\alpha d\tau$$

$$\le \delta^\alpha \cdot \|u\|_\alpha \left[\left(\frac{2}{t}\right)^{2\alpha} \ln \frac{t}{4\delta} + \ln 5 + \frac{1}{2\alpha} \right]$$

$$\le \delta^\alpha \cdot \|u\|_\alpha \left[\frac{2}{e \cdot \alpha} + \ln \frac{1}{4\delta} + \ln 5 + \frac{1}{2\alpha} \right]. \tag{16}$$

It follows from (12), (13), (14), (15) and (16) that for any $\delta > 0$

$$J_3^{(2,2)}(t) \le \|u\|_\alpha \cdot \delta^\alpha \left[\ln(e + 1/\delta) + \frac{7}{2\alpha} \right]. \tag{17}$$

Now (3) follows from (4), (5), (6), (7), (8), (11) and (17). This completes the proof of the theorem. ◄

References

[1] R.A. Aliev, *A new constructive method for solving singular integral equations*, Math. Notes, **79(6)**, 2006, 749–770.

[2] R.A. Aliev, L.Sh. Alizade, *Approximation of the Hilbert transform in the Lebesgue spaces*, J. Numer. Anal. Approx. Theory, **52(3)**, 2023, 139–154.

[3] R.A. Aliev, A.F. Amrakhova, *A constructive method for the solution of integral equations with Hilbert kernel*, Trudy Inst. Mat. Mekh. UrO RAN, **18(4)**, 2012, 14–25 (in Russian).

[4] R.A. Aliev, Ch.A. Gadjieva, *Approximation of hypersingular integral operators with Cauchy kernel*, Numerical Functional Analysis and Optimization, **37(9)**, 2016, 1055-1065.

[5] D.M. Akhmedov, K.M. Shadimetov, *Optimal quadrature formulas for approximate solution of the first kind singular integral equation with Cauchy kernel*, Stud. Univ. Babes-Bolyai Math. **67(3)**, 2022, 633–651.

[6] B. Bialecki, *Sinc quadratures for Cauchy principal value integrals. Numerical Integration*, Recent Developments, Software and Applications, edited by T.O. Espelid and A. Genz, NATO ASI Series, Series C: Mathematical and Physical Sciences, 357. Dordrecht: Kluwer Academic Publishers, 1992, 81-92.

[7] H. Boche, V. Pohl, *Calculating the Hilbert Transform on Spaces With Energy Concentration: Convergence and Divergence Regions*, IEEE Transactions on Information Theory, **65**, 2019, 586–603.

[8] M.C. De Bonis, B.D. Vecchia, G. Mastroianni, *Approximation of the Hilbert transform on the real line using Hermite zeros*, Math. Comp., **71**, 2002, 1169-1188.

[9] G. Criscuolo, G. Mastroianni, *On the Uniform Convergence of Gaussian Quadrature Rules for Cauchy Principal Value Integrals*, Numer. Math., **54**, 1989, 445-461.

[10] S.B. Damelin, K. Diethelm, *Boundedness and uniform numerical approximation of the weighted Hilbert transform on the real line*, Numer. Func. Anal. and Optimization, **22(1-2)**, 2001, 13-54.

[11] P.J. Davis, P. Rabinowitz, *Methods of Numerical Integration,* Academic Press, New York, 1984.

[12] F.D. Gakhov, *Boundary value problems, Dover Publications*, New York, 1966.

[13] T. Hasegawa, *Uniform approximations to finite Hilbert transform and its derivative*, J. of Comp. and Appl. Math., **163(1)**, 2004, 127-138.

[14] T. Hasegawaa, H. Sugiura, *Uniform approximation to finite Hilbert transform of oscillatory functions and its algorithm*, J. of Comp. and Appl. Math., **358**, 2019, 327-342.

[15] V.R. Kress, E. Martensen, *Anwendung der rechteckregel auf die reelle Hilbert transformation mit unendlichem interval*, ZAMM, **50**, 1970, 61-64.

[16] S. Kumar, *A note on quadrature formulae for Cauchy principal value integrals*, IMA Journal of Applied Mathematics, **26(4)**, 1980, 447–451.

[17] J.Li, Z.Wang, *Simpson's rule to approximate Hilbert integral and its application*, Appl. Math. and Comp., **339**, 2018, 398-409.

[18] I.K. Lifanov, *Singular Integral Equations and Discrete Vortices*, VSP, Amsterdam, 1996.

[19] M.B. Abd-el-Malek, S.S. Hanna, *The Hilbert transform of cubic splines*, Comm. in Nonlinear Science and Numer. Simulation, **80**, 2020, 104983.

[20] G. Monegato, *The Numerical Evaluation of One-Dimensional Cauchy Principal Value Integrals*, Computing, **29**, 1982, 337-354.

[21] N.I. Muskhelishvili, *Singular integral equations: Boundary Problems of Function Theory and Their Application to Mathematical Physics*, 2-nd Edition. Dover Publications, New York, 2008.

[22] I. Notarangelo, *Approximation of the Hilbert transform on the real line using Freud weights*, in: W. Gautschi et al. (Eds.), Approximation and Computation – In Honor of Gradimir V. Milovanovic, Springer Optimization and Its Applications, **42**, Springer, 2011, 233-252.

[23] Sh. Olver, *Computing the Hilbert transform and its inverse*, Math. Comp., **80**, 2011, 1745-1767.

[24] M.M. Panja, B.N. Mandal. *Wavelet Based Approximation Schemes for Singular Integral Equations*, CRC Press, 2020.

[25] A. Setia, *Numerical solution of various cases of Cauchy type singular integral equation*, Appl. Math. and Comp., **230**, 2014, 200-207.

[26] A. Sidi, *Analysis of errors in some recent numerical quadrature formulas for periodic singular and hypersingular integrals via regularization*, Appl. Num. Math., **81**, 2014, 30–39.

[27] A. Sidi, *Richardson extrapolation on some recent numerical quadrature formulas for singular and hypersingular integrals and its study of stability*, J. Sci. Comp., **60(1)**, 2014, 141–159.

[28] F. Stenger, *Approximations via Whittaker's cardinal function*, J. Approx. Theory, **17**, 1976, 222-240.

[29] F. Stenger, *Numerical methods based on Whittaker cardinal or Sinc functions*, SIAM Review, **23**, 1981, 165-224.

[30] F. Stenger, *Numerical methods based on Sinc and analytic functions*, Springer Series in Computational Mathematics, **20**, Springer-Verlag, 1993.

[31] X. Sun, P. Dang, *Numerical stability of circular Hilbert transform and its application to signal decomposition*, Appl. Math. and Comp., **359**, 2019, 357-373.

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