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On 2-F-Normed Spaces

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Abstract. In this paper, we construct a 2-F-normed space. The space is not only a generalization of a 2-normed space, but it is also closely related to an F-normed space, in the sense that for any given 2-F-normed space, we can derive an appropriate F-normed space. Furthermore, we can prove some properties of a 2-F-normed space through those of its derived F-normed space.

Key Words and Phrases: 2-norm, F-norm, convergent sequence, sequence spaces.

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1. Introduction

The concept of metric was first introduced by Fréchet in 1906 as a generalization of the distance between two points on the real line $\mathbb{R}$ [15]. Metric is a highly important and fundamental concept in geometry, analysis, and other fields. It is a non-negative real-valued function that defines the distance between any two elements in any set. Geometrically, the distance between two points represents the "length" of the line segment connecting those two points. Meanwhile, when three points are considered, it can be imagined that a triangle is formed. Based on this observation, some researchers extended the concept of metric or distance function to an area function, commonly known as a generalized metric. One particular type of generalized metric that has been widely studied is a 2-metric. The concept of a 2-metric was first introduced by Gähler [6] in 1963, motivated by the problem of a triangle area in $\mathbb{R}^3$.

The metric is an abstraction of the distance between any two points x and y in $\mathbb{R}$ which is defined by $|x - y|$. Banach et al. (1922) introduced the concept of a norm as a generalization of the absolute value. A linear space equipped with a norm is called a normed space. The notion of norms goes hand in hand with

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metric in the Euclidean space. In a normed space, we can define a metric by using the norm. In 1964, Gähler [7] introduced the idea of a 2-norm that induces a 2-metric. Let X be a real vector space of dimension d, where $2 \leq d \leq \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following four conditions:

1. $\|x, y\| = 0$ if and only if x and y are linearly dependent.
2. $\|x, y\| = \|y, x\|$, for all $x, y \in X$.
3. $\|\alpha x, y\| = |\alpha|\|x, y\|$, for any $\alpha \in \mathbb{R}$ and for all $x, y \in X$.
4. $\|x + y, z\| \leq \|x, z\| + \|y, z\|$, for all $x, y, z \in X$.

A linear space equipped by 2-norm is called a 2-normed space. The 2-norm of two vectors can be interpreted as the area of a parallelogram spanned by the two vectors.

In the late 1960s, Gähler also extended the concept to n-norm for $n \geq 2$. The Gähler's idea of n-norm, $n \geq 2$, inspired many mathematicians to explore some aspects related to normed spaces of dimension n (see, for instance, [10, 11, 8, 4, 19, 2, 1, 17]).

One of the conditions that must be satisfied by a norm is the absolute homogeneous property. Fréchet weakened this absolute homogeneous property, resulting in a new function called the Fréchet norm (or abbreviated as F-norm). Some researchers use the term "paranorm" for F-norm. Let X be a real vector space. A function $\|\cdot\|_F : X \to \mathbb{R}$ is called F-norm on X if

1. $\|x\|_F = 0$ if and only if $x = \theta$.
2. $\|-x\|_F = \|x\|_F$, for all $x \in X$.
3. $\|x + y\|_F \leq \|x\|_F + \|y\|_F$, for all $x, y \in X$.
4. If (α_n) is a sequence of scalars with $\alpha_n \to \alpha$ as $n \to \infty$ for some $\alpha \in \mathbb{R}$ and (x_n) is a sequence in X with $\|x_n - x\|_F \to 0$ as $n \to \infty$ for some $x \in X$, then $\|\alpha_n x_n - \alpha x\|_F \to 0$ as $n \to \infty$.

A linear space equipped by F-norm is called an F-normed space. A complete F-normed space is further called an F-space (Fréchet space), a term introduced by Banach in honor of M. Maurice Fréchet. The F-spaces have been studied by several mathematicians, including [12, 13, 21, 20, 18, 16, 5, 3]. The F-spaces play a significant role in the theory of sequence spaces, summability theory, and the characterization of matrix mappings between sequence spaces.

Following the notion of 2-norm and F-norm, in this paper we introduce a new norm that we call 2-F-norm. We prove that a 2-F-norm is truly a generalization of a 2-norm. Gunawan and Mashadi [8] have derived an $(n-1)$-norm from any given n-norm with $n \geq 2$ and state that every n-normed space is a normed space. Inspired by ideas in [8], we construct some F-norms from any given 2-F-norm and we prove that every 2-F-normed space is an F-normed space. Furthermore, we also prove the characterization of the convergence of sequences on the 2-F-normed space.

2. Main Results

We begin this section by the definition of a 2-F-norm. The definition is motivated by the definition of 2-norm and F-norm.

Definition 1. *Let X be a real vector space of dimension d, where $2 \leq d \leq \infty$. A 2-F-norm on X is a function $\|\cdot,\cdot\|_F : X \times X \to \mathbb{R}$ which satisfies the following five conditions:*

1. *$\|x, y\|_F = 0$ if and only if x and y are linearly dependent.*
2. *$\|x, y\|_F = \|y, x\|_F$, for all $x, y \in X$.*
3. *$\|-x, y\|_F = \|x, y\|_F$, for all $x, y \in X$.*
4. *$\|x + y, z\|_F \leq \|x, z\|_F + \|y, z\|_F$, for all $x, y, z \in X$.*
5. *If (α_n) is a sequence of scalars with $\alpha_n \to \alpha$ as $n \to \infty$ for some $\alpha \in \mathbb{R}$ and (x_n) is a sequence in X with $\|x_n - x, y\|_F \to 0$ as $n \to \infty$ for every $y \in X$, for some $x \in X$, then $\|\alpha_n x_n - \alpha x, y\|_F \to 0$ as $n \to \infty$ for every $y \in X$.*

A linear space equipped by 2-F-norm is called a 2-F-normed space. Gunawan [9] has shown an example of a 2-normed space ℓ_p. The space $\ell_p, 1 \leq p < \infty$, containing all sequences of real numbers $x = (x_k)$ for which $\sum\limits_{k=1}^{\infty} |x_k|^p < \infty$, is a 2-normed space with respect to 2-norm $\|\cdot,\cdot\|_p$ with the following definition:

$$\|x, y\|_p = \left[\frac{1}{2}\sum_j \sum_k \left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|^p\right]^{\frac{1}{p}},$$

for every $x, y \in \ell_p$. Inspired by the results in [9], we will give an example of a 2-F-norm in the space ℓ_1, containing all sequences of real numbers $x = (x_k)$ for which $\sum\limits_{k=1}^{\infty} |x_k| < \infty$.

Example 1. *A function* $\|\cdot,\cdot\|_F^1 : \ell_1 \times \ell_1 \to \mathbb{R}$ *defined as*

$$\|x,y\|_F^1 = \sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right|\right\}, \tag{1}$$

for all $x, y \in \ell_1$ *is a 2-*F*-norm in* ℓ_1.

Proof.

1. Let $x, y \in \ell_1$ be such that x and y are linearly dependent. We consider two cases. If $y = \theta$, then $\|x,y\|_F^1 = 0$. If $y \neq \theta$, then $x = \delta y$, for a scalar δ. Therefore,
$$\|x,y\|_F^1 = \sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} \delta y_j & \delta y_k \\ y_j & y_k\end{pmatrix}\right|\right\} = 0.$$
Conversely, if $\|x,y\|_F^1 = 0$, then $\sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right|\right\} = 0$.
This implies that for every $j \geq 1$, we get $\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right| = 0$.
Therefore, for every $j \geq 1, k \geq 1$, we have $\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right| = 0$.
So we conclude that x and y are linearly dependent.

2. For each $x, y \in \ell_1$, by properties of determinants, it is clear that
$$\|x,y\|_F^1 = \|y,x\|_F^1.$$

3. For each $x, y \in \ell_1$, observe that
$$\begin{aligned}\|-x,y\|_F^1 &= \sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} -x_j & -x_k \\ y_j & y_k\end{pmatrix}\right|\right\} \\ &= \sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}|-1|\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right|\right\} \\ &= \|x,y\|_F^1.\end{aligned}$$

4. For any $x, y, z \in \ell_1$, we have
$$\|x+z,y\|_F^1 = \sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} x_j + z_j & x_k + z_k \\ y_j & y_k\end{pmatrix}\right|\right\}$$

$$= \sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix} + \det\begin{pmatrix} z_j & z_k \\ y_j & y_k\end{pmatrix}\right|\right\}$$
$$\leq \sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left\{\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right| + \left|\det\begin{pmatrix} z_j & z_k \\ y_j & y_k\end{pmatrix}\right|\right\}\right\}$$
$$\leq \sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right|\right\} + \sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} z_j & z_k \\ y_j & y_k\end{pmatrix}\right|\right\}$$
$$= \|x,y\|_F^1 + \|z,y\|_F^1.$$

5. Let (α_n) be a sequence of scalars with $\alpha_n \to \alpha$, as $n \to \infty$. Then (α_n) is bounded, which means that there exists $M > 0$ such that $|\alpha_n| \leq M$, for any $n \in \mathbb{N}$. Observe that for every $y \in \ell_1$ and for any $n \in \mathbb{N}$, we have

$$\|\alpha_n x^n - \alpha x, y\|_F^1 = \sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} \alpha_n x_j^n - \alpha x_j & \alpha_n x_k^n - \alpha x_k \\ y_j & y_k\end{pmatrix}\right|\right\}$$
$$= \sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} \alpha_n(x_j^n - x_j) + (\alpha_n-\alpha)x_j & \alpha_n(x_k^n - x_k) + (\alpha_n-\alpha)x_k \\ y_j & y_k\end{pmatrix}\right|\right\}$$
$$\leq \sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} \alpha_n(x_j^n - x_j) & \alpha_n(x_k^n - x_k) \\ y_j & y_k\end{pmatrix}\right|\right\}$$
$$+ \sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} (\alpha_n-\alpha)x_j & (\alpha_n-\alpha)x_k \\ y_j & y_k\end{pmatrix}\right|\right\}$$
$$= |\alpha_n|\sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} x_j^n - x_j & x_k^n - x_k \\ y_j & y_k\end{pmatrix}\right|\right\}$$
$$+ |\alpha_n-\alpha|\sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right|\right\}$$
$$= |\alpha_n|\|x^n - x, y\|_F^1 + |\alpha_n - \alpha|\|x,y\|_F^1$$
$$\leq M\|x^n - x, y\|_F^1 + |\alpha_n - \alpha|\|x,y\|_F^1.$$

Because of $\|x_n - x, y\|_F^1 \to 0$ and $|\alpha_n - \alpha| \to 0$, as $n \to \infty$, we have

$$\|\alpha_n x_n - \alpha x, y\|_F^1 \to 0,$$

as $n \to \infty$, for every $y \in \ell_1$.

Hence, $\|\cdot,\cdot\|_F^1$ is a 2-F-norm on ℓ_1. ◄

It is well known that every norm is an F-norm. Similarly, we prove that every 2-norm is a 2-F-norm.

Theorem 1. *Let X be a real vector space of dimension $d \geq 2$. Every 2-norm on X is a 2-F-norm on X.*

Proof. Let $\|\cdot,\cdot\|$ be a 2-norm on X. Then function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ satisfies the following properties:

B1 $\|x, y\| = 0$ if and only if x and y are linearly dependent.

B2 $\|x, y\| = \|y, x\|$, for all $x, y \in X$.

B3 $\|\alpha x, y\| = |\alpha|\|x, y\|$, for all $x, y \in X$ and $\alpha \in \mathbb{R}$.

B4 $\|x + y, z\| \leq \|x, z\| + \|y, z\|$, for all $x, y, z \in X$.

It is clear that $\|\cdot,\cdot\|$ satisfies the four properties of 2-F-norm. It now remains to verify the fifth. Let (α_n) be a sequence of scalars with $\alpha_n \to \alpha$, as $n \to \infty$. Then (α_n) is bounded, which means that there exists $M > 0$ such that $|\alpha_n| \leq M$, for any $n \in \mathbb{N}$. Observe that for every $y \in \ell_1$ and for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|\alpha_n x_n - \alpha x, y\| &= \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x, y\| \\
&\leq \|\alpha_n x_n - \alpha_n x, y\| + \|\alpha_n x - \alpha x, y\| \\
&= |\alpha_n|\|x_n - x, y\| + |\alpha_n - \alpha|\|x, y\| \\
&\leq M\|x_n - x, y\| + |\alpha_n - \alpha|\|x, y\|.
\end{aligned}$$

Because of $\|x_n - x, y\|_F^1 \to 0$ and $|\alpha_n - \alpha| \to 0$, as $n \to \infty$, for every $y \in X$ we have

$$\|\alpha_n x_n - \alpha x, y\|_F^1 \to 0, , \quad \text{as} \quad n \to \infty.$$

Hence, $\|\cdot,\cdot\|$ is a 2-F-norm. ◄

Based on Theorem 1, it is concluded that every 2-normed space is a 2-F-normed space. However, the converse is not always true. Below is a counter example.

Example 2. *For every $x, y \in \ell_1$, we define a function $\|\cdot,\cdot\|_F^* : \ell_1 \times \ell_1 \to \mathbb{R}$ by*

$$\|x, y\|_F^* = \left(\sup_{j \geq 1} \sum_{k=1}^{\infty} \left| \det \begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} \right| \right)^{\frac{1}{2}}. \tag{2}$$

The function $\|\cdot,\cdot\|_F^*$ is a 2-F-norm on ℓ_1.

Proof.

1. Let $x, y \in \ell_1$ be such that x and y are linearly dependent. We consider two cases. If $y = \theta$, then $\|x, y\|_F^* = 0$. If $y \neq \theta$, then $x = \delta y$, for a scalar δ. Therefore,
$$\|x, y\|_F^* = \left(\sup_{j \geq 1} \sum_{k=1}^{\infty} \left| \det \begin{pmatrix} \delta y_j & \delta y_k \\ y_j & y_k \end{pmatrix} \right| \right)^{\frac{1}{2}} = 0.$$
Conversely, if $\|x, y\|_F^* = 0$, then $\sup\limits_{j \geq 1} \sum\limits_{k=1}^{\infty} \left| \det \begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} \right| = 0$.
This implies that for every $j \geq 1$, we get $\sum\limits_{k=1}^{\infty} \left| \det \begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} \right| = 0$.
Therefore, for every $j \geq 1, k \geq 1$, we have $\left| \det \begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} \right| = 0$.
So we conclude that x and y are linearly dependent.

2. For each $x, y \in \ell_1$, by properties of determinants, it is clear that
$$\|x, y\|_F^* = \|y, x\|_F^*.$$

3. For each $x, y \in \ell_1$, observe that
$$\begin{aligned} \|-x, y\|_F^* &= \left(\sup_{j \geq 1} \sum_{k=1}^{\infty} \left| \det \begin{pmatrix} -x_j & -x_k \\ y_j & y_k \end{pmatrix} \right| \right)^{\frac{1}{2}} \\ &= \left(\sup_{j \geq 1} \sum_{k=1}^{\infty} |-1| \left| \det \begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} \right| \right)^{\frac{1}{2}} \\ &= \|x, y\|_F^*. \end{aligned}$$

4. For any $x, y, z \in \ell_1$, we have
$$\begin{aligned} \|x + y, z\|_F^* &= \left(\sup_{j \geq 1} \sum_{k=1}^{\infty} \left| \det \begin{pmatrix} x_j + y_j & x_k + y_k \\ z_j & z_k \end{pmatrix} \right| \right)^{\frac{1}{2}} \\ &= \left(\sup_{j \geq 1} \sum_{k=1}^{\infty} \left| \det \begin{pmatrix} x_j & x_k \\ z_j & z_k \end{pmatrix} + \det \begin{pmatrix} y_j & y_k \\ z_j & z_k \end{pmatrix} \right| \right)^{\frac{1}{2}} \\ &\leq \left(\sup_{j \geq 1} \sum_{k=1}^{\infty} \left| \det \begin{pmatrix} x_j & x_k \\ z_j & z_k \end{pmatrix} \right| + \left| \det \begin{pmatrix} y_j & y_k \\ z_j & z_k \end{pmatrix} \right| \right)^{\frac{1}{2}} \end{aligned}$$

$$\leq \left(\sup_{j\geq 1}\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} x_j & x_k \\ z_j & z_k\end{pmatrix}\right|\right)^{\frac{1}{2}} + \left(\sup_{j\geq 1}\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} y_j & y_k \\ z_j & z_k\end{pmatrix}\right|\right)^{\frac{1}{2}}$$
$$= \|x, z\|_F^* + \|y, z\|_F^*.$$

5. Let (α_n) be a sequence of scalars with $\alpha_n \to \alpha$, as $n \to \infty$. Then (α_n) is bounded, which means that there exists $M > 0$ such that $|\alpha_n| \leq M$, for any $n \in \mathbb{N}$. Observe that for every $y \in \ell_1$ and for any $n \in \mathbb{N}$, we have

$$\|t_n x^n - tx, y\|_F^* = \left(\sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} t_n x_j^n - tx_j & t_n x_k^n - tx_k \\ y_j & y_k\end{pmatrix}\right|\right\}\right)^{\frac{1}{2}}$$
$$= \left(\sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} t_n x_j^n - t_n x_j + t_n x_j - tx_j & t_n x_k^n - t_n x_k + t_n x_k - tx_k \\ y_j & y_k\end{pmatrix}\right|\right\}\right)^{\frac{1}{2}}$$
$$\leq \left(\sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} t_n(x_j^n - x_j) & t_n(x_k^n - x_k) \\ y_j & y_k\end{pmatrix}\right|\right\}\right)^{\frac{1}{2}}$$
$$+ \left(\sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} (t_n - t)x_j & (t_n - t)x_k \\ y_j & y_k\end{pmatrix}\right|\right\}\right)^{\frac{1}{2}}$$
$$= |t_n|^{\frac{1}{2}}\left(\sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} x_j^n - x_j & x_k^n - x_k \\ y_j & y_k\end{pmatrix}\right|\right\}\right)^{\frac{1}{2}}$$
$$+ |t_n - t|^{\frac{1}{2}}\left(\sup_{j\geq 1}\left\{\sum_{k=1}^{\infty}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right|\right\}\right)^{\frac{1}{2}}$$
$$= |t_n|^{\frac{1}{2}}\|x^n - x, y\|_F^* + |t_n - t|^{\frac{1}{2}}\|x, y\|_F^*$$
$$\leq M^{\frac{1}{2}}\|x^n - x, y\|_F^* + |t_n - t|^{\frac{1}{2}}\|x, y\|_F^*.$$

Because of $\|x_n - x, y\|_F^1 \to 0$ and $|\alpha_n - \alpha| \to 0$, as $n \to \infty$, we have

$$\|\alpha_n x_n - \alpha x, y\|_F^1 \to 0,$$

as $n \to \infty$, for every $y \in \ell_1$.

Hence, $\|\cdot,\cdot\|_F^1$ is a 2-F-norm on ℓ_1. Next, it will be shown that $\|\cdot,\cdot\|_F^*$ is not a 2-norm in ℓ_1. In particular, it will be shown that $\|\cdot,\cdot\|_F^*$ does not satisfy homogeneous properties. We choose $\alpha = 2$, $x = (1, 0, 0, ...) \in \ell_1$, and $y = (0, 1, 0, 0, ...) \in$

ℓ_1. We see that

$$\|2x, y\|_F^* = \left(\sup_{j\geq 1}\sum_{k=1}^{\infty}\left|\det\begin{pmatrix}2x_j & 2x_k\\ y_j & y_k\end{pmatrix}\right|\right)^{\frac{1}{2}} = \sqrt{2}.$$

Meanwhile,

$$\|x, y\|_F^* = \left(\sup_{j\geq 1}\sum_{k=1}^{\infty}\left|\det\begin{pmatrix}x_j & x_k\\ y_j & y_k\end{pmatrix}\right|\right)^{\frac{1}{2}} = 1.$$

It implies that, $\|2x, y\|_F^* \neq 2\|x, y\|_F^*$. This shows that, $\|\cdot,\cdot\|_F^*$ does not satisfy homogeneous properties. Hence, $\|\cdot,\cdot\|_F^*$ is not a 2-norm in ℓ_1. ◀

Based on the explanation above, 2-F-normed spaces are more general than 2-normed spaces. An $(n-1)$-norm derived from any given n-norm with $n \geq 2$ is given in [8]. Similarly, we construct some F-norms from any given 2-F-norm in following theorems.

Theorem 2. *Let $(X, \|\cdot,\cdot\|_F)$ be a 2-F-normed space and $\{a_1, a_2\}$ be a linearly independent set in X. Then, with respect to $\{a_1, a_2\}$, we define a function $\|\cdot\|_{(\infty,\{a_1,a_2\})} : X \to \mathbb{R}$ by*

$$\|x\|_{(\infty,\{a_1,a_2\})} = \max\{\|x, a_1\|_F, \|x, a_2\|_F\} \tag{3}$$

for all $x \in X$. Function $\|\cdot\|_{(\infty,\{a_1,a_2\})}$ is an F-norm.

Proof.

1. If $x = \theta$, then $\|\theta\|_{(\infty,\{a_1,a_2\})} = 0$. Conversely, if $\|x\|_{(\infty,\{a_1,a_2\})} = 0$, then it will be shown that $x = \theta$. Suppose that $x \neq \theta$.
   Because of $\|x\|_{(\infty,\{a_1,a_2\})} = 0$, we have $\max\{\|x, a_1\|_F, \|x, a_2\|_F\} = 0$, so that $\|x, a_i\|_F = 0, i = 1, 2$. It implies that, x, a_i, are linearly dependent for every $i = 1, 2$. Consequently, there are scalars $\alpha \neq 0, \beta \neq 0$ such that $x = \alpha a_1$ and $x = \beta a_2$. We obtain $a_1 = \dfrac{\beta}{\alpha} a_2$. Hence, a_1, a_2 are linearly dependent. There is a contradiction with the assumption that a_1, a_2 are linearly independent. Thus, we conclude that $x = \theta$.

2. For every $x \in X$ we get

$$\begin{aligned}\|-x\|_{(\infty,\{a_1,a_2\})} &= \max\{\|-x, a_1\|_F, \|-x, a_2\|_F\}\\ &= \max\{\|x, a_1\|_F, \|x, a_2\|_F\}\\ &= \|x\|_{(\infty,\{a_1,a_2\})}.\end{aligned}$$

3. For every $x, y \in X$ we get

$$\begin{aligned}\|x+y\|_{(\infty,\{a_1,a_2\})} &= \max\{\|x+y,a_1\|_F, \|x+y,a_2\|_F\}\\ &\le \max\{\|x,a_1\|_F + \|y,a_1\|_F, \|x,a_2\|_F + \|y,a_2\|_F\}\\ &\le \max\{\|x,a_1\|_F, \|x,a_2\|_F\} + \max\{\|y,a_1\|_F, \|y,a_2\|_F\}\\ &= \|x\|_{(\infty,\{a_1,a_2\})} + \|y\|_{(\infty,\{a_1,a_2\})}.\end{aligned}$$

4. Let (x_n) be any sequence in X such that $\|x_n - x\|_{(\infty,\{a_1,a_2\})} \to 0, n \to \infty$, for some $x \in X$ and (t_n) be a sequence of real numbers that converges to $t \in \mathbb{R}$. Because of $\|x_n - x\|_{(\infty,\{a_1,a_2\})} \to 0, n \to \infty$ and

$$\|x_n - x\|_{(\infty,\{a_1,a_2\})} = \max\{\|x_n - x, a_1\|_F, \|x_n - x, a_2\|_F\},$$

we have $\|x_n - x, a_i\|_F \to 0, n \to \infty, i = 1, 2$. Then

$$\|t_n x_n - tx, a_i\|_F \to 0, n \to \infty, i = 1, 2.$$

It follows that,

$$\|t_n x_n - tx\|_{(\infty,\{a_1,a_2\})} = \max\{\|t_n x_n - tx, a_1\|_F, \|t_n x_n - tx, a_2\|_F\} \to 0.$$

Therefore $\|\cdot\|_{(\infty,\{a_1,a_2\})}$ defines an F-norm on X. ◄

Theorem 3. *Let $(X, \|\cdot,\cdot\|_F)$ be a 2-F-normed space and $\{a_1, a_2\}$ be a linearly independent set in X. Then, with respect to $\{a_1, a_2\}$, we define a function $\|\cdot\|^*_{(p,\{a_1,a_2\})} : X \to \mathbb{R}$ by*

$$\|x\|^*_{(p,\{a_1,a_2\})} = \{(\|x,a_1\|_F)^p + (\|x,a_2\|_F)^p\}^{\frac{1}{p}}, \tag{4}$$

for all $x \in X$, for $1 \le p < \infty$. Function $\|\cdot\|^*_{(p,\{a_1,a_2\})}$ is an F-norm.

Proof.

1. If $x = \theta$, then $\|x\|^*_{(p,\{a_1,a_2\})} = 0$. Conversely, if $\|x\|^*_{(p,\{a_1,a_2\})} = 0$, then it will be shown that $x = \theta$. Suppose that $x \ne \theta$. Because of $\|x\|^*_{(p,\{a_1,a_2\})} = 0$, we have $\{(\|x,a_1\|_F)^p + (\|x,a_2\|_F)^p\}^{\frac{1}{p}} = 0$, so that $\|x, a_i\|_F = 0, i = 1, 2$. Therefore, x, a_i, are linearly dependent for every $i = 1, 2$. Consequently, there are scalars $\alpha \ne 0, \beta \ne 0$ such that $x = \alpha a_1$ and $x = \beta a_2$. We obtain $a_1 = \dfrac{\beta}{\alpha} a_2$. Hence, a_1, a_2 are linearly dependent. There is a contradiction with the assumption that a_1, a_2 are linearly independent. Thus, we conclude that $x = \theta$.

2. For each $x \in X$ we have

$$\begin{aligned} \|-x\|^*_{(p,\{a_1,a_2\})} &= \{(\|-x,a_1\|_F)^p + (\|-x,a_2\|_F)^p\}^{\frac{1}{p}} \\ &= \{(\|x,a_1\|_F)^p + (\|x,a_2\|_F)^p\}^{\frac{1}{p}} \\ &= \|x\|^*_{(p,\{a_1,a_2\})}. \end{aligned}$$

3. For each $x, y \in X$ we have

$$\begin{aligned} \|x+y\|^*_{(p,\{a_1,a_2\})} &= \{(\|x+y,a_1\|_F)^p + (\|x+y,a_2\|_F)^p\}^{\frac{1}{p}} \\ &\leq \{(\|x,a_1\|_F)^p + (\|y,a_1\|_F)^p + (\|x,a_2\|_F)^p + (\|y,a_2\|_F)^p\}^{\frac{1}{p}} \\ &\leq \{(\|x,a_1\|_F)^p + (\|x,a_2\|_F)^p\}^{\frac{1}{p}} + \{(\|y,a_1\|_F)^p + (\|y,a_2\|_F)^p\}^{\frac{1}{p}} \\ &= \|x\|^*_{(p,\{a_1,a_2\})} + \|y\|^*_{(p,\{a_1,a_2\})}. \end{aligned}$$

4. Let (x_n) be any sequence in X such that $\|x_n - x\|_{(\infty,\{a_1,a_2\})} \to 0, n \to \infty$, for some $x \in X$ and (t_n) be a sequence of real numbers that converges to $t \in \mathbb{R}$. Because of $\|x_n - x\|^*_{(p,\{a_1,a_2\})} \to 0, n \to \infty$ and

$$\|x_n - x\|^*_{(p,\{a_1,a_2\})} = \{(\|x_n - x, a_1\|_F)^p + (\|x_n - x, a_2\|_F)^p\}^{\frac{1}{p}},$$

we have $\|x_n - x, a_i\|_F \to 0, n \to \infty, i = 1, 2$. Then

$$\|t_n x_n - tx, a_i\|_F \to 0, n \to \infty, i = 1, 2.$$

It follows that,

$$\|t_n x_n - tx\|^*_{(p,\{a_1,a_2\})} = \{(\|t_n x_n - tx, a_1\|_F)^p + (\|t_n x_n - tx, a_2\|_F)^p\}^{\frac{1}{p}} \to 0.$$

Therefore, $\|\cdot\|^*_{(p,\{a_1,a_2\})}$ for $1 < p < \infty$ defines an F-norm on X. ◄

Further, we will prove that the F-norm $\|\cdot\|_{(\infty,\{a_1,a_2\})}$ is equivalent to the F-norm $\|\cdot\|^*_{(p,\{a_1,a_2\})}$, for $1 \leq p < \infty$.

Theorem 4. *Let $(X, \|\cdot,\cdot\|_F)$ be a 2-F-normed space. For all $1 \leq p < \infty$, there exist scalars $A > 0$ and $B > 0$ such that*

$$A\|x\|_{(\infty,\{a_1,a_2\})} \leq \|x\|^*_{(p,\{a_1,a_2\})} \leq B\|x\|_{(\infty,\{a_1,a_2\})}, \tag{5}$$

for every $x \in X$. Furthermore, for all $1 \leq p, q < \infty$, there exist scalars $C > 0$ and $D > 0$ such that

$$C\|x\|^*_{(p,\{a_1,a_2\})} \leq \|x\|^*_{(q,\{a_1,a_2\})} \leq D\|x\|^*_{(p,\{a_1,a_2\})}, \tag{6}$$

for every $x \in X$.

Proof. For any $x \in X$, we notice that

$$(\|x, a_1\|_F)^p + (\|x, a_2\|_F)^p \leq 2^p \max\{(\|x, a_1\|_F)^p, (\|x, a_2\|_F)^p\}$$

holds for every $1 \leq p < \infty$. It implies that,

$$\begin{aligned}\{(\|x, a_1\|_F)^p + (\|x, a_2\|_F)^p\}^{\frac{1}{p}} &\leq (2^p \max\{(\|x, a_1\|_F)^p, (\|x, a_2\|_F)^p\})^{\frac{1}{p}} \\ &= 2(\max\{(\|x, a_1\|_F)^p, (\|x, a_2\|_F)^p\})^{\frac{1}{p}} \\ &= 2\max\{(\|x, a_1\|_F), (\|x, a_2\|_F)\}.\end{aligned}$$

Then

$$\|x\|^*_{(p,\{a_1,a_2\})} \leq 2\|x\|_{(\infty,\{a_1,a_2\})}.$$

Further, we observe that the relation

$$\begin{aligned}(\|x\|_{(\infty,\{a_1,a_2\})})^p &= (\max\{\|x, a_1\|_F, \|x, a_2\|_F\})^p \\ &= \max\{(\|x, a_1\|_F)^p, (\|x, a_2\|_F)^p\} \\ &\leq (\|x, a_1\|_F)^p + (\|x, a_2\|_F)^p\end{aligned}$$

holds for every $1 \leq p < \infty$. Therefore,

$$\|x\|_{(\infty,\{a_1,a_2\})} \leq \{(\|x, a_1\|_F)^p + (\|x, a_2\|_F)^p\}^{\frac{1}{p}} = \|x\|^*_{(p,\{a_1,a_2\})}.$$

Hence, for every $1 \leq p < \infty$ we get

$$\|x\|_{(\infty,\{a_1,a_2\})} \leq \|x\|^*_{(p,\{a_1,a_2\})} \leq 2\|x\|_{(\infty,\{a_1,a_2\})}.$$

Thus, we obtain that for every $1 \leq p < \infty$, there exist $A = 1 > 0$ and $B = 2 > 0$ such that

$$A\|x\|_{(\infty,\{a_1,a_2\})} \leq \|x\|^*_{(p,\{a_1,a_2\})} \leq B\|x\|_{(\infty,\{a_1,a_2\})}$$

for every $x \in X$. Furthermore, for every $1 \leq q < \infty$ we have

$$\|x\|_{(\infty,\{a_1,a_2\})} \leq \|x\|^*_{(q,\{a_1,a_2\})} \leq 2\|x\|_{(\infty,\{a_1,a_2\})}.$$

Notice that $\|x\|^*_{(p,\{a_1,a_2\})} \leq 2\|x\|_{(\infty,\{a_1,a_2\})}$. It means, $\frac{1}{2}\|x\|^*_{(p,\{a_1,a_2\})} \leq \|x\|_{(\infty,\{a_1,a_2\})}$.
It follows that, $\frac{1}{2}\|x\|^*_{(p,\{a_1,a_2\})} \leq \|x\|^*_{(q,\{a_1,a_2\})}$.
At the same time, we also have

$$\|x\|^*_{(q,\{a_1,a_2\})} \leq 2\|x\|_{(\infty,\{a_1,a_2\})} \leq 2\|x\|^*_{(p,\{a_1,a_2\})}.$$

Hence, we obtain

$$\frac{1}{2}\|x\|^*_{(p,\{a_1,a_2\})} \leq \|x\|^*_{(q,\{a_1,a_2\})} \leq 2\|x\|^*_{(p,\{a_1,a_2\})},$$

for all $x \in X$. This shows that for every $1 \leq p, q < \infty$, there exist $C = \dfrac{1}{2} > 0$ and $D = 2 > 0$ such that

$$C\|x\|^*_{(p,\{a_1,a_2\})} \leq \|x\|^*_{(q,\{a_1,a_2\})} \leq D\|x\|^*_{(p,\{a_1,a_2\})},$$

for every $x \in X$. ◄

Based on Theorem 4, it can be concluded that for every $1 \leq p < \infty$, the two F-norms $\|\cdot\|_{(\infty,\{a_1,a_2\})}$ and $\|\cdot\|^*_{(p,\{a_1,a_2\})}$ are mutually equivalent. Likewise, for every $1 \leq p, q < \infty$, the two F-norms $\|\cdot\|^*_{(p,\{a_1,a_2\})}$ and $\|\cdot\|^*_{(q,\{a_1,a_2\})}$ are mutually equivalent.

The definitions of a convergent sequence, a Cauchy sequence, and the completeness in n-normed space are given in [8]. Likewise, we define a convergent sequence, a Cauchy sequence, and a completeness with respect to 2-F-norm.

Definition 2. *Let $(X, \|\cdot,\cdot\|_F)$ be a 2-F-normed space.*
*A sequence (x_n) in X is said to **converge** to an $x \in X$ if for all $y \in X$,*

$$\|x_n - x, y\|_F \to 0, \quad as \ \ n \to \infty.$$

*A sequence (x_n) in X is called **Cauchy** if for all $y \in X$,*

$$\|x_n - x_m, y\|_F \to 0, \quad as \ \ n \to \infty.$$

*A 2-F-normed space $(X, \|\cdot,\cdot\|_F)$ is said to be **complete** if every Cauchy sequence in $(X, \|\cdot,\cdot\|_F)$ converges to an $x \in X$. A complete 2-F-normed space is then called a **2-F-space**.*

In a normed space and also in a 2-normed space, we know that every convergent sequence is a Cauchy sequence. We can prove that this fact also hold in a 2-F-normed space.

In the following theorem, we give a characterization of the convergence of a sequence in a 2-F-normed space using its derived norm.

Theorem 5. *Let $(X, \|\cdot,\cdot\|_F)$ be a 2-F-normed space of dimension d, where $2 \leq d < \infty$.*
A sequence in X is convergent in 2-F-norm $\|\cdot,\cdot\|_F$ if and only if it is convergent in F-norm $\|\cdot\|_{(\infty,\{a_1,a_2\})}$ with respect to $\{a_1, a_2\}$ for every linearly independent $\{a_1, a_2\}$ in X.

Proof. ($\Longrightarrow$) Let (x_n) be a sequence in X which converges to $x \in X$ in $\|\cdot,\cdot\|_F$. Then, for arbitrary linearly independent set $\{a_1, a_2\}$ in X, we get $\|x_n - x, a_i\|_F \to 0$, as $n \to \infty$, $i = 1, 2$. Because of

$$\|x_n - x\|_{(\infty,\{a_1,a_2\})} = \max\{\|x_n - x, a_1\|_F, \|x_n - x, a_2\|_F\},$$

we have

$$\|x_n - x\|_{(\infty,\{a_1,a_2\})} \to 0,$$

as $n \to \infty$. It follows that (x_n) converges to $x \in X$ in F-norm $\|\cdot\|_{(\infty,\{a_1,a_2\})}$.

($\Longleftarrow$) Let (x_n) be a sequence converging to $x \in X$ with respect to F-norm $\|\cdot\|_{(\infty,\{a_1,a_2\})}$ for every linearly independent $\{a_1, a_2\}$ in X. For any $y \in X$, we will prove that $\|x_n - x, y\|_F \to 0$, as $n \to \infty$.

If $y = \theta$, then it is clear. If $y \neq \theta$, then we can take $z \in X$ which is linearly independent with y. Notice that as $n \to \infty$ we get

$$\|x_n - x, y\|_F \leq \max\{\|x_n - x, y\|_F, \|x_n - x, z\|_F\} = \|x_n - x\|_{(\infty,\{y,z\})} \to 0.$$

Therefore, for any $y \in X$ we obtain $\|x_n - x, y\|_F \to 0$, as $n \to \infty$, that is, (x_n) converges to $x \in X$ with respect to 2-F-norm. ◀

Based on Theorem 5, it can be concluded that the convergence of the sequence on the finite-dimensional 2-F-normed space is equivalent to the convergence of the sequence to the F-norm $\|\cdot\|_{(\infty,\{a_1,a_2\})}$. Analogously, Theorem 5 also holds for Cauchy sequences. Therefore, these properties can be utilized to investigate the completeness of finite-dimensional 2-F-normed spaces. Meanwhile, to show the completeness in the infinite-dimensional 2-F-normed spaces, we will show the completeness in $(\ell_1, \|\cdot,\cdot\|_F^1)$.

Based on Example 1, the space ℓ_1 is a 2-F-normed space for

$$\|x, y\|_F^1 = \sup_{j \geq 1} \left\{ \sum_{k=1}^{\infty} \left| \det \begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} \right| \right\},$$

for every $x, y \in \ell_1$. Further, by Theorem 3, a function $\|\cdot\|^*_{(1,\{a_1,a_2\})} : \ell_1 \to \mathbb{R}$ defined by

$$\|x\|^*_{(1,\{a_1,a_2\})} = \|x, a_1\|_F^1 + \|x, a_2\|_F^1, \tag{7}$$

for all $x \in \ell_1$, where $\{a_1, a_2\}$ is a linearly independent set in ℓ_1, is an F-norm in ℓ_1.

If we take $a_1 = (1, 0, 0, ...) \in \ell_1$ and $a_2 = (0, 1, 0, ...) \in \ell_1$, then

$$\|x, a_1\|_F^1 = \sup_{j \geq 1} \left\{ \sum_{k=1}^{\infty} \left| \det \begin{pmatrix} x_j & x_k \\ a_{1_j} & a_{1_k} \end{pmatrix} \right| \right\} = \sum_{k \in \mathbb{N} - \{1\}} |x_k|$$

and

$$\|x, a_2\|_F^1 = \sup_{j \geq 1} \left\{ \sum_{k=1}^{\infty} \left| \det \begin{pmatrix} x_j & x_k \\ a_{2_j} & a_{2_k} \end{pmatrix} \right| \right\} = \sum_{k \in \mathbb{N} - \{2\}} |x_k|.$$

Consequently

$$\|x\|^*_{(1,\{a_1,a_2\})} = \sum_{k \in \mathbb{N} - \{1\}} |x_k| + \sum_{k \in \mathbb{N} - \{2\}} |x_k| = |x_1| + |x_2| + 2 \sum_{k=3}^{\infty} |x_k|. \tag{8}$$

It is well known that ℓ_1 is complete with respect to F-norm

$$\|x\|_1 = \sum_{k=1}^{\infty} |x_k|. \tag{9}$$

We also note that for all $x \in \ell_1$, we have

$$\|x\|_1 \leq \|x\|^*_{(1,\{a_1,a_2\})} \leq 2\|x\|_1. \tag{10}$$

In other words, $\| \cdot \|^*_{(1,\{a_1,a_2\})}$ is equivalent to $\| \cdot \|_1$. Furthermore, by Theorem 4, $\| \cdot \|_{(\infty,\{a_1,a_2\})}$ is equivalent to $\| \cdot \|_1$. Before we show the completeness of $(\ell_1, \|\cdot,\cdot\|_F^1)$, we need to prove the following theorem.

Theorem 6. *Let $(\ell_1, \|\cdot,\cdot\|_F^1)$ be a 2-F-normed space. For every $x, y \in \ell_1$, we have*

$$\|x, y\|_F^1 \leq 2\|x\|_1 \|y\|_1.$$

Proof. For any $x, y \in \ell_1$, we obtain

$$\begin{aligned}
\|x, y\|_F^1 &= \sup_{j \geq 1} \left\{ \sum_{k=1}^{\infty} \left| \det \begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} \right| \right\} \\
&= \sup_{j \geq 1} \left\{ \sum_{k=1}^{\infty} |x_j y_k - x_k y_j| \right\} \\
&\leq \sup_{j \geq 1} \left\{ \sum_{k=1}^{\infty} \left(|x_j||y_k| + |x_k||y_j| \right) \right\} \\
&\leq \sup_{j \geq 1} \left\{ \sum_{k=1}^{\infty} |x_j||y_k| \right\} + \sup_{j \geq 1} \left\{ \sum_{k=1}^{\infty} |x_k||y_j| \right\} \\
&= \sup_{j \geq 1} |x_j| \sum_{k=1}^{\infty} |y_k| + \sup_{j \geq 1} |y_j| \sum_{k=1}^{\infty} |x_k|
\end{aligned}$$

$$\leq \sum_{j=1}^{\infty} |x_j| \sum_{k=1}^{\infty} |y_k| + \sum_{j=1}^{\infty} |y_j| \sum_{k=1}^{\infty} |x_k|$$
$$= 2\|x\|_1 \|y\|_1.$$

◄

Based on these facts, we will show the completeness properties of $(\ell_1, \|\cdot,\cdot\|_F^1)$.

Theorem 7. *The space ℓ_1 is complete with respect to 2-F-norm $\|\cdot,\cdot\|_F^1$.*

Proof. Let $(x^{(n)})$ be any Cauchy sequence in $(\ell_1, \|\cdot,\cdot\|_F^1)$. Then, for arbitrary linearly independent set $\{a_1, a_2\}$ in X, we get $\|x_n - x, a_i\|_F \to 0$, as $n \to \infty$, $i = 1, 2$. Based on formula (7), we get $\|x^{(n)} - x^{(m)}\|^*_{(1,\{a_1,a_2\})} \to 0$ as $n, m \to \infty$. Because of (10), we have $\|x^{(n)} - x^{(m)}\|_1 \to 0$ as $n, m \to \infty$. Therefore, $(x^{(n)})$ is a Cauchy sequence in $(\ell_1, \|\cdot\|_1)$. Since $(\ell_1, \|\cdot\|_1)$ is complete, $(x^{(n)})$ must converge in $(\ell_1, \|\cdot\|_1)$. Then, by Theorem 6, $(x^{(n)})$ also converges in $(\ell_1, \|\cdot,\cdot\|_F^1)$. Hence, $(\ell_1, \|\cdot,\cdot\|_F^1)$ is complete. ◄

3. Concluding Remarks

In this paper, we have constructed a 2-F-normed space, as a generalization of a 2-normed space. We have also proved some basic and topological properties of this space.

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