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Generalized Hölder Estimates via Generalized Morrey Norms for Kolmogorov Operators

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Abstract. We obtain generalized Hölder estimates for Kolmogorov operators on $\mathbb{R}^3$ by establishing several estimates for singular integrals in generalized Morrey spaces.

Key Words and Phrases: Kolmogorov operator, homogeneous type space, singular integral operators, generalized Morrey space, generalized Hölder estimate.

2010 Mathematics Subject Classifications: 35R03, 35B45, 42B20

1. Introduction and main results

The Kolmogorov equation was first introduced by Kolmogorov in 1934 to study the time evolution of the density of a Brownian test particle in the phase space. It is a linear strongly degenerate second order PDE whose diffusion part is governed by the Laplace operator in a subset of the variables (velocity variables) coupled with a transport term that contains the directions of missing ellipticity (position variables). Such a drift term makes the equation non-symmetric, but at the same time it is responsible for the hypoelliptic properties of the operator.

Let us consider a Kolmogorov operator in $\mathbb{R}^3$:

$$\mathcal{L} = \partial^2_{x_1x_1} + x_1\partial_{x_2} - \partial_t. \tag{1}$$

Kolmogorov in [20] presented an explicit fundamental solution, smooth outside the pole, for the ultraparabolic operator $\mathcal{L}$, which, despite its degeneracy, possesses a fundamental solution Γ smooth outside the pole, this fact implying the hypoellipticity of $\mathcal{L}$. Actually,

$$\mathcal{L}\big((x_1, x_2, t), (y_1, y_2, \tau)\big)$$

$$= \begin{cases} \frac{\sqrt{3}}{2\pi(t-\tau)^2} \exp\Big(- \frac{x_1^2+x_1y_1+y_1^2}{t-\tau} - \frac{3(x_1+y_1)(x_2-y_2)}{(t-\tau)^2} - \frac{3\big(x_2-y_2\big)^2}{(t-\tau)^3}\Big) & \text{for } t > \tau, \\ 0 & \text{for } t \le \tau. \end{cases}$$

This phenomenon is well understood in the framework of the theory of Hörmander operators; actually, this operator can be written as $\mathcal{L}u = X_1^2 u + X_0 u$ with

$$X_1 = \partial_{x_1}, \quad X_0 = -\big(x\partial_{x_2} + \partial_t\big),$$

and since $\big[X_1, X_0\big] = -\partial_t$, we see that $X_1, X_0, \big[X_1, X_0\big]$ span $\mathbb{R}^3$ at every point of the space, hence Hörmander's condition is satisfied. This operator is explicitly quoted as a motivating example in the introduction of Hörmander's paper [19] and, as was shown, is part of a large class of operators of Hörmander type which represent interesting physical models.

It is known that $\mathcal{L}$ is a degenerate operator which appears in many research fields. For instance, the Kolmogorov equation

$$\partial^2_{x_1x_1} u + x_1\partial_{x_2} u - \partial_t u = 0, \quad (x_1, x_2, t) \in \mathbb{R}^3 \tag{2}$$

occurs in the financial problem (see [3, 12]), in the kinetic theory (see [8, 22]) as well as in the visual perception problem (see [26]).

The second order part in (2) is strongly degenerate due to the presence in it of the only term $\partial^2_{x_1x_1}$. However, Kolmogorov constructed in 1934 an explicit fundamental solution of (2) which is a C^∞ function outside the diagonal [24]. This implies that (2) is hypoelliptic, i.e. every distributional solution to (2) in an open subset Ω of $\mathbb{R}^3$, actually is a $C^\infty(\Omega)$ function.

We know that $\mathcal{L}$ is a class of Kolmogorov-Fokker-Planck ultraparabolic operators. Due to its importance in physics and in mathematical finance, it has been extensively studied (see [5, 6, 13, 15, 21, 29, 30]). The authors in [13, 21, 29, 30] proved an invariant Harnack inequality for the non-negative solutions of the equation $\mathcal{L}u = 0$. The local L^p estimates have been studied in [5] and [6]. Based on the theory of singular integral, Polidoro and Ragusa in [31] obtained Morrey-type imbedding results and gave a local Holder continuity of the solution. In this paper, we obtain generalized Hölder estimates for Kolmogorov operators $\mathcal{L}$ on $\mathbb{R}^3$.

Morrey spaces and their properties play an important role in the study of local behavior of solutions to elliptic partial differential equations, refer to [25, 28]. In [1, 4, 9], the authors showed the boundedness in Morrey spaces for some important operators in harmonic analysis such as Hardy-Littlewood operators, Calderón-Zygmund singular integral operators and fractional integral operators. Moreover, various Morrey spaces are defined in the process of study. In [16, 24, 27], the authors introduced and studied the boundedness of the classical operators in generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see, also [2, 17, 18, 32]), etc.

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 1. *(Generalized Morrey space). Let* $1 \le p < \infty$ *and* $\varphi(x,r)$ *be a positive measurable function on* $\mathbb{R}^3 \times (0,\infty)$. *The generalized Morrey space* $M^{p,\varphi}(\mathbb{R}^3)$ *is defined as a set of all functions* $f \in L^p_{\rm loc}(\mathbb{R}^3)$ *equipped with the finite norm*

$$\|f\|_{M^{p,\varphi}(\mathbb{R}^3)} = \sup_{x\in\mathbb{R}^3, r>0} \frac{r^{-\frac{6}{p}}}{\varphi(x,r)} \|f\|_{L^p(B(x,r))}.$$

Also, the weak generalized Morrey space $WM^{p,\varphi}(\mathbb{R}^3)$ *is defined as a set of all functions* $f \in L^p_{\rm loc}(\mathbb{R}^3)$ *equipped with the finite norm*

$$\|f\|_{WM^{p,\varphi}(\mathbb{R}^3)} = \sup_{x\in\mathbb{R}^3, r>0} \frac{r^{-\frac{6}{p}}}{\varphi(x,r)} \|f\|_{WL^p(B(x,r))}.$$

Remark 1. (1) *If* $\varphi(x,r) = r^{\frac{\lambda-4}{p}}$ *with* $0 < \lambda < 4$, *then* $M^{p,\varphi}(\mathbb{R}^3) = L^{p,\lambda}(\mathbb{R}^3)$ *is the classical Morrey space and* $WM^{p,\varphi}(\mathbb{R}^3) = WL^{p,\lambda}(\mathbb{R}^3)$ *is the weak Morrey space.*

(2) *If* $\varphi(x,r) \equiv r^{-\frac{6}{p}}$, *then* $M^{p,\varphi}(\mathbb{R}^3) = L^p(\mathbb{R}^3)$ *is the Lebesgue space and* $WM^{p,\varphi}(\mathbb{R}^3) = WL^p(\mathbb{R}^3)$ *is the weak Lebesgue space.*

Lemma 1. *[14] Let* $\varphi(x,r)$ *be a positive measurable function on* $\mathbb{R}^3 \times (0,\infty)$.

(i) If

$$\sup_{t<r<\infty} \frac{r^{-\frac{6}{p}}}{\varphi(x,r)} = \infty \quad \textit{for some } t>0 \textit{ and for all } x \in \mathbb{R}^3,$$

then $M^{p,\varphi}(\mathbb{R}^3) = \Theta$.

(ii) If

$$\sup_{0<r<\tau} \varphi(x,r)^{-1} = \infty \quad \textit{for some } \tau>0 \textit{ and for all } x \in \mathbb{R}^3,$$

then $M^{p,\varphi}(\mathbb{R}^3) = \Theta$.

Remark 2. *[14] We denote by* Ω_p *the sets of all positive measurable functions* φ *on* $\mathbb{R}^3 \times (0,\infty)$ *such that for all* $r > 0$,

$$\sup_{x\in\mathbb{R}^3} \left\| \frac{r^{-\frac{6}{p}}}{\varphi(x,r)} \right\|_{L^\infty(t,\infty)} < \infty, \quad \textit{and} \quad \sup_{x\in\mathbb{R}^3} \left\| \varphi(x,r)^{-1} \right\|_{L^\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 1, we always assume that $\varphi \in \Omega_p$.

Define

$$[u]_{C^{\omega}(\mathbb{R}^3)} = \sup_{x,z\in\mathbb{R}^3, x\neq z} \frac{|u(x)-u(z)|}{\omega\big(\|x^{-1}\circ z\|\big)},$$

and set $C^{0,\omega}(\mathbb{R}^3)$ for the space with the all functions $u : \mathbb{R}^3 \to \mathbb{R}$ with the finite norm

$$\|u\|_{C^{\omega}(\mathbb{R}^3)} = \|u\|_{L_\infty(\mathbb{R}^3)} + [u]_{C^{\omega}(\mathbb{R}^3)}.$$

In the case $\omega(t) = t^{\alpha}$, $0 < \alpha \le 1$, we get the Hölder spaces $C^{\alpha}(\mathbb{R}^3)$.

The main results in this paper are as follows.

Theorem 1. *Let* $1 < p < \infty$ *and* $\varphi = \varphi(x, r) \in \Omega_p$ *satisfy the condition*

$$\int_0^1 \varphi(x,r)\, r\, dr + \int_1^\infty \varphi(x,r)\, dr < \infty.$$

Then there exists a positive constant C*, depending only on* p*,* φ *and the operator* $\mathcal{L}$*, such that for every* $u \in C_0^\infty(\mathbb{R}^3)$*,*

$$|u(x) - u(z)| \le C\|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^3)} \times \Big(\int_0^{\|x^{-1}\circ z\|} \varphi(x,r)\, r\, dr + \|x^{-1}\circ z\| \int_{\|x^{-1}\circ z\|}^\infty \varphi(x,r)\, dr\Big)$$

for every $x, z \in \mathbb{R}^3$*,* $x \neq z$*, where* $\circ$ *is the group law given in Section* 2*.*

Let $1 < p < \infty$ *and* $\varphi = \varphi(x, r) \in \Omega_p$ *satisfy the condition*

$$\int_0^1 \varphi(x,r)\, dr + \int_1^\infty \varphi(x,r)\, \frac{dr}{r} < \infty.$$

Then there exists a positive constant C*, depending only on* p*,* φ *and the operator* $\mathcal{L}$*, such that for every* $u \in C_0^\infty(\mathbb{R}^3)$*,*

$$|\partial_{x_1}u(x) - \partial_{x_1}u(z)| \le C\|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^3)} \times \Big(\int_0^{\|x^{-1}\circ z\|} \varphi(x,r)\, dr + \|x^{-1}\circ z\| \int_{\|x^{-1}\circ z\|}^\infty \varphi(x,r)\, \frac{dr}{r}\Big)$$

for every $x, y \in \mathbb{R}^3$*,* $x \neq z$*.*

Corollary 1. *Let $1 < p < \infty$ and $\varphi = \varphi(x,r) \in \Omega_p$ satisfy the condition*

$$\int_0^\delta \varphi(x,r)\, r\, dr + \delta \int_\delta^\infty \varphi(x,r)\, dr \lesssim \varphi(x,\delta)\, \delta^2$$

for all x and $\delta > 0$. Then there exists a positive constant C, depending only on p, φ and the operator $\mathcal{L}$, such that for every $u \in C_0^\infty(\mathbb{R}^3)$,

$$|u(x) - u(z)| \leq C \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^3)}\, \varphi(x, \|x^{-1} \circ z\|)\ \|x^{-1} \circ z\|^2$$

for every $x, z \in \mathbb{R}^3$, $x \neq z$, where $\circ$ is the group law given in Section 2. Moreover,

$$\|u\|_{C^{\varphi(\cdot,r)\, r^2}(\mathbb{R}^3)} \lesssim \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^3)}.$$

Let $1 < p < \infty$ and $\varphi = \varphi(x,r) \in \Omega_p$ satisfy the condition

$$\int_0^\delta \varphi(x,r)\, dr + \delta \int_\delta^\infty \varphi(x,r)\, \frac{dr}{r} \lesssim \varphi(x,\delta)\, \delta$$

for all x and $\delta > 0$. Then there exists a positive constant C, depending only on p, φ and the operator $\mathcal{L}$, such that for every $u \in C_0^\infty(\mathbb{R}^3)$,

$$|\partial_{x_j} u(x) - \partial_{x_j} u(z)| \leq C \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^"+1)}\, \varphi(x, \|x^{-1} \circ z\|)\ \|x^{-1} \circ z\|$$

for every $x, z \in \mathbb{R}^3$, $x \neq z$. Moreover,

$$\||\partial_{x_1} u\|_{C^{\varphi(\cdot,r)\, r}(\mathbb{R}^3)} \lesssim \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^3)}.$$

Note that for $\varphi(x,r) = |B(x,r)|^{\frac{\lambda-1}{p}}$, from Theorem 1 we get the following result proven in [10].

Corollary 2. *[10, Theorem 1.2] If $2p + \lambda > 6$, $p + \lambda < 6$ and $\theta = \frac{2p+\lambda-6}{p}$, then there exists a positive constant C, depending only on p, λ and the operator $\mathcal{L}$, such that for every $u \in C_0^\infty(\mathbb{R}^3)$,*

$$|u(x) - u(z)| \leq C \|\mathcal{L}u\|_{L^{p,\lambda}(\mathbb{R}^3)}\ \|x^{-1} \circ x\|^\theta$$

for every $x, z \in \mathbb{R}^3$, $z \neq w$, where $\circ$ is the group law given in Section 2. Moreover,

$$\|u\|_{C^\theta(\mathbb{R}^3)} \lesssim \|\mathcal{L}u\|_{M^{p,\lambda}(\mathbb{R}^3)}.$$

If $p+\lambda > 6$ and $\delta = \frac{p+\lambda-6}{p}$, then there exists a positive constant C, depending only on p, λ and the operator $\mathcal{L}$, such that for every $u \in C_0^\infty(\mathbb{R}^3)$,

$$\left|\partial_{x_1} u(x) - \partial_{x_1} u(z)\right| \leq C \|\mathcal{L}u\|_{L^{p,\lambda}(\mathbb{R}^3)}\ \|x^{-1} \circ z\|^\delta$$

for every $x, z \in \mathbb{R}^3$, $x \neq z$. Moreover,

$$\|\partial_{x_1} u\|_{C^\delta(\mathbb{R}^3)} \lesssim \|\mathcal{L}u\|_{M^{p,\lambda}(\mathbb{R}^3)}.$$

The paper is organized as follows. In Section 2, we introduce some preliminary and known results which will be used later. The proof of Theorem 1 is given in Section 3.

2. Preliminaries

It is proved in [21] that the operator $\mathcal{L}$ is left-invariant with respect to the Lie group $\mathcal{K} = \big(\mathbb{R}^3, \circ\big)$, whose underlying manifold is $\mathbb{R}^3$, endowed with the composition law

$$(x_1, x_2, t) \circ (x_1, x_2, \tau) = (x_1 + x_2, x_2 + y_2 - tx_1, t + \tau).$$

Note that

$$(x_1, x_2, t)^{-1} = (-x_1, -x_2 - tx_1, -t).$$

The left translation by $y = (y_1, y_2, \tau)$ given by

$$(x_1, x_2, t) \to (y_1, y_2, \tau) \circ (x_1, x_2, t),$$

is an invariant translation to the operator $\mathcal{L}$ given by

$$\delta_\lambda = diag\big(t, t^3, t^2\big),$$

where t is a positive parameter, and the homogeneous dimension of $\big(\mathbb{R}^3, \circ\big)$ with respect to the dilation δ_λ is 6.

Remark 3. *There is a natural homogeneous norm in* $\mathbb{R}^3$*, induced by dilation* $D(\lambda)$*:* $\|x\| \equiv \|(x_1, x_2, t)\| = |x_1| + |x_2|^{1/3} + |t|^{1/2}$*. Clearly, we have* $\|\delta_\lambda z\| = \lambda\|z\|,\ \lambda > 0,\ z \in \mathbb{R}^3$.

For every $x, y \in \mathbb{R}^3$, define a quasidistance by $d(x, y) = \|y^{-1} \circ x\|$. The ball with respect to d is denoted by

$$B(x, r) = B_r(x) = \{w \in \mathbb{R}^3 : d(x, y) < r\}. \tag{3}$$

Since $B(0, r) = \delta_r B(0, 1)$ and $\det(\delta_\lambda) = \lambda^6$, we also have

$$|B_r(0)| = r^6 |B_1(0)|,$$

where $|B_1(0)| = w_2$ is the Lebesgue measure of the Euclidean unit ball of $\mathbb{R}^3$. This implies that the Lebesgue measure dx is a doubling measure with respect to d, since

$$|B(x, 2r)| = 2^6 |B(x, r)|, \quad z \in \mathbb{R}^3, \quad r > 0.$$

Therefore, the space $(\mathbb{R}^3, dx, d)$ is a space of homogenous type. Recall that if f and g are functions on $\mathbb{R}^3$, their convolution $f * g$ is defined by

$$f * g(x) = \int_{\mathbb{R}^3} f(x \circ y^{-1}) g(y) dy = \int_{\mathbb{R}^3} g(y^{-1} \circ x) f(y) dy.$$

For the operator $\mathcal{L}$, the fundamental solution $\Gamma(\cdot, y)$ with pole in $y = (y_1, y_2, \tau) \in \mathbb{R}^3$ is smooth except on diagonal of $\mathbb{R}^3 \times \mathbb{R}^3$. It has the following form at $y = (0, 0, 0)$:

$$\Gamma(x) = \Gamma(x, 0) = \begin{cases} \frac{\sqrt{3}}{2\pi t^2} \exp\Big(- \frac{x_1^2}{t} - \frac{3x_1x_2}{t^2} - \frac{3x_2^2}{t^3}\Big) & \text{for } t > 0, \\ 0 & \text{for } t \le 0. \end{cases}$$

And

$$\Gamma(x, y) = \begin{cases} \frac{\sqrt{3}}{2\pi(t-\tau)^2} \exp\Big(- \frac{x_1^2 + x_1y_1 + y_1^2}{t-\tau} - \frac{3(x_1+x_2)(x_2-y_2)}{(t-\tau)^2} - \frac{3(x_2-y_2)^2}{(t-\tau)^3}\Big) & \text{for } t > \tau, \\ 0 & \text{for } t \le \tau. \end{cases}$$

Moreover, $\Gamma \in C^\infty(\mathbb{R}^3 \setminus \{0\})$.

The authors in [11] and [33] proved a representation formula:

$$u(x) = -(\mathcal{L}u * \Gamma)(x) = -\int_{\mathbb{R}^3} \Gamma(y^{-1} \circ x) \mathcal{L}u(y) dy. \tag{4}$$

The following formula was given by Bramanti in [7]:

$$\partial^2_{x_1x_1} u(x) = -P.V.\, (\mathcal{L}u * \partial^2_{x_1x_1}\Gamma)(x) + c_{11}\mathcal{L}u(x) \tag{5}$$

for every $u \in C_0^\infty(\mathbb{R}^3)$ and some constant c_{11}. The principal value in (5) is understood as

$$P.V.\, (\mathcal{L}u * \partial^2_{x_1x_1}\Gamma)(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3 \setminus B(z,\varepsilon)} (\partial^2_{x_1x_1}\Gamma)(y^{-1} \circ x) \mathcal{L}u(Y) dy.$$

Set

$$\Gamma_1(x) = \partial_{x_1}\Gamma(x), \ \Gamma_{11}(x) = \partial_{x_1}\partial_{x_1}\Gamma(x).$$

We also observe that $\Gamma(x)$ is homogeneous of degree -4 with respect to the group $(\delta_\lambda)_{\lambda>0}$, and $\Gamma_1(x)$ is homogeneous of degree -5. Recall that $\Gamma_{11}(\cdot)$ has the following properties.

Lemma 2. *([6]). One has*

(*a*) $\Gamma_{11}(\cdot) \in C^\infty(\mathbb{R}^3 \setminus \{0\})$;

(*b*) $\Gamma_{11}(\cdot)$ *is homogeneous of degree* -6*;*

(*c*) *for every* $R > r > 0$,

$$\int_{r<\|x\|<R} \Gamma_{11}(x) dx = \int_{\|z\|=1} \Gamma_{11}(x) d\sigma(x) = 0.$$

3. Generalized Hölder continuity

In this section, by demonstrating generalized Hölder estimates for two integral operators, we prove Theorem 1.

Lemma 3. *[23]. Let $K \in C^1(\mathbb{R}^3 \setminus \{0\})$ be a homogeneous function of degree $b < 1$ with respect to the group $(\delta_\lambda)_{\lambda>0}$. There exist two constants $c > 0$ and $M > 1$ such that if $\|x\| > M\|x^{-1} \circ y\|$, then*

$$|K(y) - K(x)| \le c\|x^{-1} \circ y\| \cdot \|x\|^{b-1}.$$

Lemma 4. *[23] For every $x, y, z \in \mathbb{R}^3$, the following assertions hold:*

(1) *there exists a constant $c > 0$ such that*

$$\Gamma(x^{-1} \circ y) \le \frac{c}{\|x^{-1} \circ y\|^4}, \quad \Gamma_i(x^{-1} \circ y) \le \frac{c}{\|x^{-1} \circ y\|^5}.$$

(2) *there exist two constants $c > 0$ and $M > 1$ such that if $\|x^{-1} \circ z\| \ge M\|x^{-1} \circ y\|$, then*

$$\left|\Gamma(x^{-1} \circ z) - \Gamma(x^{-1} \circ y)\right| \le \frac{c\,\|z^{-1} \circ y\|}{\|x^{-1} \circ z\|^5},$$

$$\left|\Gamma_i(x^{-1} \circ z) - \Gamma_i(x^{-1} \circ y)\right| \le \frac{c\,\|z^{-1} \circ y\|}{\|x^{-1} \circ z\|^6}.$$

Lemma 5. *Let $p \in (1, \infty)$ and $\lambda \in [0, 6)$. With fixed $z \in \mathbb{R}^3$, $\alpha \in [0, 6)$, $\beta \in (0, 6)$ and $\sigma > 0$, for every $g \in M^{p,\varphi}(\mathbb{R}^3)$, we set*

$$T'_\alpha g(x) = \int_{\|y^{-1}\circ x\| \ge \sigma\,\|z^{-1}\circ x\|} \frac{g(y)}{\|y^{-1} \circ x\|^{6-\alpha}} dy$$

and

$$T''_\beta g(x) = \int_{\|y^{-1}\circ x\| < \sigma \|z^{-1}\circ x\|} \frac{g(y)}{\|y^{-1} \circ x\|^{6-\beta}} dy.$$

If $\int_1^\infty \varphi(x, r)\, r^{\alpha-1}\, dr < \infty$, then there exists $c = c(p, \varphi, \alpha, \sigma) > 0$ such that

$$|T'_\alpha g(x)| \le c\,\|g\|_{M^{p,\varphi}(\mathbb{R}^3)} \int_{\|z^{-1}\circ x\|}^\infty \varphi(x, r)\, r^{\alpha-1}\, dr. \tag{6}$$

Moreover, if $\int_0^1 \varphi(x, r)\, r^{\beta-1}\, dr < \infty$, then there exists $c = c(p, \varphi, \beta, \sigma) > 0$ such that

$$|T''_\beta g(x)| \le c\,\|g\|_{M^{p,\varphi}(\mathbb{R}^3)} \int_0^{\|z^{-1}\circ x\|} \varphi(x, r)\, r^{\beta-1}\, dt. \tag{7}$$

Proof. Observing that

$$
\begin{aligned}
|T'_\alpha g(x)| &\le \sum_{k=1}^{\infty} \int_{2^{k-1}\sigma\|z^{-1}\circ x\| \le \|y^{-1}\circ x\| < 2^k\sigma\|z^{-1}\circ x\|} \frac{|g(y)|}{\|y^{-1}\circ x\|^{6-\alpha}}\, dy \\
&\le \sum_{k=1}^{\infty} \Big(\frac{2}{2^k\sigma\|z^{-1}\circ x\|}\Big)^{6-\alpha} \int_{B_{2^k c_1\sigma\|z^{-1}\circ x\|}(x)} |g(y)|\, dy \\
&\le \sum_{k=1}^{\infty} \Big(\frac{2}{2^k\sigma\|z^{-1}\circ x\|}\Big)^{6-\alpha} \|g\|_{L^p\left(B_{2^k c_1\sigma\|z^{-1}\circ x\|}(x)\right)} \left|B_{2^k c_1\sigma\|z^{-1}\circ x\|}(x)\right|^{\frac{1}{p'}} \\
&\le \sum_{k=1}^{\infty} \Big(\frac{2}{2^k\sigma\|z^{-1}\circ x\|}\Big)^{\frac{6}{p}-\alpha} \|g\|_{L^p\left(B_{2^k c_1\sigma\|z^{-1}\circ x\|}(x)\right)} \lesssim \|g\|_{M^{p,\varphi}(\mathbb{R}^3)} \\
&\times \sum_{k=1}^{\infty} \Big(\frac{2}{2^k\sigma\|z^{-1}\circ x\|}\Big)^{\frac{6}{p}-\alpha} \big(2^k\sigma\|z^{-1}\circ x\|\big)^{\frac{6}{p}} \varphi\big(x, 2^k c_1\sigma\|z^{-1}\circ x\|\big) \\
&\lesssim \|g\|_{M^{p,\varphi}(\mathbb{R}^3)} \sum_{k=1}^{\infty} \Big(2^k\sigma\|z^{-1}\circ x\|\Big)^{\alpha} \varphi\big(x, 2^k\sigma\|z^{-1}\circ x\|\big) \\
&\lesssim \|g\|_{M^{p,\varphi}(\mathbb{R}^3)} \int_{\|z^{-1}\circ x\|}^{\infty} \varphi(x,r)\, r^{\alpha-1}\, dr,
\end{aligned}
$$

we see that (6) is true, since the above series is convergent.

Similarly, by integrating over the set

$$
\{y \in \mathbb{R}^3 : 2^{-k}\sigma\|z^{-1}\circ x\| \le \|y^{-1}\circ x\| < 2^{1-k}\sigma\|z^{-1}\circ x\|\},
$$

we get

$$
\begin{aligned}
|T''_\beta g(x)| &\le \sum_{k=1}^{\infty} \int_{2^{-k}\sigma\|z^{-1}\circ x\| \le \|y^{-1}\circ x\| < 2^{1-k}\sigma\|z^{-1}\circ x\|} \frac{|g(y)|}{\|y^{-1}\circ x\|^{6-\beta}}\, dy \\
&\le \sum_{k=1}^{\infty} \Big(\frac{2}{2^{1-k}\sigma\|z^{-1}\circ x\|}\Big)^{6-\beta} \int_{B_{2^{1-k} c_1\sigma\|z^{-1}\circ x\|}(x)} |g(y)|\, dy \\
&\le \sum_{k=1}^{\infty} \Big(\frac{2}{2^{1-k}\sigma\|z^{-1}\circ x\|}\Big)^{6-\beta} \|g\|_{L^p\left(B_{2^{1-k} c_1\sigma\|z^{-1}\circ x\|}(x)\right)} \left|B_{2^{1-k} c_1\sigma\|z^{-1}\circ x\|}(x)\right|^{\frac{1}{p'}} \\
&\le \sum_{k=1}^{\infty} \Big(\frac{2}{2^{1-k}\sigma\|z^{-1}\circ x\|}\Big)^{\frac{6}{p}-\beta} \|g\|_{L^p\left(B_{2^{1-k} c_1\sigma\|z^{-1}\circ x\|}(x)\right)} \lesssim \|g\|_{M^{p,\varphi}(\mathbb{R}^3)} \\
&\times \sum_{k=1}^{\infty} \Big(\frac{1}{2^{-k}\sigma\|z^{-1}\circ x\|}\Big)^{\frac{6}{p}-\beta} \big(2^{-k}\sigma\|z^{-1}\circ x\|\big)^{\frac{6}{p}} \varphi\big(x, 2^{-k}\sigma\|z^{-1}\circ x\|\big)
\end{aligned}
$$

$$\lesssim \|g\|_{M^{p,\varphi}(\mathbb{R}^3)} \sum_{k=1}^{\infty} \left(2^{-k}\sigma\|z^{-1}\circ x\|\right)^{\beta} \varphi\big(x, 2^{-k}\sigma\|z^{-1}\circ z\|\big)$$

$$\lesssim \|g\|_{M^{p,\varphi}(\mathbb{R}^3)} \int_0^{\|z^{-1}\circ x\|} \varphi(x,r)\, r^{\beta-1}\, dr.$$

As the above series is convergent, (7) is proved. ◄

Proof of Theorem 1. For $u \in C_0^\infty(\mathbb{R}^3)$, by Lemmas 4 and 5, there exist $M, c > 0$ such that

$$\begin{aligned} |u(x)-u(z)| &\leq \int_{\mathbb{R}^3} |\Gamma(y^{-1}\circ x) - \Gamma(y^{-1}\circ z)||\mathcal{L}(y)|dy \\ &\lesssim \int_{\|y^{-1}\circ z\|\geq M\|x^{-1}\circ z\|} \frac{\|x^{-1}\circ z\|}{\|y^{-1}\circ x\|^5}|\mathcal{L}u(y)|dy \\ &+ \int_{\|y^{-1}\circ x\|<M\|x^{-1}\circ z\|} \frac{1}{\|y^{-1}\circ x\|^4}|\mathcal{L}u(y)|dy \\ &+ \int_{\|y^{-1}\circ x\|<M\|x^{-1}\circ z\|} \frac{1}{\|y^{-1}\circ z\|^4}|\mathcal{L}u(y)|dy \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

By applying Lemma 5 and choosing $\alpha = 1$ and $\sigma = M/c_1$, we obtain the existence of a positive constant c such that

$$|I_1| \lesssim \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^3)}\, \|x^{-1}\circ z\| \int_{\|x^{-1}\circ z\|}^{\infty} \varphi(x,r)\, dr. \tag{8}$$

Choosing $\beta = 2$ and $\sigma = Mc_1$ in Lemma 5, we obtain the existence of a positive constant c such that

$$|I_2| \lesssim \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^3)} \int_0^{\|x^{-1}\circ z\|} \varphi(x,r)\, r\, dr. \tag{9}$$

Choosing $\beta = 2$ and $\sigma = c_2(1+M)$ in Lemma 5, we obtain the existence of a positive constant c such that

$$|I_3| \lesssim \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^3)} \int_0^{\|x^{-1}\circ z\|} \varphi(x,r)\, r\, dr. \tag{10}$$

Hence, by (8), (9) and (10), it is easy to obtain

$$|u(x)-u(z)| \leq C\|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^3)}$$

$$\times \Big(\int_0^{\|x^{-1}\circ z\|} \varphi(x,r)\, r\, dr + \|x^{-1}\circ z\| \int_{\|x^{-1}\circ z\|}^{\infty} \varphi(x,r)\, dr \Big),$$

where C is a positive constant, $x, z \in \mathbb{R}^3$, $x \neq z$.

By (4), we write

$$\partial_{x_1} u(x) = -\int_{\mathbb{R}^3} \Gamma_1(y^{-1}\circ x)\mathcal{L}u(y)dy$$

for every $x \in \mathbb{R}^3$. Analogously, by Lemmas 4 and 5, we obtain the existence of $M, c > 0$ such that

$$\begin{aligned}
|\partial_{x_1} u(x) - \partial_{x_1} u(z)| &\le \int_{\mathbb{R}^3} |\Gamma_1(y^{-1}\circ x) - \Gamma_1(y^{-1}\circ z)||\mathcal{L}u(y)|dy \\
&\le \int_{\|y^{-1}\circ x\| \ge M\|x^{-1}\circ z\|} \frac{c\|x^{-1}\circ z\|}{\|y^{-1}\circ x\|^6} |\mathcal{L}u(y)|dy \\
&+ \int_{\|y^{-1}\circ x\| < M\|x^{-1}\circ z\|} \frac{c}{\|y^{-1}\circ x\|^5} |\mathcal{L}u(y)|dy \\
&+ \int_{\|y^{-1}\circ x\| < M\|x^{-1}\circ z\|} \frac{c}{\|y^{-1}\circ z\|^5} |\mathcal{L}u(y)|dy \\
&\equiv I_1' + I_2' + I_3'.
\end{aligned}$$

By applying Lemma 5 and choosing $\alpha = 0$ and $\sigma = M/c_1$, we obtain the existence of a positive constant c such that

$$|I_1'| \lesssim \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^3)} \, \|x^{-1}\circ z\| \int_{\|x^{-1}\circ z\|}^{\infty} \varphi(x,r)\, \frac{dr}{r}. \tag{11}$$

Choosing $\beta = 1$ and $\sigma = Mc_1$ in Lemma 5, we obtain the existence of a positive constant c such that

$$|I_2'| \le c\, \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^3)} \int_0^{\|x^{-1}\circ z\|} \varphi(x,r)\, dr. \tag{12}$$

Choosing $\beta = 1$ and $\sigma = c_2(1+M)$ in Lemma 5, we obtain the existence of a positive constant c such that

$$|I_3'| \le c\|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^3)} \int_0^{\|x^{-1}\circ z\|} \varphi(x,r)\, dr. \tag{13}$$

Hence, by (11), (12) and (13), we derive

$$\big|\partial_{x_1} u(x) - \partial_{x_1} u(z)\big| \le C\, \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^3)}$$

$$\times \Big(\int_0^{\|x^{-1}\circ z\|} \varphi(x,r)\, dr + \|x^{-1}\circ z\| \int_{\|x^{-1}\circ z\|}^{\infty} \varphi(x,r)\, \frac{dr}{r} \Big),$$

where C is a positive constant, $x, z \in \mathbb{R}^3$, $x \neq z$. This completes the proof. ◀

Proof of Corollary 2. If we take $\varphi(x,r) = |B(x,r)|^{\frac{\lambda-1}{p}}$ in Theorem 5, then we get

$$\int_{\|z^{-1}\circ x\|}^{\infty} \varphi(x,r)\, r^{\alpha-1}\, dr = \int_{\|z^{-1}\circ x\|}^{\infty} r^{\frac{\lambda-6}{p}+\alpha-1}\, dr = \|z^{-1}\circ x\|^{\frac{\lambda-6}{p}+\alpha}$$

and

$$\int_1^{\infty} \varphi(x,r)\, r^{\alpha-1}\, dr = \int_1^{\infty} r^{\frac{\lambda-6}{p}+\alpha-1}\, dr < \infty \Leftrightarrow \frac{\lambda-6}{p} + \alpha > 0$$
$$\Leftrightarrow \lambda + p\alpha < 6.$$

Also,

$$\int_0^{\|z^{-1}\circ x\|} \varphi(x,r)\, r^{\beta-1}\, dr = \int_0^{\|z^{-1}\circ x\|} r^{\frac{\lambda-6}{p}+\beta-1}\, dr = \|z^{-1}\circ x\|^{\frac{\lambda-6}{p}+\beta}$$

and

$$\int_0^{1} \varphi(x,r)\, r^{\beta-1}\, dr = \int_0^{1} r^{\frac{\lambda-6}{p}+\beta-1}\, dr < \infty \Leftrightarrow \frac{\lambda-6}{p} + \beta > 0$$
$$\Leftrightarrow \lambda + p\beta > 6.$$

This completes the proof. ◀

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