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Existence and Continuous Dependence of Nonlocal Final Value Problem With Caputo-Hadamard Derivative

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Abstract. In this paper, we demonstrate the existence of a global solution to the problems with the final condition containing the Caputo-Hadamard derivative of order $p \in (0, 1)$. In particular, the uniqueness of the solution is proven when the source function satisfies the Lipschitz condition. To obtain these results, we employ topological degree theory in conjunction with the condensation condition of the operator, which corresponds to a measure of noncompactness. A few examples are provided to demonstrate the utility of this approach.

Key Words and Phrases: Hadamard derivative, Caputo–Hadamard derivative, fractional equation, Cauchy fractional problem.

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1. Introduction

Fraction calculations appeared very early; until now, they have been constantly developing because of their applicability in practice. Anomalous dynamic phenomena occur in physics, chemical biology, and even optimal control, among other fields (see [4, 2, 5, 7, 1, 3, 6] and references therein). The theory of fractional calculus developed brilliantly through the contributions of many mathematicians, including Oldham and Spanier (1974), Samko, Kilbas and Marichev (1993), Podlubny (1999), etc. Many practical applications and several different fractional derivatives, such as Riemann-Liouville fractional derivative, Caputo fractional derivative [8, 9], Riesz fractional derivative [9], and Hadamard, Hadamard-Type fractional derivative [5, 7, 10] and their properties, have been introduced.

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The Caputo-Hadamard derivative with kernel was introduced in recent years, and it has piqued the interest of some mathematicians [14, 13, 12, 11]. Intending to contribute to the enrichment of this field, we are interested in partial differential equations containing Caputo-Hadamard derivatives with fractional order.

Let X be a Banach space with the norm $|\cdot|$, and $J = [a, T]$. We denote by $C(J, X)$ the Banach space of continuous functions from J into X with the norm $\|u\| = \sup_{s\in J} |u(t)|$.

In this paper, we first consider the equation containing the fractional Caputo-Hadamard derivative of order $q \in (0, 1)$:

$$\begin{cases} {}^{C}_{H}D^{q}_{T^-}u(t) = f(t, u(t)), \text{ a.e. } t \in J, \\ u(T) = -g(u) + \xi_T, \end{cases} \tag{1}$$

where $\xi_T \in X$, $f : (a, T] \times X \to X$ and $g : C(J, X) \to X$ are the given functions satisfying the following conditions:

(f): f is $L^{\frac{1}{p_1}}$-Caratheodory, that is,

(f_1): $f(t, .) : X \to X$ is continuous for a.e. $t \in (a, T)$ and for $x \in X$, the function $f(., x) : J \to X$ is measurable;

(f_2): there exist $p_1 \in [0, q)$ and $\beta_1 \in L^{\frac{1}{p_1}}(J, \mathbb{R}_+)$ such that

$$|f(t, w)| \leq (|w| + 1)\,\beta_1(t) \ \text{ a.e on } J,$$

for all $w \in X$;

(g):

(i): either

(i_1): g is Lipscitz with a constant $K_g \in [0, 1)$ or

(i_2): g is compact.

(ii): there exists $c_g > 0$ such that

$$|g(u)| \leq c_g(\|u\| + 1) \text{ for all } u \in C(J, X).$$

Next, we consider the problem of finding $u = u(t, x)$ satisfying

$$\begin{cases} {}^{C}_{H}D^{q}_{T^-}u(t) = \mathcal{L}u(t) + \mathcal{F}(t, u(t)), \text{ a.e. } t \in J, \\ u(T) = -g(u) + \xi_T, \end{cases} \tag{2}$$

where $\mathcal{L}$ is a bounded linear operator from X to X, $\mathcal{F} : J \times X \to X$ and $g : C(J, X) \to X$ satisfy the conditions (g) and

($\mathcal{F}$):

(i): $\mathcal{F}(t, .) : X \to X$ is continuous for a.e. $t \in (a, T)$ and for $x \in X$, the function $\mathcal{F}(., x) : J \to X$ is measurable and

(ii): there exists $\beta \in L^{1,\gamma}((a,T),\mathbb{R}_+)$ for some $\gamma > -q$ such that

$$|\mathcal{F}(t,w)| \leq \beta(t)(|w|+1),$$

for a.e. $t \in (a,T)$, for all $w \in X$, where

$$L^{1,\gamma}((c,b),X) = \left\{ d \in L^1((c,b),X) : \left(\ln \frac{b}{(.)}\right)^{-\gamma} d(.) \in L^\infty((c,b),X) \right\}, \text{ and}$$

$$\|h(.)\|_{L^{1,\gamma}((c,b),X)} = \left\| \left(\ln \frac{b}{(.)}\right)^{-\gamma} h(.) \right\|_{L^\infty((c,b),X)},$$

with $\|u\|_{L^\infty((c,b),X)} = \inf\{C > 0 : |u(t)| \leq C, \text{ a.e. } t \in (c,b)\}$.

Several articles related to this study are listed below. Ma-Li [15] established some of the fundamental properties of Hadamard-type fractional operators, such as semigroup and reciprocal properties. The authors propose well-posed conditions for HTFDEs of fractional order $\rho \in (0,1)$:

$$\begin{aligned} {}_H\mathcal{D}^\rho_{a^+,\mu} u(t) &= g(t,x) \\ u(a) &= u_a. \end{aligned}$$

Gohar et al. in [11], investigated the fractional differential equation of Caputo-Hadamard

$${}^C_H\mathcal{D}^\rho_{a^+} u(t) = f(t,u(t)),$$

with the initial condition $\lim_{t\to a^+} u(t) = u_a \in \mathbb{R}$. They used Ascoli's theorem to prove the existence of the local solution. When f satisfies the Lipschitz condition, the authors achieve the uniqueness of the solution.

In [12], Li et al. studied the explosion and global existence of solutions to the space-time fractional diffusion equation

$$\begin{aligned} {}^C_H\mathcal{D}^\rho_{a^+} u(t,x) + (-\Delta)^\sigma u(t,x) &= |u(t,x)|^{q-1}, \ x \in \mathbb{R}^N, t > a > 0 \\ u(t,x) &= u_a(x), \ \ x \in \mathbb{R}^n, \end{aligned}$$

where $\rho \in (0,1)$, $0 < \sigma < 1$.

Allow us to share a few technical remarks and contributions to our work.

- The solution and source function f take values in the general Banach space X rather than the set of real numbers $\mathbb{R}$. This is useful when applied to problems.

- The final condition depends on the given source function g. In the particular case of $g = 0$, we obtain the well-known results.

- Using the Lebesgue-dominated convergence theorem to determine $\lim_{t\to t_*} \int_E \Phi_t(s)ds$, we frequently have difficulty locating the upper-bounded integrable function of the family $\{\Phi_t(s)\}_{t\in T}$. We solve this problem by demonstrating that the family $\{\int_E \Phi_t(s)ds\}_{t\in T}$ is bounded and deriving a convergent subsequence.

These issues have not yet been researched, as far as we know. The method used is the theory of topological degree for condensing mapping to prove the existence of its fixed points.

The remainder of this paper is structured as follows. The following section goes over the definitions of the Hadamard integral and Hadamard derivative, as well as the properties that will be used later. Section 3 discusses the existence of solutions to the Caputo-Hadamard fractional differential equations (1) and (2). The final section contains illustrative examples of applications to problems with final non-local conditions.

2. Preliminaries

Denote by $b(Y)$ (resp., $cc(Y)$, $ccb(Y)$) the family of all nonempty and bounded (resp., convex and closed, convex-closed and bounded) subsets of Banach space Y. Throughout this article, without explanation, X is a Banach space with norm $|.|$ and the order generated by the cone P, that is, $P \in cc(X)$, $\gamma P \subset P$ for all $\gamma \geq 0$ (we do not require that $P \cap (-P) = \{0\}$, and in particular, we can choose $P = Y$), and we write $x \leq_1 y$ iff $y - x \in P$. We always consider P to be the normal cone in the Banach space X, that is, there exists $\mathcal{N} > 0$ such that $x \leq_1 y$ implies $|x| \leq \mathcal{N}|y|$ $\forall u, v \in P$. Let $0 < a < T$, $J = [a, T]$. Denote $\mathcal{P} = \{u \in C(J, X) : u(t) \in P \ \forall t \in J\}$. Then $\mathcal{P}$ is a normal cone in $C(J, X)$ and wewrite $u \preceq v$ iff $v(t) - u(t) \in P$ for all $t \in J$. We define $J_c = [c, T]$ for $c \in [a, T)$ and $C(J_c, X) = \{u : J_c \to X |\ u \text{ is continuous}\}$. Then $C(J_c, X)$ is the Banach space with the norm $\|u\|_{C(J_c,X)} = \sup_{s\in J_c} |u(t)|$. In addition, it is clear that $\|u\|_{C(J_c,X)} \leq \|u\|_{C(J,X)} = \|u\|$ for all $u \in C(I, X)$. The characteristic function of $A \subset J$ is denoted by χ_A.

2.1. Caputo-Hadamard fractional derivative

Let $u : J \to X$ and $c < d$. In this article, we use the concept of the integral of u in terms of the Bochner integral, and we already know that u is Bochner integrable on (c, d) if and only if $\int_c^d |u(s)|ds < \infty$.

Definition 1. *Let $q \in (0,\infty)$, $a < b < \infty$, $q > 0$. The right Hadamard-type fractional integral ${}_H D_{b^-}^{-q} f$ and the right Hadamard-type fractional derivative ${}_H D_{b^-}^{q} f$ of order q, are defined by*

$${}_H D_{b^-}^{-q} f(t) = \frac{1}{\Gamma(q)} \int_t^b \frac{1}{s} \left(\ln \frac{s}{t}\right)^{q-1} f(s)ds, \ \ (a \leq t < b)$$

and

$$\begin{aligned} {}_H D_{b^-}^{q} f(t) &= (-\delta)^n \left({}_H D_{b^-}^{-(n-q)} f\right)(t) \\ &= \left(-\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-q)} \int_t^b \frac{1}{s} \left(\ln \frac{s}{t}\right)^{n-q-1} f(s)ds, \ \ (a \leq t < b) \end{aligned}$$

respectively, where $n = [q]+1$ and $\delta = t\frac{d}{dt}$.

Lemma 1. [16, Page 116, Property 2.28] *Let $q > 0$ and $x \in L^\mu((a,b),X)$ for some $\mu \in [1,\infty]$. We have*

$$\left({}_H D_{b^-}^{q} \left({}_H D_{b^-}^{-q}\right) x\right)(t) = x(t).$$

Lemma 2. [16, Page 117, Lemma 2.35] *Let $0 < a < b < \infty$ and $q > 0$. If $x \in \mathcal{J}_{b^-}^{q}(L^\mu((a,b),X))$ for some $\mu \in [1,\infty]$, then*

$${}_H D_{b^-}^{-q}({}_H D_{b^-}^{q} x)(t) = x(t),$$

where $\mathcal{J}_{b^-}^{q}(L^\mu((a,b),X)) = \left\{{}_H D_{b^-}^{-q}\psi : \psi \in L^\mu((a,b),X)\right\}$.

Definition 2. *The Caputo-Hadamard derivative of f with order $q > 0$ is defined by*

$${}_H^C\mathcal{D}_{b^-}^{q} f(t) =_H D_{b^-}^{q} \left[f(t) - \sum_{k=0}^{n-1} \frac{\delta^k f(b)}{k!} \left(\ln \frac{b}{t}\right)^k\right],$$

where $n = [p]+1$. In particular, if $0 < p < 1$ and $f \in C(J,X)$, then

$${}_H^C\mathcal{D}_{b^-}^{p} f(t) =_H D_{b^-}^{p}[f(t) - f(b)].$$

2.2. Fixed point index

A map $\alpha : b(X) \to X$ is called a *measure noncompactness* (in short, M.N.C.) if $\alpha(\overline{co}(\omega)) = \alpha(\omega)$ for all $\omega \in b(X)$. An operator $\mathbb{T} : X \to X$ is said to be *condensing* to α (in short, α-condensing) if $\omega \in b(X)$ with $\alpha(\omega) \leq \alpha(\mathbb{T}(\omega))$. Then ω is relative compact in X ([17, see Definition 2.1.1]).

Let $G \subset X$ and $\epsilon > 0$. A subset L of E is said to be ϵ-net of G if $G \subset \bigcup_{x\in L} \{y \in E : \|x-y\| < \epsilon\}$. If L is finite, then L is called a *finite* ϵ*-net*. We use the M.N.C. α defined by $\alpha(G) = \inf\{\epsilon > 0 : G$ has a finite ϵ-net$\}$.

Let D be an open subset of the Banach space Y, $D \in b(Y)$, $0 \in \Omega$, and $P \in cc(Y)$ with $\Omega \cap P \neq \emptyset$. Assume that $T : \overline{D} \cap P :\to P$ is a continuous and α-condensing operator. If $x \neq T(x)$ for all $x \in \partial D \cap P$, then the fixed point index $i_P(T, D)$ of T is well defined. The useful properties of this topological degree are shown in [18, Theorem 2.1]. We need the following results, which were proved in [19].

Proposition 1. [19, Proposition 2.4] *Let $D \ni 0$ be an open bounded subset of Y and $P \in cc(Y)$ with $D \cap P \neq \emptyset$. Assume that $T : \overline{D} \cap P :\to P$ is a continuous and α-condensing operator. Then we have the following properties:*

[(i)]

1. *if $\rho u \neq T(u)$ for all $(u, \rho) \in (\partial D \cap P) \times [1, \infty)$, then $i_P(T, D) = 1$.*
2. *if there is $u_0 \in P\backslash\{0\}$ satisfying $u \neq T(u) + \rho u_0$ for all $(u, \rho) \in (\partial D \cap P) \times [0, \infty)$, then $i_P(T, D) = 0$.*

Proposition 2. [18, Theorem 2.1] *Assume that $P \in cc(X)$, D is an open subset of Y, and $T : \overline{D} \cap P \to P$ is a continuous and α-condensing operator such that $u \neq T(u)$ for all $u \in \partial D \cap P$. Then,*

[(i)]

1. $\mathrm{Fix}(T) \neq \emptyset$ *if $i_P(T, D) \neq 0$, where* $\mathrm{Fix}(T) := \{x : T(x) = x\}$;
2. *if $D = D_1 \cup D_2$, where $D_1, D_2 \subset Y$ with $D_1 \cap D_2 = \emptyset$, such that $u \neq \mathbb{T}(u)$ for $u \in (\partial D_1 \cup \partial D_2) \cap P$, then*

$$i_P(T, D) = i_P(T, D_1) + i_P(T, D_2).$$

We use the above result in the following form:

Lemma 3. [20, Consequences of Theorem 4.1] *Let $\{\phi_n\}$ be a sequence in $L^p(J, \mathbb{R})$ ($p \geq 1$) such that*

[(i)]

1. *$\phi_n(s) \searrow 0$ (resp., $\nearrow 0$), a.e. $s \in J$;*
2. *there exists $c > 0$ such that $\left(\int_J |\phi_n(s)|^p ds\right)^{\frac{1}{p}} \leq c$ for all $n \in \mathbb{N}$.*

Then $\left(\int_J |\phi_n(s)|^p ds\right)^{\frac{1}{p}} \to 0$, as $n \to \infty$.

3. Main results

3.1. Integral formulations

The set of the absolutely continuous functions from J into X is denoted by $AC(J,X)$.

Proposition 3. *Assume that* (C1) and (C2) *hold. A function $u \in AC(J,X)$ is an integral solution of Problem* (1) *(resp.,* (2)*) iff u satisfies*

$$\begin{cases} u(t) = -g(u) + \xi_T +_H D_{T^-}^{-q} f(t,u(t)), \ \text{a.e. } t \in J, \\ u(T) = -g(u) + \xi_T, \end{cases} \tag{3}$$

(resp.,

$$\begin{cases} u(t) = -g(u) + \xi_T +_H D_{T^-}^{-q} \mathcal{L}u(t) +_H D_{T^-}^{-q} \mathcal{F}(t,u(t)), \ \text{a.e. } t \in J, \\ u(T) = -g(u) + \xi_T. \ ) \end{cases} \tag{4}$$

Proof. We consider equation (2), and the remaining equation is argued similarly. Suppose that $u \in C(J,X)$ satisfiying (4). Obviously, for $t \in [a,T)$, we have

$$\left| {}_H D_{T^-}^{-q} u(t) \right| \leq \infty \text{ and}$$
$$\left| {}_H D_{T^-}^{-q} \mathcal{F}(t,u(t)) \right| \leq \infty.$$

Therefore, functions $s \mapsto \frac{1}{s}\left(\ln \frac{s}{t}\right)^{p-1} \mathcal{L}u(s)$ and $s \mapsto \frac{1}{s}\left(\ln \frac{s}{t}\right)^{p-1} \mathcal{F}(s,u(s))$ are Bochner integrable on $[t,T]$.

From (4), it follows that

$$u(t) - u(T) =_H D_{T^-}^{-q} \mathcal{L}u(t) +_H D_{T^-}^{-q} \mathcal{F}(t,u(t)).$$

By using Lemma 1, we obtain

$${}_H D_{T^-}^{q} [u(t) - u(T)] = \mathcal{L}u(t) + \mathcal{F}(t,u(t)).$$

Thus, u is a solution of (2).

Reversely, if $u \in AC(J,X)$ satisfies (2), then there exists $\varphi \in L^1(J,X)$ such that $u(t) = u(T) - \int_t^T \varphi(s) ds$. Hence, $u(t) - u(T) =_H D_a^{-q} \phi(s)$ with $\phi(s) = -\Gamma(q) s \left(\ln \frac{s}{t}\right)^{-q+1} \varphi(s)$. Thus, $u(.) - u(T) \in \mathcal{J}_{T^-}^{q}(L^1(J,X))$. Applying Lemma 2, ${}_H D_{T^-}^{q} \left({}_H D_{T^-}^{-q}\right) [u(t) - u(T)] = u(t) - u(T)$. It implies (4). ◄

From the above results, we have the following definitions:

Definition 3. *A function $u \in C(J,X)$ is said to be an integral solution to Problem* (1) *(resp.,* (2)*) if it satisfies the following conditions:*

(i) $u(T) = -g(u) + \xi_T$,

(ii) $u(t) = -g(u) + \xi_T + {}_H D_{T^-}^{-q} f(t,u(t)) \;\; t \in [a,T)$
(resp., $u(t) = -g(u) + \xi_T + {}_H D_{T^-}^{-q} \mathcal{L}u(t) + {}_H D_{T^-}^{-q} \mathcal{F}(t,u(t)) \;\; t \in [a,T)$*).*

3.2. Boundedness settings

We define the following operators to establish the existence of a solution to problem (1):
$U, F, \mathbb{T} : C(J,X) \to C(J,X)$, $u \in C(J,X)$,

$$
\begin{aligned}
U(u)(t) &= -g(u) + \xi_T, \;\; t \in J; \\
F(u)(t) &= {}_H D_{T^-}^{-q} f(t,u(t)) \quad \text{for } a \le t < T; \quad \text{and } F(u)(T) = 0; \\
\mathbb{T}(u) &= U(u) + F(u).
\end{aligned}
$$

Then, the problem (1) has a solulion if and only if $\mathrm{Fix}(\mathbb{T}) \neq \emptyset$.

To prove the existence of a solution to problem (2), we consider the following operators:
$\mathbb{U}, \mathbb{F}, \mathbb{G} : C(J,X) \to C(J,X)$,

$$
\begin{aligned}
\mathbb{U}(u)(t) &= -g(u) + \xi_T, \;\; t \in J; \\
\mathbb{F}(u)(t) &= {}_H D_{T^-}^{-q} \mathcal{L}u(t) + {}_H D_{T^-}^{-q} \mathcal{F}(t,u(t)) \quad \text{for } a \le t < T; \quad \text{and } \mathbb{F}(u)(T) = 0; \\
\mathbb{G}(u) &= \mathbb{U}(u) + \mathbb{F}(u).
\end{aligned}
$$

The problem (2) has a solulion if and only if $\mathbb{G}$ has a fixed point in $C(J,X)$.

Proposition 4. *Let $a < b \le T$. We have the following assertions:*

(i) *If the condition (f) holds, then*

$$
\left| {}_H D_{b^-}^{-q} f(t,u(t)) \right| \le C_1 (b-t)^{q-p_1} (\|u\|_{C(J,X)} + 1)
$$

for all $u \in C(J,X)$ and $a \le t < b$, where $C_1 = \frac{(1-p_1)^{1-p_1}}{\Gamma(q) a^q (q-p_1)^{1-p_1}} \|\beta_1\|_{L^{\frac{1}{p_1}}(J,X)}$.

(ii) *If the condition ($\mathcal{F}$) is true, then*

$$
\begin{aligned}
|{}_H D_{b^-}^{-q} \mathcal{L}u(t)| &\le C_0 (b-t)^q \|u\|_{C(J,X)}; \\
|{}_H D_{b^-}^{-q} \mathcal{F}(t,u(t))| &\le C_2 (b-t)^{q+\gamma} (1 + \|u\|_{C(J,X)}),
\end{aligned}
$$

for all $u \in C(J,X)$, $a \le t < b$, here $C_0 = \frac{\|\mathcal{L}\|_\dagger}{a^q \Gamma(q+1)}$, $C_2 = \frac{\|\beta\|_{L^{1,\gamma}(J,X)}}{a^{q+\gamma}} \frac{\Gamma(\gamma+1)}{\Gamma(q+\gamma+1)}$.

(iii) *If the conditions* (f)-$(\mathcal{F})$ *are true, then*

$$\|F(u)\|_{C(J,X)} \leq C_1(T-a)^{q-p_1}\left(\|u\|_{C(J,X)}+1\right), \tag{5}$$

$$\|\mathbb{F}(u)\|_{C(J,X)} \leq C_0(T-a)^{q}\|u\|_{C(J,X)} + C_2(T-a)^{q+\gamma}\left(\|u\|_{C(J,X)}+1\right) \tag{6}$$

for all $u \in C(J,X)$*; and the operators* F *and* $\mathbb{F}$ *are compact.*

Proof. (i): Using Holder's inequality, we have

$$\begin{aligned}\left|{}_H D_{b^-}^{-q} f(t,u(t))\right| &\leq \frac{(\|u\|_{C([t,b],X)}+1)}{\Gamma(q)}\int_t^b \frac{1}{s}\left(\ln\frac{s}{t}\right)^{q-1}\beta_1(s)ds \\ &\leq \frac{(\|u\|_{C(J,X)}+1)}{\Gamma(q)}\|\beta_1\|_{L^{\frac{1}{p_1}}(J,X)}\left(\int_t^b\left(\frac{1}{s}\left(\ln\frac{s}{t}\right)^{q-1}\right)^{\frac{1}{1-p_1}}ds\right)^{1-p_1}.\end{aligned} \tag{7}$$

Using variable substitution $z=\ln\frac{s}{t}$, we obtain

$$\left(\int_t^b\left(\frac{1}{s}\left(\ln\frac{s}{t}\right)^{q-1}\right)^{\frac{1}{1-p_1}}ds\right)^{1-p_1} \leq \frac{(1-p_1)^{1-p_1}}{a^q(q-p_1)^{1-p_1}}(b-t)^{q-p_1}. \tag{8}$$

From (7) and (8), we get (i).
(ii): Denote by $\|.\|_\dagger$ the norm defined on the Banach space of the linear operators from X into X. For $u \in C(J,X)$, we have

$$\begin{aligned}|\mathcal{L}u(t)| &\leq \|\mathcal{L}\|_\dagger |u(t)| \\ &\leq \|\mathcal{L}\|_\dagger.\|u\|_{C(J,X)}.\end{aligned}$$

This gives

$$\begin{aligned}|{}_H D_{b^-}^{-q}\mathcal{L}u(t)| &\leq \frac{\|\mathcal{L}\|_\dagger.\|u\|_{C(J,X)}}{\Gamma(q)}\int_t^{b^-}\frac{1}{s}\left(\ln\frac{s}{t}\right)^{q-1}ds \\ &= \frac{\|\mathcal{L}\|_\dagger.\|u\|_{C(J,X)}}{q\Gamma(q)}\left(\ln\frac{b}{t}\right)^q \\ &\leq \frac{\|\mathcal{L}\|_\dagger.\|u\|_{C(J,X)}}{a^q\Gamma(q+1)}(b-t)^q.\end{aligned} \tag{9}$$

From (ii) of condition $(\mathcal{F})$, using Holder's inequality and changing variable $z=\frac{\ln\frac{s}{t}}{\ln\frac{b}{t}}$, we get

$$|{}_H D_{b^-}^{-q}\mathcal{F}(t,u)| \leq \frac{(\|u\|_{C(J,X)}+1)}{\Gamma(q)}\int_t^b\frac{1}{s}\left(\ln\frac{b}{s}\right)^{q-1}\beta(s)ds$$

$$\le \frac{(\|u\|_{C(J,X)}+1)}{\Gamma(q)}\|\beta\|_{L^{1,\gamma}((a,b),X)}\int_t^b \frac{1}{s}\left(\ln\frac{s}{t}\right)^{q-1}\left(\ln\frac{b}{s}\right)^{\gamma}ds$$
$$\le \frac{(\|u\|_{C(J,X)}+1)}{\Gamma(q)}\|\beta\|_{L^{1,\gamma}((a,b),X)}\frac{1}{a^{q+\gamma}}\frac{\Gamma(q)\Gamma(\gamma+1)}{\Gamma(q+\gamma+1)}(b-t)^{q+\gamma}$$
$$\le \frac{\|\beta\|_{L^{1,\gamma}(J,X)}}{a^{q+\gamma}}\frac{\Gamma(\gamma+1)}{\Gamma(q+\gamma+1)}(b-t)^{q+\gamma}(\|u\|_{C(J,X)}+1). \tag{10}$$

From (9)-(10), we derive (ii).

(iii): For $u\in C(J,X)$, to show $F(u)\in C(J,X)$, instead of considering the continuity of $F(u)$ at the point $t_0\in J$, we show that the family $\mathcal{M}:=F(\mathcal{B}_r(J,X))$ is equicontinuous at t_0, where $\mathcal{B}_r(J,X):=\{u\in C(J,X):\|u\|\le r\}$.

Let $t_0\in J$, $a\le t<t_0\le T$ (resp., $a\le t_0<t<T$) and $u\in\mathcal{B}_r(J,X)$. For short, denote $\Phi_t(s):=\frac{1}{s}\left(\ln\frac{s}{t}\right)^{q-1}$. From the representation

$$\int_t^T \Phi_t(s)f(s,u(s))ds=\int_t^{t_0}\Phi_t(s)f(s,u(s))ds+\int_{t_0}^T\Phi_t(s)f(s,u(s))ds$$

(resp., $\displaystyle\int_{t_0}^T \Phi_{t_0}(s)f(s,u(s))ds=\int_{t_0}^{t}\Phi_{t_0}(s)f(s,u(s))ds+\int_{t}^T\Phi_{t_0}(s)f(s,u(s))ds$),

we have

$$|F(u)(t)-F(u)(t_0)|\le A(t,t_0)+B(t,t_0), \tag{11}$$

where

$$A(t,t_0):=\left|{}_HD_{t_0^-}^{-q}f(t,u(t))\right|\le C_1(\|u\|_{C(J,X)}+1)(t_0-t)^{q-p_1}$$
$$\le C_1(r+1)(t_0-t)^{q-p_1}\to 0 \text{ as } t\to t_0^- \text{ (independently of } u); \tag{12}$$

(resp., $A(t,t_0)\le C_1(r+1)(t-t_0)^{q-p_1}$), (13)

and

$$B(t,t_0):=\frac{1}{\Gamma(q)}\int_{t_0}^T|\Phi_t(s)-\Phi_{t_0}(s)|\,|f(s,u(s))|ds$$
$$\le c_1\left(\int_{t_0}^T|\Phi_t(s)-\Phi_{t_0}(s)|^{\frac{1}{1-p_1}}ds\right)^{1-p_1}, \tag{14}$$

where $c_1=\frac{1}{\Gamma(q)}(\|\beta_1\|_{L^{\frac{1}{p_1}}(J,X)}+1)$. Noting that $|\Phi_t(.)-\Phi_{t_0}(.)|^{\frac{1}{1-p_1}}\le|\Phi_{t_0}(.)|^{\frac{1}{1-p_1}}\in L^1(J_{t_0},\mathbb{R}_+)$ and $|\Phi_t(s)-\Phi_{t_0}(s)|\to 0$, as $t\to t_0^-$, and using Lebesgue's dominated convergence theorem, we get $B(t,t_0)\to 0$ independently of u. Hence, $A(t,t_0)+B(t,t_0)\to 0$ independently of u, as $t\to t_0^-$.

In case $a \le t_0 < t < T$, we have

$$\begin{aligned}
B(t,t_0) &:= \frac{1}{\Gamma(q)}\int_t^T |\Phi_t(s)-\Phi_{t_0}(s)||f(s,u(s))|ds \\
&\le c_1\left(\int_t^T |\Phi_t(s)-\Phi_{t_0}(s)|^{\frac{1}{1-p_1}}ds\right)^{1-p_1} \\
&=: c_1\left(\int_a^T L(t)(s)^{\frac{1}{1-p_1}}ds\right)^{1-p_1}, \qquad (15)
\end{aligned}$$

where $L(t)(s) := \chi_{(t,T]}(s)|\Phi_t(s)-\Phi_{t_0}(s)|$. A direct calculation yields

$$\begin{aligned}
0 \le \left(\int_a^T L(t)(s)^{\frac{1}{1-p_1}}ds\right)^{1-p_1} &\le \left(\int_t^T |\Phi_t(s)|^{\frac{1}{1-p_1}}ds\right)^{1-p_1} \\
&= \left(\int_t^T \left(\frac{1}{s}\left(\ln\frac{s}{t}\right)^{q-1}\right)^{\frac{1}{1-p_1}}ds\right)^{1-p_1} \\
&\le c < \infty, \qquad (16)
\end{aligned}$$

where $c = \frac{(1-p_1)^{1-p_1}(T-a)^{q-p_1}}{a^q(q-p_1)^{1-p_1}}$. In addition, for every $s \in (a,T)$, $\lim_{t\to t_0^+} L(t)(s) = 0$. From (15)-(16), using Lemma 3, we get $\lim_{t\to t_0^+} B(t,t_0) \to 0$. Thus, $A(t,t_0) + B(t,t_0) \to 0$ as $t \to t_0^+$. Therefore, we conclude that $\mathcal{M}$ is equicontinuous.

In addition, we immediately see that $\mathcal{M}$ is pointwise bounded by (i). Thus, applying Arzela-Ascoli theorem we deduce that $\mathcal{M}$ is relatively compact.

We now show that F is continuous from $C(J,X)$ to $C(J,X)$. Assume that $\{u_n\} \subset C(J,X)$ with $u_n \to u$. Since

$$\begin{aligned}
|f(s,u_n(s)) - f(s,u(s))| &\le |f(s,u_n(s))| + |f(s,u(s))| \\
&\le \beta_1(s)\left(\|u_n\| + 1 + \|u\| + 1\right) \\
&\le \beta_1(s)\left(2\|u\| + 3\right) \in L^{\frac{1}{p_1}}(J,\mathbb{R}_+)
\end{aligned}$$

and $|f(s,u_n(s)) - f(s,u(s))| \to 0$ as $n \to \infty$ for a.e. $s \in (a,T)$, we have

$$\begin{aligned}
|F(u_n)(t) - F(u)(t)| &\le \frac{1}{\Gamma(q)}\int_t^T \left|\frac{1}{s}\left(\ln\frac{s}{t}\right)^{q-1}\right| |f(t,u_n(t)) - f(t,u(t))|ds \\
&\le c\left(\int_t^T |f(s,u_n(s)) - f(s,u(s))|^{\frac{1}{p_1}}\right)^{p_1} \\
&\le c\left(\int_a^T |f(s,u_n(s)) - f(s,u(s))|^{\frac{1}{p_1}}\right)^{p_1} \to 0
\end{aligned}$$

(here $c = \frac{(1-p_1)^{1-p_1}}{\Gamma(q)a^q(q-p_1)^{1-p_1}}(T-a)^{q-p_1}$), independently of t, as $n \to \infty$. Therefore, we obtain the continuity and compactness of F. Evaluating (5) and (6) is clear by using (ii).

Define $G, K : C(J,X) \to C(J,X)$ by

$$\begin{aligned} &G(u)(t) = {}_H D_{T^-}^{-q} \mathcal{F}(t, u(t)), \\ &H(u)(t) = {}_H D_{T^-}^{-q} \mathcal{L}u(t) \quad \text{with } a \le t < T; \\ &G(u)(T) = H(u)(T) = 0 \end{aligned}$$

for $u \in C(J,X)$. We prove that G and H are compact in the same way that we proved that the operator F is compact. Thus, $\mathbb{F} = H + G$ is compact. The proof is complete. ◄

Lemma 4. (i) *If the conditions* (f) *and* (g) *hold, then* $\mathbb{T}$ *is* α*-condensing.*

(ii) *If the assumptions* $(\mathcal{F})$ *and* (g) *are true, then* $\mathbb{G}$ *is* α*-condensing.*

Proof. (i): It is clear that operators $U, F, \mathbb{T} : C(J,X) \to C(J,X)$ are continuous and bounded. Let $\mathcal{N} \in b(C(J,X))$. Consider condition (i) of (g). If (i_1) is true, we get

$$\begin{aligned} \|U(x) - U(y)\| &= |g(x) - g(y)| \\ &\le K_g \|x - y\| \;\; \forall x, y \in C(J,X). \end{aligned}$$

Consequently, $\alpha(U(\mathcal{N})) \le K_g \alpha(\mathcal{N})$. Therefore,

$$\begin{aligned} \alpha(\mathbb{T}(\mathcal{N})) &= \alpha\left(U(\mathcal{N}) + F(\mathcal{N})\right) \\ &\le \alpha(U(\mathcal{N})) + \alpha(F(\mathcal{N})) \\ &= \alpha(U(\mathcal{N})) \\ &\le K_g \alpha(\mathcal{N}). \end{aligned}$$

Hence, $\mathbb{T}$ is an α-condensing operator.

If (i_2) is true, since g is continuous, it implies that $\mathbb{T}$ is continuous by Proposition 4. Let $\{u_n\} \subset \mathcal{N}$, $y_n = g(u_n)$. Since g and F are compact, we can assume that $y_n \to y$ and $F(u_n) \to z$, so $\mathcal{T}(u_n) \to -y + \xi_T + z$. Therefore $\mathbb{T}$ is compact. Thus, it is α-condensing. Claim (ii) is proven similarly. ◄

Proposition 5. *Assume that the following conditions are satisfied:* (f), (g), $c_g + C_1(T-a)^{q-p_1} < 1$, $f(J,P) \subset P$, $\mathcal{L}(P) \subset P$ *and* $-g(u) + \xi_T \in P$ *for all* $u \in \mathcal{P}$. *We have*

(i) *$i_{\mathcal{P}}(\mathbb{T}, \mathcal{B}_r(J)) = 1$ with $r > 0$ large enough;*

(ii) *if there exists $\xi_* \in P\backslash\{0\}$ such that $|-g(u) + \xi_T| \geq |\xi_*|$ for all $u \in \mathcal{P}\backslash\{0\}$, then $i_{\mathcal{P}}(\mathbb{T}, \mathcal{B}_r(J)) = 0$ with $r > 0$ small enough,*

where $\mathcal{B}_r(J) = \{x \in C(J, X) : \|x\|_{C(J,X)} < r\}$, $C_1 = \frac{(1-p_1)^{1-p_1}}{\Gamma(q)a^q(q-p_1)^{1-p_1}}\|\beta_1\|_{L^{\frac{1}{p_1}}(J,X)}$.

Proof. Choose $\gamma_0 \in (0, \infty)$ such that

$$c_1 := c_g + C_1(T - a)^{q-p_1} < \gamma_0 < 1.$$

It is clear that $\mathcal{B}_r(J)$ is open in $C(J, X)$ and $\mathbb{T}(\mathcal{P}) \subset \mathcal{P}$. To get (i), we will prove that

$$\rho u \neq \mathbb{T}(u) \quad \text{for all } (u, \rho) \in (\partial B_r(J) \cap \mathcal{P}) \times [1, \infty) \text{ for sufficiently large } r > 0. \tag{17}$$

Assume that (17) is false. Then we can choose sequences $\rho_n \geq 1$ and $u_n \in \mathcal{P}$ such that

$$\rho_n u_n = \mathbb{T}(u_n) \text{ and } \|u_n\| \to \infty.$$

Using Proposition 4, we derive

$$\begin{aligned}\rho_n\|u_n\| &\leq \|\mathbb{T}(u_n)\| \\ &\leq |\xi_T| + c_1(\|u_n\| + 1).\end{aligned}$$

It implies that

$$\rho_n \leq \frac{|\xi_T| + c_1(\|u_n\| + 1)}{\|u_n\|}. \tag{18}$$

Letting $n \to \infty$ in (18), we get a contradiction. Therefore, (17) holds for $r > 0$. Thus, by applying Proposition 1, we get $i_{\mathcal{P}}(\mathbb{T}, \mathcal{B}_r(J)) = 1$ for sufficiently large $r > 0$.

To obtain (ii) we will show that

$$\rho u \neq \mathbb{T}(u) + \rho u_* \quad \text{for all } (u, \rho) \in (\partial\mathcal{B}_r(J) \cap \mathcal{P}) \times [0, \infty) \text{ with } r > 0 \text{ small enough}, \tag{19}$$

where $u_*(t) = \xi_*$ for all $t \in J$. Assume that (19) is not true. There are sequences $\rho_n \geq 0$, $u_n \in \mathcal{P}$ satisfying

$$\|u_n\| \to 0 \quad \text{and} \quad u_n = \mathbb{T}(u_n) + \rho_n u_*.$$

This implies that

$$u_n \succeq U(u_n) + \rho_n u_* \succeq U(u_*).$$

Thus

$$u_n(t) \geq_1 U(u_n)(t) + \rho_n u_*(t) \geq_1 -g(u_n) + \xi_T \geq_1 \xi_* \ \forall t \in J.$$

Since $\mathcal{P}$ is a normal cone,

$$\|u_n\| \geq \mathcal{N}^{-1}|\xi_*| > 0.$$

This is impossible, so we obtain (19).

Using Proposition 1, we derive $i_{\mathcal{P}}(\mathbb{T}, \mathcal{B}_r(J)) = 0$ with $r > 0$ small enough. The proof is finished. ◀

Proposition 6. *Assume that the following conditions are fulfilled:* $(\mathcal{F})$, (g), $\mathcal{F}(J,P) \subset P$, $\mathcal{L}(P) \subset P$, $c_g + C_0(T-a)^q + C_2(T-a)^{q+\gamma} < 1$, *and* $-g(u)+\xi_T \in P$ *for all* $u \in \mathcal{P}$, *where* $C_0 = \frac{\|\mathcal{L}\|_\dagger}{a^q\Gamma(q+1)}$, $C_2 = \frac{\|\beta\|_{L^{1,\gamma}(J,X)}}{a^{q+\gamma}}\frac{\Gamma(\gamma+1)}{\Gamma(q+\gamma+1)}$. *Then*

(i) *there exists* $r_0 > 0$ *such that* $i_{\mathcal{P}}(\mathbb{G}, \mathcal{B}_r(J)) = 1$ *for all* $r \geq r_0$*;*

(ii) *if there exists* $\xi_* \in P\backslash\{0\}$ *such that* $|-g(u)+\xi_T| \geq |\xi_*|$ *for all* $u \in \mathcal{P}\backslash\{0\}$, *then there is* $r_0 > 0$ *such that* $i_{\mathcal{P}}(\mathbb{G}, \mathcal{B}_r(J)) = 0$ *with* $0 < r \leq r_0$.

Proof. Denote

$$c_0 := C_0(T-a)^q,$$
$$c_2 := C_2(T-a)^{q+\gamma}.$$

Choose γ_0 satisfying

$$c_\dagger := c_g + c_0 + c_2 < \gamma_0 < 1.$$

Let $\rho \in [1,\infty)$ and $u \in C(J,X)$. If $\rho u = G(u)$, using (6), we arrive at

$$\|\rho u\| \leq |\xi_T| + c_g(\|u\|+1) + c_0\|u\| + c_2(\|u\|+1).$$

Thus, we get

$$\rho \leq \frac{|\xi_T| + c_g + c_\dagger\|u\| + c_2}{\|u\|}.$$

From this estimate, we get (17), so we have assertion (i). To obtain assertion (ii), we make the same argument as in the proof of Proposition 5. ◀

3.3. Global/Local solution existence theorems

Theorem 1. *Assume* (f)*-*(g) *and* $c_g + C_1(T-a)^{q-p_1} < 1$*, where* $C_1 = \frac{(1-p_1)^{1-p_1}}{\Gamma(q)a^q(q-p_1)^{1-p_1}}\|\beta_1\|_{L^{\frac{1}{p_1}}(J,X)}$*. Then Problem* (1) *has at least one solution* $w \in C(J,X)$*, and*

$$w(t) = -g(w) + \xi_T + \frac{1}{\Gamma(q)}\int_t^T \frac{1}{s}\left(\log\frac{s}{t}\right)^{q-1} f(s,w(s))ds,\ t \in [a,T).$$

Furthermore, if there exists $\xi_* \in C(J,X)\backslash\{0\}$ *such that* $|-g(u)+\xi_T| \geq |\xi_*|$ *for all* $u \neq 0$*, then equation* (1) *has a nontrivial solution in* $\mathcal{C}(J,X)$*.*

Proof. It is clear that the assumptions of Proposition 5 are satisfied with $P := X$ and $\mathcal{P} := C(J,X)$. Therefore, there exists $R > 0$ satisfying $i_{\mathcal{P}}(\mathbb{T}, \mathcal{B}_R(J)) = 1$. Applying Proposition 2, we derive $\mathrm{Fix}(\mathbb{T}) \neq \emptyset$. This implies that Problem (1) has at least one solution $u \in \mathcal{P}$. Furthermore, by (ii) of Proposition 5 we can choose r with $0 < r < R$ such that $i_{\mathcal{P}}(\mathbb{T}, \mathcal{B}_r(J)) = 0$. Applying Proposition 2 with $D_1 = \mathcal{B}_r(J)$, $D_2 = \mathcal{B}_R(J)\backslash\overline{\mathcal{B}_r(J)}$, we see that the problem has a solution u in D_2. The theorem is proved. ◄

Theorem 2. *Assume* $(\mathcal{F})$*-*(g)*, and*

$$c_g + \frac{\|\mathcal{L}\|_\dagger}{a^q\Gamma(q+1)}(T-a)^q + \frac{\|\beta\|_{L^{1,\gamma}(J,X)}}{a^{q+\gamma}}\frac{\Gamma(\gamma+1)}{\Gamma(q+\gamma+1)}(T-a)^{q+\gamma} < 1.$$

Then Problem (2) *has at least one solution* $w \in C(J,X)$*, and*

$$w(t) = -g(w) + \xi_T + {}_H D_{T^-}^{-q}\mathcal{L}w(t) + {}_H D_{T^-}^{-q}\mathcal{F}(t,w(t)),\ t \in [a,T),$$

where

$${}_H D_{T^-}^{-q}\mathcal{L}w(t) = \frac{1}{\Gamma(q)}\int_t^T \frac{1}{s}\left(\log\frac{s}{t}\right)^{q-1}\mathcal{L}u(s)ds,$$
$${}_H D_{T^-}^{-q}\mathcal{F}(t,w(t)) = \frac{1}{\Gamma(q)}\int_t^T \frac{1}{s}\left(\log\frac{s}{t}\right)^{q-1}\mathcal{F}(s,u(s))ds.$$

Furthermore, if there exists $\xi_* \in C(J,X)\backslash\{0\}$ *such that* $|-g(u)+\xi_T| \geq |\xi_*|$ *for all* $u \neq 0$*, then* (2) *has a nontrivial solution in* $C(J,X)$*.*

Proof. The proof is argued in the same way as the one of Theorem 1, by checking the assumptions of Proposition 6. ◄

Remark 1. *Assumption* (g) *can be replaced with the following assumption* (g_1)*:*
(i): g *is Lipschitz with a constant* $K_g \in [0,1)$*;*
(ii): $g(0) = 0$*.*

3.4. Uniqueness of solution

In this section, we establish sufficient conditions for equation (1) to have a unique solution.

The operator $f : J \times X \to X$ is said to be uniformly Lipschitz with respect to the second variable on J if there exists $L > 0$ such that

$$|f(t,x) - f(t,y)| \le L|x-y| \text{ for all } t \in J, x, y \in X. \tag{20}$$

Theorem 3. *Assume that*

(i) *f satisfies the Lipschitz condition* (20)*, and*

(ii) *there exists $K_g > 0$ such that*

$$|g(u) - g(v)| \le K_g\|u - v\| \ \forall u, v \in C(J,X)$$

and $K_g + \frac{L(T-a)^q}{a^q\Gamma(q+1)} < 1$.

Then, the problem (1) *has a unique solution.*

Proof. Set

$$\mathcal{B}(r) = \{u \in C(J,X) : \|u - \mathbb{T}(\theta)\| \le r\} \ \ (r > 0),$$

where θ is a zero of $C(J,X)$. It is clear $\mathcal{B}(r)$ is closed in $C(J,X)$. For $u, v \in \mathcal{B}(r)$ (r will be chosen later), we have

$$\begin{aligned} |\mathbb{T}u(t) - \mathbb{T}v(t)| &\le |g(u) - g(v)| + |Fu(t) - Fv(t)| \\ &\le K_g\|u - v\| + \frac{1}{\Gamma(q+1)}\int_t^T \Phi_t(s)\,|f(s,u(s)) - f(s,v(s))|\,ds \\ &\le \left(K_g + \frac{L(T-a)^q}{a^q\Gamma(q+1)}\right)\|u - v\|. \end{aligned} \tag{21}$$

This implies

$$\|\mathbb{T}u - \mathbb{T}v\| \le \left(K_g + \frac{L(T-a)^q}{a^q\Gamma(q+1)}\right)\|u - v\|.$$

From (21) with $v = \theta$, we get

$$\|\mathbb{T}(u) - \mathbb{T}(\theta)\| \le \left(K_g + \frac{L(T-a)^q}{a^q\Gamma(q+1)}\right)\|u - \mathbb{T}(\theta)\| + \|\mathbb{T}(\theta)\|$$

$$\leq \left(K_g + \frac{L(T-a)^q}{a^q\Gamma(q+1)}\right) r + \|\mathbb{T}(\theta)\|. \tag{22}$$

From the estimate (22), we can choose

$$r \geq \frac{\|\mathbb{T}(\theta)\|}{1 - \left(K_g + \frac{L(T-a)^q}{a^q\Gamma(q+1)}\right)}$$

and have $\mathbb{T}(\mathcal{B}(r)) \subset \mathcal{B}(r)$. From Banach's fixed point theorem, it foloows that $\mathbb{T}$ has a unique fixed point in $\mathcal{B}(c, r)$. The proof is completed. ◄

Theorem 4. *Assume that the following assumptions are fulfilled:*

(i) *$\mathcal{F}$ satisfies the Lipschitz condition* (20)*, and*

(ii) *there exists $K_g > 0$ such that*

$$|g(u) - g(v)| \leq K_g\|u - v\| \ \forall u, v \in C(J, X)$$

and $K_g + (\|\mathcal{L}\|_\dagger + L)\frac{(T-a)^q}{a^q\Gamma(q+1)} < 1.$

Then, the problem (2) *has a unique solution.*

Proof. The proof is argued in the same way as the one of Theorem 3 with

$$\mathcal{B}(r) := \{u \in C(J, X) : \|u - \mathbb{G}(\theta)\| \leq r\},$$

where $r > \frac{\|\mathbb{G}(\theta)\|}{1-\left(k_g+(\|\mathcal{L}\|_\dagger+L)\frac{(T-a)^q}{a^q\Gamma(q+1)}\right)}$. ◄

4. Illustrative examples

This section presents three examples of applying abstract results to final-value problems that contain functions that observe the state of past solutions. In the first example, we get the solution to the fractional diffusion equation. In the second example, we illustrate the case where the problem has a unique solution. These non-local conditions $u(T) + g(u) + u_T$, or $u(a) + g(u) + u_a$ can be applied in physics with effective initial conditions that outperform classical initial conditions. For instance, $g(u) = \sum_{j=1}^{m} c_j u(t_j)$, c_j $(j = 1, ..., m)$ are constants, $u(t)$ reflects the state of process u at time t. The third example shows that by constructing the function space and choosing the appropriate function g, we can use the abstract results.

Example 1. *Let $q \in (0,1)$, $\kappa > 0$, $\Omega = [0,\pi]$, $T > 0$, $\mathcal{K} \in L^2(\Omega \times \Omega, \mathbb{R})$ with*

$$\|\mathcal{K}(.,.)\|_{L^2(\Omega^2,\mathbb{R})} < \frac{1}{(m+1)}.$$

We consider the problem of finding a number $b \in [a,T)$ and a function $w(x,t)$ such that

$$\begin{cases} {}^C_H\partial^q_t w(x,t) + \kappa(-\Delta)^{\sigma_1}\, {}^C_H\partial^q_t w(x,t) = (-\Delta)^{\sigma_2} w(x,t) + F(t, w(x,t)), \\ \qquad\qquad\qquad\qquad \text{on } [0,\pi] \times [b,T); \\ w(x,T) + \displaystyle\sum_{j=0}^{m} \int_\Omega \mathcal{K}(x,v) w(v,t_j,) dv = \xi_T(x), \qquad x \in \Omega, \end{cases} \tag{23}$$

where $0 < \sigma_2 < \sigma_1 < 1$, $t_j = b + \frac{j(T-b)}{m+1}$ $j = 0,1,...,m$, ${}^C_H\partial^q_t u$ is a fractional Caputo-Hadamard derivative of u with order q, $(-\Delta)^\sigma$ is a fractional Laplace operator of order $\sigma \in (0,1)$ defined by

$$(-\Delta)^\sigma u(x) = \frac{\sigma}{\Gamma(1-\sigma)} \int_0^\infty t^{-(1+\sigma)}(u(x) - v(x,t))dt,$$

where $v(t,x)$ is a solution of equations

$$\frac{\partial}{\partial t} v - \Delta v = 0 \text{ on } [0,\infty) \times \mathbb{R}, \text{ and } v(0,x) = u(x);$$

and Δ is the Laplace operator (see [22, 21] and references therein).

Denote $u(t)(x) := w(x,t)$, $X = L^2(\Omega,\mathbb{R})$, $\xi_T \in X$. Then X is a Banach space with the norm

$$|\omega| = \left(\int_\Omega |\omega(x)|^2 dx\right)^{\frac{1}{2}}$$

and the operator $\mathcal{B}_{\sigma_1} := I + \kappa(-\Delta)^{\sigma_1}$ has the inverse operator, which is denoted by $\mathcal{B}^{-1}_{\sigma_1}$. The problem (23) can be rewritten as the following equation in X:

$$\begin{cases} {}^C_H\mathcal{D}^q_{a^+} u(t) = \mathcal{L}u(t) + \mathcal{F}(t,u(t)), \ \ t \in [a,T) \text{ on } \Omega; \\ u(T) = -g(u) + \xi_T \qquad\qquad x \in \Omega, \end{cases} \tag{24}$$

where $\mathcal{L} := \mathcal{B}^{-1}_{\sigma_1}(-\Delta)^{\sigma_2}$, $\mathcal{F}(t,u(t)) := \mathcal{B}^{-1}_{\sigma_1} F(t,u(t))$ and

$$g(u) = \sum_{j=0}^{m} \mathcal{A}u(t_j),\ \mathcal{A}v(.) = \int_\Omega \mathcal{K}(.,w)v(w)dw \ \ \text{for } v \in X.$$

Let $(\lambda, \phi_\lambda) \in (0, \infty) \times E$ *be an eigen-pair of* $(-\Delta)$, *(i.e.,* $(-\Delta)\phi_\lambda = \lambda\phi_\lambda$*). Then* $((1+\kappa\lambda^{\sigma_1})^{-1}, \phi_\lambda)$ *is an eigen-pair of* $\mathcal{B}_{\sigma_1}^{-1}$*. Let* $\{e_n\}_{n\geq 1}$ *be an orthonormal basis of* $L^2(\Omega, \mathbb{R})$ *and* $\{\lambda_n\}_{n\geq 1}$ *be a sequence of the eigenvalues of* $(-\Delta)$ *corresponding to* $\{e_n\}_{n\geq 1}$ *with* $0 < \lambda_1 < \lambda_2, ...,$ $\lim \lambda_n = \infty$*. Furthermore, by representing*

$$v = \sum_{j=1}^{\infty} \langle u, e_n \rangle e_n \quad \textit{for } v \in L^2(\Omega, \mathbb{R}),$$

we get

$$\mathcal{B}_{\sigma_1}^{-1}(-\Delta)^{\sigma_2} v = \sum_{j=1}^{\infty} \frac{\lambda_j^{\sigma_2}}{1+\kappa\lambda_j^{\sigma_1}} \langle v, e_n \rangle e_n.$$

This implies that there exists $C_* > 0$ *such that*

$$|\mathcal{B}_{\sigma_1}^{-1}(-\Delta)^{\sigma_2} v| \leq C_* |v| \quad \textit{for all } v \in L^2(\Omega, \mathbb{R}).$$

Hence, $\mathcal{L}$ *is bounded.*

To illustrate the problem, we consider specific functions below. Let $q = \frac{1}{3}$, $\gamma = -\frac{1}{4}$*. Denote* $c_0 = \frac{\|\mathcal{L}\|_\dagger}{a^q \Gamma(q+1)}$, $c_2 \frac{\|\beta\|_{L^{1,\gamma}(J,X)}}{a^{q+\gamma}} \frac{\Gamma(\gamma+1)}{\Gamma(q+\gamma+1)}$, $K_g = \pi(m+1)\|\mathcal{K}(.,.)\|_{C(\Omega^2,\mathbb{R})}$*. Since* $c_g := K_g < 1$*, there exists* $b \in [a, T)$ *such that*

$$c_g + c_0(T-b)^q + c_2(T-b)^{q+\gamma} < 1.$$

Denote

$$F(t, u(t))(x) = \left(\ln \frac{T}{t}\right)^{-\frac{1}{4}} (|\sin(u(t)(x))| + 1), \;\; x \in [0, \pi];$$

$$\beta(t) = \left(\ln \frac{T}{t}\right)^{-\frac{1}{4}}, \; t \in [a, T).$$

- *Let* $u, v \in C(J, X)$*. Since* $b = t_0 < t_1 < ... < t_m < T$*, we have*

$$\begin{aligned} |g(u)(x) - g(v)(x)| &= \left| \sum_{j=0}^{m} \int_\Omega K(x,y) u(t_j)(y) dy - \sum_{j=0}^{m} \int_\Omega \mathcal{K}(x,y) v(t_j)(y) dy \right| \\ &\leq \sum_{j=0}^{m} \int_\Omega |\ell(x,y)| |u(t_j)(y) - v(t_j)(y)| dy \end{aligned}$$

$$\leq \sum_{j=0}^{m} \left(\int_{\Omega} |\mathcal{K}(x,y)|^2 dy \right)^{1/2} . \left(\int_{\Omega} |u(t_j)(y) - v(t_j)(y)|^2 dy \right)^{1/2}$$
$$\leq \sum_{j=1}^{m} \|\mathcal{K}(x,.)\|_{L^1(\Omega,\mathbb{R})}.|u(t_j) - v(t_j)|$$
$$\leq \sum_{j=1}^{m} \|\mathcal{K}(x,.)\|_{L^1(\Omega,\mathbb{R})}.\|u - v\|_{C(J_b,X)}$$
$$\leq (m+1)\|\mathcal{K}\|_{L^2(\Omega^2,\mathbb{R})}\|u - v\|_{C(J_b,X)}.$$

It follows

$$|g(u) - g(v)| \leq (m+1)K_g\|u - v\|_{C(J_b,X)},$$

where $K_g = \|\mathcal{K}\|_{L^2(\Omega^2,\mathbb{R})}$*. Thus the condition (g) is satisfied with* $c_g := K_g$*.*

• *It is simple to demonstrate that the function* $\mathcal{F}$ *satisfies the condition (i) of* $(\mathcal{F})$*.*

• *Finally, we verify the assumption (ii) on the function* $(\mathcal{F})$*. Since* $\mathcal{B}_{\sigma_1}^{-1}$ *is bounded, we get*

$$|\mathcal{B}^{-1}F(t, u(t))| \leq C^{\dagger}|F(t, u(t))|.$$

For $t \in [b, T)$*, we get*

$$|\mathcal{F}(t, u(t))| \leq C^{\dagger}|F(t, u(t))|$$
$$\leq 2C^{\dagger}\beta(t)(1 + \|u\|_{C(J_b,X)}).$$

Then $2C^{\dagger}\beta \in L^{1,\gamma}(J_b, \mathbb{R}_+)$*. The conditions* $(\mathcal{F})$*-(g) hold. Using Theorem 2 (with a replaces by b), we see that the problem* (24) *has at least one solution in* $C(J_b, X)$*.*

Example 2. *Consider the problem of finding* $w : [a, T] \times [0, \pi] \to \mathbb{R}$ *that satisfies*

$$\begin{cases} {}_{H}^{C}D_{T^-}^{q} w(t,x) = \dfrac{\phi(w(t,x))}{1+(T-t)^{\alpha}}, & t \in J := [a,T); \\ w(T,x) + \displaystyle\sum_{j=0}^{m} \int_0^{\pi} \ell(x,y) w(t_j,y) dy = \xi_T(x), & x \in [0,\pi], \end{cases} \tag{25}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ *is a Lipschitz function with the coefficient* k_ϕ *(to be determined later);* $\ell(.,.) \in L^2([0,\pi] \times [0,\pi], \mathbb{R})$*,* $0 < a \leq t_0 < ... < t_m < T$*,* $m \in \mathbb{N}$*,* $q \in (0,1)$*,* $\alpha > 0$*, and* $\xi_T \in L^2(\Omega, \mathbb{R})$*.*

Denote $\Omega = [0, \pi]$*. We define the functional space* $X = L^2(\Omega, \mathbb{R})$ *with norm* $|\ell| = \left(\int_\Omega |\ell(x)|^2 dx\right)^{1/2}$*. We define the function* $u \in C(J, X)$*,* $u : J \to X$*, by*

$$u(t)(x) = w(t, x) \ \ (t, x) \in J \times \Omega.$$

The source functions $f_k : J \times E \to E$, $f : J \times X \to X$ *are defined by*

$$f(t,v)(x) = \frac{\phi(v(x))}{1 + (T-t)^\alpha}$$

for $(t,v) \in J \times X$.

We define $g : C(J,X) \to X$ *by*

$$g(u) = \sum_{j=0}^{m} \mathcal{A}u(t_j), \quad \text{where } \mathcal{A}v(x) = \int_\Omega \mathcal{K}(x,\tau)v(\tau)d\tau.$$

Then the problem (25) *is rewritten in the form* (1).

- *For* $u, v \in X$, *we have*

$$\begin{aligned}|f(t,u) - f(t,v)| &= \frac{1}{1+(T-t)^\alpha}\left(\int_\Omega |\phi(u(x)) - \phi(v(x))|^2 dx\right)^{1/2} \\ &\le k_\phi |u - v|.\end{aligned}$$

Therefore, f *satisfies the condition (i) of Theorem 3 with* $L = k_\phi$.

- *Check the assumption (ii). For* $u, v \in C(J,X)$, *we have*

$$\begin{aligned}|g(u)(x) - g(v)(x)| &= \left|\sum_{j=0}^{m}\int_\Omega \ell(x,y)u(t_j)(y)dy - \sum_{j=0}^{m}\int_\Omega \ell(x,y)v(t_j)(y)dy\right| \\ &\le \sum_{j=0}^{m}\int_\Omega |\ell(x,y)||u(t_j)(y) - v(t_j)(y)|dy \\ &\le \sum_{j=0}^{m}\left(\int_\Omega |\ell(x,y)|^2 dy\right)^{1/2} . \left(\int_\Omega |u(t_j)(y) - v(t_j)(y)|^2 dy\right)^{1/2} \\ &\le (m+1)\|\ell\|_{L^2(\Omega^2,\mathbb{R})}\|u - v\|.\end{aligned}$$

This means

$$|g(u) - g(v)| \le (m+1)K_g\|u - v\|,$$

where $K_g = \|\ell\|_{L^2(\Omega^2,\mathbb{R})}$. *We can choose* ℓ *and* ϕ *satisfying*

$$\begin{aligned}&\|\ell\|_{L^2(\Omega^2,\mathbb{R})} = \left(\int_{\Omega^2} |\mathcal{K}(x,y)|^2 dxdy\right)^{\frac{1}{2}} < \frac{1}{m+1} \quad \text{and} \\ &K_g + \frac{k_\phi (T-a)^{q-\alpha}}{a^q \Gamma(q+1)} < 1.\end{aligned}$$

Then all assumptions of Theorem 3 are fulfilled with $c_g := K_g$, *so the equation* (25) *has a unique solution.*

Example 3. *Let $T > a > 0$. Consider the following infinite system of fractional equations:*

$$ {}^{C}_{H}\partial^{\frac{2}{3}}_{T-} w_n(t) = \frac{\sin(w_n(t)) + 1}{n(T-t)^{\frac{1}{3}}} \quad \textit{for } t \in [a, T) \tag{26} $$

$$ w_n(T) = \frac{T}{n} \quad \textit{for all } n = 1, 2, ..., \tag{27} $$

where ${}^{C}_{H}\partial^{q}_{T-}$ is a fractional Caputo-Hadamard derivative of order q. We define the Banach space

$$ X = \left\{ x = (x_1, x_2, ...) \in \mathbb{R}^{\infty} : \lim_{n\to\infty} x_n = 0 \right\} \textit{ with the norm } |x| = \sup_{n\geq 1} |x_n|. $$

For $n \geq 1$, $u_n \in C(J, \mathbb{R})$, $u \in C(J, X)$, $f_n : J \times \mathbb{R} \to \mathbb{R}$ and $f : J \times X \to X$ we define

$$ \begin{aligned} u(t) &= (u_1(t), u_2(t), ...); \\ f_n(t, x) &= \frac{\sin(x) + 1}{n(T-t)^{\frac{1}{3}}}; \\ f(t, w) &= (f_1(t, w_1), f_2(t, w_2), ...). \end{aligned} $$

For $u \in C(J, X)$, we define $g(u) = 0$ (zero of X), and $\xi_T = \left(\frac{T}{1}, \frac{T}{2}, ...\right) \in X$.

Then the system of equations (26)-(27) is equivalent to fractional equation (1). Since

$$ \begin{aligned} |f_n(t, u_n(t))| &\leq \frac{|\sin(u_n(t))| + 1}{n(T-t)^{\frac{1}{3}}} \\ &\leq \frac{|u_n(t)| + 1}{n(T-t)^{\frac{1}{3}}} \\ &\leq \frac{(\|u\|_{C(J_c,X)} + 1)}{n(T-t)^{\frac{1}{3}}} \\ &\leq \frac{2}{n(T-t)^{\frac{1}{3}}} \to 0, \text{ as } n \to \infty \text{ for all } t \in J_c, \ c \in [a, T), \end{aligned} $$

$f(t, u(t)) \in X$ and

$$ \begin{aligned} |f(t, u(t))| &= \sup_{n\geq 1} |f_n(t, u_n(t))| \\ &\leq \frac{2(\|u\|_{C(J_c,X)} + 1)}{(T-t)^{\frac{1}{3}}} \quad \text{for all } t \in J_c, \ c \in [a, T). \end{aligned} $$

Hence, the conditions (f) and (g) are satisfied with $q = \frac{2}{3}$, $p_1 = \frac{1}{2}$, $\beta_1(t) = \frac{2}{(T-t)^{\frac{1}{3}}}$.

We can choose $c \in [a, T)$ such that

$$c_g + \frac{(1-p_1)^{1-p_1}}{\Gamma(q)a^q(q-p_1)^{1-p_1}} \|\beta_1\|_{L^{\frac{1}{p_1}}(J,X)} (T-c)^{q-p_1} < 1.$$

The system (26)-(27) has a local solution on $[c, T]$ thanks to Theorem 1.

5. Conclusion

The fundamental issue with fractional operators and their generalized versions is to correctly define them in the appropriate function space. In this paper, we define general function spaces by connecting Lebesgue integrals and Bochner integrals. In these spaces, the continuity of the absolute solution as well as its derivative is not required. We get the existence of global solutions of differential equations with fractional order; however, these solutions do not have good properties such as uniqueness. The problems are studied under assumptions that are not too strict.

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