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Existence of Nodal Solutions of Some Nonlinear Sturm-Liouville Problems With a Parameter in the Boundary Condition

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Abstract. In this paper, we consider nonlinear boundary value problem for ordinary differential equations of second order with a parameter in the equation and the boundary condition. We define the interval of this parameter, in which there are two different solutions of considered problem that have a fixed number of simple nodal zeros. Our approach is based upon global bifurcation techniques.

Key Words and Phrases: nonlinear problem, nodal solution, global bifurcation, global continua, oscillatory property.

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1. Introduction

Let f and g be real-valued continuous functions on $\mathbb{R}$ that satisfy the following conditions: there exist a positive constant $M > 0$, a sufficiently small positive number τ_0 and a sufficiently large positive number $\varkappa_0$ such that

$$\left|\frac{f(s)}{s}\right| \leq M \ \text{ for } \ 0 < \ |s| \ < \tau_0 \ \text{ and } \ |s| \ > \varkappa_0; \tag{1}$$

there exist constants $g_0 > 0$ and $g_\infty > 0$ such that

$$g_0 = \lim_{|s|\to 0} \frac{g(s)}{s} \ \text{ and } \ g_\infty = \lim_{|s|\to +\infty} \frac{g(s)}{s}\,. \tag{2}$$

We consider the nonlinear boundary value problem for the equation

$$\ell(y) \equiv -\left(p\,(x)y'(x)\right)' + q(x)y(x) = \varrho\, r(x)h\left(y(x)\right),\ x \in (0,1), \tag{3}$$

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subject to the boundary conditions

$$b_0 y(0) - d_0 p(0) y'(0) = 0, \tag{4}$$

$$(a_1 \varrho g_0 + b_1) y(1) - (c_1 \varrho g_0 + d_1) p(1) y'(1) = 0. \tag{5}$$

Here p is a positive continuously differentiable function on $[0,1]$, q is a continuous function on $[0,1]$, ϱ is a real parameter, $r(x)$ is a positive continuous function on $[0,1]$, the nonlinear term has the form $h = f + g$, and b_0, d_0, b_1, d_1 are real constants such that $|b_0| + |d_0| > 0$ and $a_1 d_1 - b_1 c_1 > 0$.

The nonlinear Sturm-Liouville problems arise when modeling various problems in mechanics and physics (see [6, 7, 19, 21] and references therein).

Nonlinear Sturm-Liouville problems with specified nodal properties have been studied by many authors (see, for example, [8, 9, 11, 12, 13, 14, 15, 16]). In these papers, using the global bifurcation techniques ([10, 17, 18, 20]) and analytical methods, the existence of solutions to the considered problems which have a fixed number of simple nodal zeros has been shown. The sufficient conditions established by the authors of these papers that provide the existence of nodal solutions are valid only in the case where the eigenvalues of the corresponding linear spectral problem are positive. Moreover, on the nonlinear term g, the condition $sg(s) > 0$ for $s \in \mathbb{R}$, $s \neq 0$ is imposed (with the exception of [8]).

Problem (3)-(5) for $f \equiv 0$ was considered in the recent paper [5], which determined the interval of the parameter ϱ, where this problem has nodal solutions. Note that in [5] the above condition is not imposed on g.

In this paper, we determine the interval of ϱ, where the problem (3)-(5) has solutions that have a fixed number of simple nodal zeros in $(0,1)$.

The rest of the article is organized as follows. In Section 2, we study global bifurcation from zero and at infinity of some auxiliary nonlinear Sturm-Liouville problem. We show the existence of two families of global components of the set of nontrivial solutions of this problem, meeting the intervals of the lines $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{\infty\}$ and contained in classes of functions with oscillatory properties of eigenfunctions of the corresponding linear spectral problem. In Section 3, using these global bifurcation results, we determine the interval of ϱ, in which the nonlinear problem (3)-(5) has solutions belonging to these classes. It should be noted that the eigenvalues of the corresponding linear Sturm-Liouville eigenvalue problem with a spectral parameter in the boundary condition can be negative and the function g does not satisfy the condition $sg(s) > 0$ for $s \in \mathbb{R}$, $s \neq 0$.

2. Preliminaries and some auxiliary results

Let $(b.c.)_0$ be the set of functions that satisfy the boundary condition (4) and let $(b.c.)_\lambda$, $\lambda \in \mathbb{R}$, be the set of functions that satisfy the boundary conditions (4) and

$$(a_1\lambda + b_1)y(1) - (c_1\lambda + d_1)p\,(1)y'(1) = 0.$$

By E we denote the Banach space $C^1[0,1] \cap (b.c.)_0$ equipped with the norm $||y||_1 = ||y||_\infty + ||y'||_\infty$, where $||y||_\infty = \max\limits_{x\in[0,\,1]} |y\,(x)|$, and by S we denote the subset of E given as follows:

$$S = \left\{ y \in E : |y(x)| + |y'(x)| > 0,\, x \in [0,1] \right\}.$$

Moreover, ν will denote either $+$ or $-$; $-\,\nu$ will denote the opposite sign to ν.

In [2, 3], for each $k \in \mathbb{N}$ and each ν, the authors constructed the sets S_k^ν of functions in E which have the oscillatory properties of eigenfunctions of the linear Sturm-Liouville problem

$$\begin{cases} \ell(y)(x) = \lambda r(x)y(x),\, x \in (0,1), \\ b_0 y(0) - d_0 p\,(0)y'(0) = 0, \\ (a_1\lambda + b_1)y(1) - (c_1\lambda + d_1)p\,(1)y'(1) = 0, \end{cases} \tag{6}$$

the eigenvalues of which are real and simple, and form an infinitely increasing sequence $\{\lambda_k\}_{k=1}^\infty$. Note that the sets S_k^+, S_k^- and $S_k = S_k^+ \cup S_k^-$ are pairwise disjoint open subsets of E. Moreover, if $y \in \partial S_k^\nu$ (∂S_k), then y has at least one double zero in $[0,1]$ (see [12]).

Remark 1. In what follows, we shall assume that $\varrho \neq 0$ and is contained in some bounded interval of the real axis.

By (2), the function g has the following representations:

$$g(s) = g_0 s + \varkappa_0(s) \text{ and } g(s) = g_\infty s + \varkappa_\infty(s), \tag{7}$$

where

$$\varkappa_0(s) = o(|s|) \text{ as } |s| \to 0 \text{ and } \varkappa_\infty(s) = o(|s|) \text{ as } |s| \to \infty, \tag{8}$$

respectively.

By the first relation of (7), we can rewrite problem (3)-(5) in the form

$$\begin{cases} \ell(y)(x) = \varrho\, g_0 r(x)y(x) + \varrho\, r(x)f(y(x)) + \varrho\, r(x)\varkappa_0(y(x)),\, x \in (0,1), \\ y \in (b.c.)_{\varrho\, g_0}. \end{cases} \tag{9}$$

Along with problem (9), we will consider the following nonlinear eigenvalue problem:

$$\begin{cases} \ell(y)(x) = \lambda\varrho\, g_0 r(x)y(x) + \varrho\, r(x)f(y(x)) + \varrho\, r(x)\varkappa_0(y(x)),\, x \in (0,1), \\ y \in (b.c.)_\lambda^\varrho, \end{cases} \tag{10}$$

where $(b.c.)^{\varrho}_{\lambda}$ is the set of functions satisfying the boundary conditions (4) and

$$(a_1\lambda\varrho\, g_0 + b_1)y(1) - (c_1\lambda\varrho\, g_0 + d_1)p\,(1)y'(1) = 0. \tag{11}$$

By the first relation of (8), problem (10) is a bifurcation from zero problem. On the other hand, it follows from relations (7) that

$$\varkappa_0(s) = (g_\infty - g_0)s + \varkappa_\infty(s) \text{ for } s \in \mathbb{R}. \tag{12}$$

In view of (12), problem (10) takes the form

$$\begin{cases} \ell(y)(x) = \lambda\varrho\, g_0 r(x)y(x) + \varrho(g_\infty - g_0)r(x)y(x) + \\ \qquad\qquad \varrho\, r(x)f(y(x)) + \varrho\, r(x)\varkappa_\infty(y(x)),\ x \in (0,1), \\ y \in (b.c.)^{\varrho}_{\lambda}, \end{cases} \tag{13}$$

which, by the second relation of (8), is a bifurcation from infinity problem.

We introduce the following notations:

$$\begin{gathered} f^*(x,y,v,\lambda) = \tfrac{1}{g_0} f(y),\ g_0^*(x,y,v,\lambda) = \tfrac{1}{g_0}\varkappa_0(y), \\ g_\infty^*(x,y,v,\lambda) = \tfrac{1}{g_0}\,\varkappa_\infty(y). \end{gathered} \tag{14}$$

Then, a bifurcation from zero problem (10) and a bifurcation at infinity problem (13) can be rewritten in the following forms:

$$\begin{cases} \frac{1}{\varrho\, g_0 r(x)}\,\ell(y) = \lambda y + f^*(x,y,y',\lambda) + g_0^*(x,y,y',\lambda),\ x \in (0,1), \\ y \in (b.c.)^{\varrho}_{\lambda}, \end{cases} \tag{15}$$

and

$$\begin{cases} \frac{1}{\varrho\, g_0 r(x)}\ell(y) = \lambda y + \left(\frac{g_\infty}{g_0} - 1\right) y\ + f^*(x,y,y',\lambda) + \\ \qquad\qquad g_\infty^*(x,y,y',\lambda),\ x \in (0,1), \\ y \in (b.c.)^{\varrho}_{\lambda}, \end{cases} \tag{16}$$

respectively.

Note that if $0 < |y| + |s| < \tau_0$ and $y \neq 0$, then $0 < |y| < \tau_0$. Hence in view of the first relation of (14), by (1) we get

$$\left|\frac{f^*(x,y,v,\lambda)}{y}\right| \leq \frac{M}{g_0} \ \text{ for any } \ (x,y,v,\lambda) \in [0,1]\times\mathbb{R}^3,\ y \neq 0,\ |y|+|v| < \tau_0. \tag{17}$$

By the first relation of (2), for any sufficiently small $\epsilon > 0$ there exists a sufficiently small $\delta_\epsilon > 0$ such that

$$\frac{|\varkappa_0(y)|}{|y|} < \epsilon g_0 \ \text{ for any } \ s \in \mathbb{R},\ 0 < |y| < \delta_\epsilon. \tag{18}$$

Then, by the second relation of (14), for any $(x, y, v, \lambda) \in [0,1] \times \mathbb{R}^3$, $0 < |y| + |v| < \delta_\epsilon$, $y \neq 0$, we get

$$\frac{|g_0^*(x,y,v,\lambda)|}{|y|+|v|} = \frac{1}{g_0}\frac{|\varkappa_0(y)|}{|y|+|v|} < \frac{1}{g_0}\frac{|\varkappa_0(y)|}{|y|} < \epsilon,$$

which shows that

$$g_0^*(x,y,v,\lambda) = o(|y|+|v|) \text{ as } |y|+|v| \to 0, \tag{19}$$

uniformly in $(x, \lambda) \in [0,1] \times \mathbb{R}$.

Let C denote the set of nontrivial solutions of problem (10) (or (15)), and let

$$I_k = \left[\lambda_k^* - \frac{M}{g_0}, \lambda_k^* + \frac{M}{g_0}\right],$$

where λ_k^* is the kth eigenvalue of the linear spectral problem

$$\begin{cases} \ell(y)(x) = \lambda \varrho\, g_0 r(x) y(x),\ x \in (0,1), \\ y \in (b.c.)_\lambda^o. \end{cases} \tag{20}$$

In view of (6) it follows from (20) that $\lambda_k^* = \frac{\lambda_k}{\varrho\, g_0}$, and consequently,

$$I_k = \left[\frac{\lambda_k}{\varrho\, g_0} - \frac{M}{g_0}, \frac{\lambda_k}{\varrho\, g_0} + \frac{M}{g_0}\right].$$

Since conditions (17) and (19) are satisfied, it follows from [3] (see also [1, Lemmas 2-4 and Theorem 4]) that for problem (15) the following global bifurcation result holds.

Theorem 1. [3, Lemma 3.2 and Theorem 3.1] *For each $k \in \mathbb{N}$ and each ν there exists a subset C_k^ν of the set C such that $C_k^\nu \cup (I_k \times \{0\})$ is closed and connected, and*

(i_1) $C_k^\nu \subset \mathbb{R} \times S_k^\nu$;

(ii_1) C_k^ν is unbounded in $\mathbb{R} \times E$ (in this case either C_k^ν meets (λ, ∞) for some $\lambda \in \mathbb{R}$, or the projection of C_k^ν onto $\mathbb{R} \times \{0\}$ is unbounded).

Let $\varepsilon_0 > 0$ be the fixed sufficiently small number such that $\varepsilon_0 < \frac{\tau_0}{2}$. By $\xi(s)$, $0 \leq \xi(s) \leq 1$, we define the continuous function on $\mathbb{R}$ given by

$$\xi(s) = \begin{cases} 1 & \text{if } |s| < \tau_0 - \varepsilon_0, \\ 0 & \text{if } \tau_0 \leq |s| \leq \tau_\infty, \\ 1 & \text{if } |s| > \tau_\infty + \varepsilon_0. \end{cases} \tag{21}$$

Since

$$f(s) = \xi(s)f(s) + (1 - \xi(s))f(s),\ s \in \mathbb{R},$$

(13) (or (16)) can be rewritten in the form

$$\begin{cases} \frac{1}{\varrho\, g_0 r(x)}\ell(y) = \lambda y + \left(\frac{g_\infty}{g_0} - 1\right) y\ + f^{**}(x, y, y', \lambda) + \\ \qquad\qquad g^{**}_\infty(x, y, y', \lambda),\ x \in (0, 1), \\ y \in (b.c.)^o_\lambda, \end{cases} \tag{22}$$

where

$$f^{**}(x, y, s, \lambda) = \xi(y)f^*(x, y, s, \lambda) = \frac{1}{g_0}\,\xi(y)f(y),\ \ g^{**}(x, y, s, \lambda) = \\ \frac{1}{g_0}\,(1 - \xi(y))f(y) + g^*(x, y, s, \lambda) = \frac{1}{g_0}\,\left((1 - \xi(y))f(y) + \varkappa_\infty(y)\right). \tag{23}$$

In view of (21) and (23), by (1) we get

$$\left|\frac{f^{**}(x, y, v, \lambda)}{y}\right| \le \frac{M}{g_0}\ \text{ for any }\ (x, y, v, \lambda) \in [0, 1] \times \mathbb{R}^3,\ \ y \ne 0. \tag{24}$$

Moreover, it follows from the second part of (8) that for any sufficiently small $\epsilon > 0$ there is a sufficiently large $\Delta_\epsilon > \tau_\infty + \varepsilon_0 > 0$ such that

$$\left|\frac{\varkappa_\infty(s)}{s}\right| < \epsilon g_0\ \text{ for }\ s \in \mathbb{R},\ |s| > \Delta_\epsilon. \tag{25}$$

Since $\varkappa_\infty(s)$ is a continuous function on $\mathbb{R}$, it follows that there exists a positive constant κ_ϵ such that

$$|\varkappa_\infty(s)| < \kappa_\epsilon\ \text{ for any }\ s \in \mathbb{R},\ |s| \le \Delta_\epsilon. \tag{26}$$

Let Δ^*_ϵ be a sufficiently large positive number such that

$$\Delta^*_\epsilon > \Delta_\epsilon\ \text{ and }\ \kappa_\epsilon < \epsilon g_0 \Delta^*_\epsilon. \tag{27}$$

For any $(y, v) \in \mathbb{R}^2$ such that $|y| + |v| > \Delta^*_\epsilon$, by (25)-(27), we have

$$\frac{|g^*_\infty(x, y, v, \lambda)|}{|y| + |v|} = \frac{1}{g_0}\frac{|\varkappa_\infty(y)|}{|y| + |v|} \le \frac{1}{g_0}\frac{|\varkappa_\infty(y)|}{|y|} < \epsilon\ \text{ for any }\ y \in \mathbb{R},\ |y| > \Delta_\epsilon,$$

and

$$\frac{|g^*_\infty(x, y, v, \lambda)|}{|y| + |v|} = \frac{1}{g_0}\frac{|\varkappa_\infty(y)|}{|y| + |v|} < \frac{1}{g_0}\frac{\kappa_\epsilon}{\Delta^*_\epsilon} < \epsilon\ \text{ for any }\ y \in \mathbb{R},\ |y| \le \Delta_\epsilon.$$

Thus, the last two relations show that

$$g_{\infty}^{*}(x,y,v,\lambda) = o(|y|+|v|) \text{ as } |y|+|v| \to \infty, \tag{28}$$

uniformly for $(x,\lambda) \in [0,1]\times\mathbb{R}$.

Next, in view of $f \in C(\mathbb{R})$, by the relation

$$(1-\xi(s))f(s) = \begin{cases} 0 & \text{if } |s| < \tau_0 - \varepsilon_0, \\ f(s) & \text{if } \tau_0 \le |s| \le \tau_\infty, \\ 0 & \text{if } |s| > \tau_\infty + \varepsilon_0, \end{cases} \tag{29}$$

there exists a positive constant κ_1 such that

$$|(1-\xi(s))f(s)| \le \kappa_1 \text{ for any } s \in \mathbb{R},\ \tau_0 - \varepsilon_0 \le |s| \le \tau_\infty + \varepsilon_0. \tag{30}$$

Let

$$\tilde{g}_{\infty}^{*}(x,y,v,\lambda) = (1-\xi(y)f(y).$$

By $\Delta_{\epsilon,1}^{*}$ we denote the sufficiently large positive number such that

$$\Delta_{\epsilon,1}^{*} > \Delta_{\epsilon} \text{ and } \kappa_1 < \epsilon g_0 \Delta_{\epsilon,1}^{*}. \tag{31}$$

Let $(y,v) \in \mathbb{R}^2$ satisfy the inequality $|y|+|v| > \Delta_{\epsilon,1}^{*}$. Then by (29)-(31) we get

$$\frac{|\tilde{g}_{\infty}^{*}(x,y,v,\lambda)|}{|y|+|v|} = 0 < \epsilon \text{ for any } y \in \mathbb{R},\ |y| < \tau_0 - \varepsilon_0 \text{ and } |y| > \tau_\infty + \varepsilon_0,$$

and

$$\frac{|g_{\infty}^{*}(x,y,v,\lambda)|}{|y|+|v|} = \frac{1}{g_0}\frac{|(1-\xi(y))f(y)|}{|y|+|v|} \le \frac{1}{g_0}\frac{\kappa_1}{\Delta_{\epsilon,1}^{*}} < \epsilon \text{ for any } y \in \mathbb{R},$$

$$\tau_0 - \varepsilon_0 \le |y| \le \tau_\infty + \varepsilon_0,$$

which implies that

$$\tilde{g}_{\infty}^{*}(x,y,v,\lambda) = o(|y|+|v|) \text{ as } |y|+|v| \to \infty, \tag{32}$$

uniformly for $(x,\lambda) \in [0,1]\times\mathbb{R}$.

Thus, by (23), it follows from (28) and (32) that

$$g_{\infty}^{**}(x,y,v,\lambda) = o(|y|+|v|) \text{ as } |y|+|v| \to \infty, \tag{33}$$

uniformly for $(x,\lambda) \in [0,1]\times\mathbb{R}$.

Let

$$J_k = \left[\lambda_k^{**} - \frac{M}{g_0}, \lambda_k^{**} + \frac{M}{g_0}\right],$$

where λ_k^{**} is the kth eigenvalue of the linear Sturm-Liouville problem

$$\begin{cases} \frac{1}{\varrho\, g_0 r(x)} \ell(y)(x) = \left(\lambda + \frac{g_\infty}{g_0} - 1\right) y\,,\ x \in (0,1), \\ y \in (b.c.)_\lambda^o, \end{cases} \tag{34}$$

obtained from (22) by setting $f^{**} \equiv 0$ and $g^{**} \equiv 0$. From (6) and (34) it follows that $\lambda_k = \varrho\, g_0 \left(\lambda_k^{**} + \frac{g_\infty}{g_0} - 1\right)$, and consequently,

$$J_k = \left[\frac{\lambda_k}{\varrho\, g_0} - \frac{g_\infty}{g_0} + 1 - \frac{M}{g_0}, \frac{\lambda_k}{\varrho\, g_0} - \frac{g_\infty}{g_0} + 1 + \frac{M}{g_0}\right].$$

Remark 2. We add the points $\{(\lambda, \infty) : \lambda \in \mathbb{R}\}$ to the space $\mathbb{R} \times E$ and define a correspondent topology on the resulting set. In this case, the "points at infinity" (λ, ∞), $\lambda \in J_k$, are the elements of $\overline{C}$, where $\overline{C}$ is the closure of the set C.

In view of relations (24) and (33), by following the corresponding arguments in Theorem 3.1 of [2], Theorems 3.1 and 3.3 of [20], we can justify the following global bifurcation result for problem (22).

Theorem 2. *For each $k \in \mathbb{N}$ and each ν, there exists a subset D_k^ν of the set C such that $D_k^\nu \cup (I_k \times \{\infty\})$ is closed and connected, and*

(i_2) $D_k^\nu \subset \mathbb{R} \times S_k^\nu$;

(ii_2) either D_k^ν meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$, or the projection of D_k^ν onto $\mathbb{R} \times \{0\}$ is unbounded.

Remark 3. It follows from [20, Theorem 3.3] that if C_k^ν meets (λ, ∞), for some $\lambda \in \mathbb{R}$, then $\lambda \in J_k$, and if D_k^ν meets $(\lambda, 0)$, for some $\lambda \in \mathbb{R}$, then $\lambda \in I_k$.

We will need the following theorem, which plays an essential role in the proof of our main result.

Theorem 3. *For each $k \in \mathbb{N}$ and each ν, the sets C_k^ν and D_k^ν coincide.*

Proof. By Theorem 1, Theorem 2 and Remark 3, it suffices to show that the projections of the sets C_k^ν and D_k^ν onto $\mathbb{R} \times \{0\}$ are bounded.

We show that the projection of the sets C_k^ν onto $\mathbb{R} \times \{0\}$ is bounded. This statement for the set D_k^ν can be proved in a similar way.

Suppose the opposite, i.e. let the projection of the set C_k^ν onto $\mathbb{R} \times \{0\}$ be unbounded. Then there exists a sequence $\{(\tilde{\lambda}_n, \tilde{y}_n)\}_{n=1}^\infty \subset C_k^\nu$ such that

$$\tilde{\lambda}_n \to -\infty \ \text{ or } \ \tilde{\lambda}_n \to +\infty \ \text{ as } \ n \to \infty. \tag{35}$$

Since $f \in C(\mathbb{R})$, it follows that there is a positive constant M_1 such that

$$\left|\frac{f(s)}{s}\right| \leq M_1 \text{ for any } s \in \mathbb{R},\ \tau_0 < |s| < \tau_\infty. \tag{36}$$

Let $\tilde{M} = \max\{M, M_1\}$. Then from (1) and (36) we obtain

$$\left|\frac{f(s)}{s}\right| \leq \tilde{M} \text{ for any } s \in \mathbb{R},\ s \neq 0. \tag{37}$$

Let $\epsilon_0 > 0$ be a sufficiently small fixed number. Then by (18) we have

$$\left|\frac{\varkappa_0(s)}{s}\right| < \epsilon_0 g_0 \ \text{ for any } \ s \in \mathbb{R},\ 0 < |s| < \delta_{\epsilon_0}. \tag{38}$$

Moreover, by relation (12), it follows from (25) that

$$\left|\frac{\varkappa_0(s)}{s}\right| < \epsilon_0 g_0 + |g_\infty - g_0| \ \text{ for any } \ s \in \mathbb{R},\ |s| > \Delta_{\epsilon_0}. \tag{39}$$

From condition $\varkappa_0(s) \in C(\mathbb{R})$ it follows that there exists a positive constant κ_0 such that

$$\left|\frac{\varkappa_0(s)}{s}\right| < \kappa_0 \ \text{ for any } \ s \in \mathbb{R},\ \delta_{\epsilon_0} < |s| < \Delta_{\epsilon_0}. \tag{40}$$

Thus, it follows from (38)-(40) that

$$\left|\frac{\varkappa_0(s)}{s}\right| < \tilde{\kappa} \ \text{ for any } \ s \in \mathbb{R},\ s \neq 0, \tag{41}$$

where $\tilde{\kappa} = \max\{\epsilon_0 g_0 + |g_\infty - g_0|,\ \kappa_0\}$.

We define the functions $\tilde{\phi}(x)$ and $\tilde{\varphi}(x)$, $x \in [0,1]$, as follows:

$$\tilde{\phi}(x) = \begin{cases} -\frac{f(y(x))}{g_0 y(x)} & \text{if } y(x) \neq 0, \\ 0 & \text{if } y(x) = 0, \end{cases} \quad \text{and } \tilde{\varphi}(x) = \begin{cases} -\frac{\varkappa_0(y(x))}{g_0 y(x)} & \text{if } y(x) \neq 0, \\ 0 & \text{if } y(x) = 0. \end{cases} \tag{42}$$

Using (37) and (41), from (42) we obtain

$$|\tilde{\phi}(x)| \leq \frac{\tilde{M}}{g_0} \text{ and } |\tilde{\varphi}(x)| \leq \frac{\tilde{\kappa}}{g_0},\ x \in [0,1]. \tag{43}$$

By (42) it follows from (10) that $\tilde{\lambda}_n$ for each $n \in \mathbb{N}$ is the kth eigenvalue of the linear Sturm-Liouville problem

$$\begin{cases} \frac{1}{\varrho\, g_0 r(x)}\, \ell(y)(x) + (\tilde{\phi}(x) + \tilde{\varphi}(x))y(x) = \lambda y(x), \\ y \in (b.c.)^{o}_{\lambda}. \end{cases} \tag{44}$$

Then it follows from [4, formula (14)] that

$$\tilde{\lambda}_n \in \left[\lambda_k^* - \frac{\tilde{M}+\tilde{\kappa}}{g_0}, \lambda_k^* + \frac{\tilde{M}+\tilde{\kappa}}{g_0}\right] = \left[\frac{\lambda_k}{\varrho g_0} - \frac{\tilde{M}+\tilde{\kappa}}{g_0}, \frac{\lambda_k}{\varrho g_0} + \frac{\tilde{M}+\tilde{\kappa}}{g_0}\right], \, n \in \mathbb{N},$$

which contradicts the relation (35). The proof of this theorem is complete. ◄

3. The existence of nodal solutions of nonlinear problem (3)-(5)

This section is dedicated to establishing the conditions under which the problem (3)-(5) has solutions contained in classes S_k^ν for some $k \in \mathbb{N}$.

By Theorem 3 it follows from Theorems 1 and 2 that for each $k \in \mathbb{N}$ and each ν the following statements hold: $C_k^\nu \subset (\mathbb{R} \times S_k^\nu)$; C_k^ν meets $I_k \times \{0\}$ and $J_k \times \{\infty\}$; the set $C_k^\nu \cup (I_k \times \{0\}) \cup (J_k \times \{\infty\})$ is connected in $\mathbb{R} \times E$. Then, as can be seen from (10), the problem (3)-(5) (or (9)) has a solution $y \in S_k^\nu$ in the case where C_k^ν crosses the hyperplane $\{1\} \times E$.

If I_k lies in the left side of 1 and J_k lies in the right side of 1 on the real axis $\mathbb{R}$, or J_k lies in the left side of 1 and I_k lies in the right side of 1, then by the above arguments the set C_k^ν crosses the hyperplane $\{1\} \times E$. Therefore, for each fixed $k \in \mathbb{N}$ and each ν the problem (3)-(5) has a solution $y_k^\nu \in S_k^\nu$ if the following condition holds:

$$\frac{\lambda_k}{\varrho\, g_0} + \frac{M}{g_0} < 1 < \frac{\lambda_k}{\varrho\, g_0} - \frac{g_\infty}{g_0} + 1 - \frac{M}{g_0} \tag{45}$$

or

$$\frac{\lambda_k}{\varrho\, g_0} - \frac{g_\infty}{g_0} + 1 + \frac{M}{g_0} < 1 < \frac{\lambda_k}{\varrho\, g_0} - \frac{M}{g_0}. \tag{46}$$

It follows from (45) that

$$0 < M + g_\infty < \frac{\lambda_k}{\varrho} < g_0 - M. \tag{47}$$

If $\lambda_k > 0$, then $\varrho > 0$. Hence by (47) we get

$$\frac{\lambda_k}{g_0 - M} < \varrho < \frac{\lambda_k}{g_\infty + M}. \tag{48}$$

If $\lambda_k < 0$, then $\varrho < 0$, and consequently, it follows from (47) that

$$\frac{\lambda_k}{g_\infty + M} < \varrho < \frac{\lambda_k}{g_0 - M}. \tag{49}$$

From (46) we obtain

$$0 < g_0 + M < \frac{\lambda_k}{\varrho} < g_\infty - M. \tag{50}$$

If $\lambda_k > 0$, then $\varrho > 0$. In this case, by (50), we have

$$\frac{\lambda_k}{g_\infty - M} < \varrho < \frac{\lambda_k}{g_0 + M} \tag{51}$$

If $\lambda_k < 0$, then $\varrho < 0$. Hence it follows from (50) that

$$\frac{\lambda_k}{g_0 + M} < \varrho < \frac{\lambda_k}{g_\infty - M}. \tag{52}$$

Thus we can formulate the main result of this paper.

Theorem 4. *Let the conditions $g_0 > M$ and $g_\infty > M$ be satisfied and for some $k \in \mathbb{N}$ the following condition hold:*

$$\lambda_k > 0 \ \textit{and} \ \frac{\lambda_k}{g_0 - M} < \varrho < \frac{\lambda_k}{g_\infty + M}, \ \textit{or}$$

$$\lambda_k > 0 \ \textit{and} \ \frac{\lambda_k}{g_\infty - M} < \varrho < \frac{\lambda_k}{g_0 + M}, \ \textit{or}$$

$$\lambda_k < 0 \ \textit{and} \ \frac{\lambda_k}{g_0 + M} < \varrho < \frac{\lambda_k}{g_\infty - M}, \ \textit{or}$$

$$\lambda_k < 0 \ \textit{and} \ \frac{\lambda_k}{g_\infty + M} < \varrho < \frac{\lambda_k}{g_0 - M}.$$

Then problem (3)-(5) *has the solutions y_k^+ and y_k^- such that $y_k^+ \in S_k^+$ and $y_k^- \in S_k^-$, respectively.*

Remark 4. If $\lambda_k = 0$ for some $k \in \mathbb{N}$, then the statement of Theorem 4 holds and is trivial. Indeed, in this case the problem (3)-(5) for $\varrho = 0$ has two solutions $y_k^+ \in S_k^+$ and $y_k^- \in S_k^-$, which correspond to the eigenvalue $\lambda_k = 0$ of problem (6).

By following the arguments in the proof of Theorem 4, we can show that the following results are also valid.

Theorem 5. *Let the conditions $g_0 > M$ and $g_\infty \leq M$ be satisfied and for some $k \in \mathbb{N}$ the following condition hold:*

$$\lambda_k > 0 \ \textit{and} \ \frac{\lambda_k}{g_0 - M} < \varrho < \frac{\lambda_k}{g_\infty + M}, \ \textit{or}$$

$$\lambda_k < 0 \ \ and \ \ \frac{\lambda_k}{g_\infty + M} < \varrho < \frac{\lambda_k}{g_0 - M}.$$

Then the statement of Theorem 4 holds.

Theorem 6. *Let the conditions $g_0 \le M$ and $g_\infty > M$ be satisfied and for some $k \in \mathbb{N}$ the following condition hold:*

$$\lambda_k > 0 \ \ and \ \ \frac{\lambda_k}{g_\infty - M} < \varrho < \frac{\lambda_k}{g_0 + M}, \ \ or$$

$$\lambda_k < 0 \ \ and \ \ \frac{\lambda_k}{g_0 + M} < \varrho < \frac{\lambda_k}{g_\infty - M}.$$

Then the statement of Theorem 4 holds.

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