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Optimal Control Problem for a Linear Pseudoparabolic Equation with Final Condition, Degeneration and Gerasimov-Caputo Operator

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Abstract. In this paper, a linear mixed optimal control problem for a fractional analogue of a pseudoparabolic differential equation with final point integral condition and degeneration is considered in rectangular domain for $0 < \alpha \leq 1$. Fractional operator is of Gerasimov-Caputo type and the differential equation involves two spatial variables. Control function has a nonlinear form and depends on time variable. The minimization of quality functional is considered. The Fourier series method is used and the necessary optimality conditions for nonlinear control are stated using the Kilbas-Saigo function. The determination of the optimal control function is reduced to the solution of a complicated functional integral equation, involving the product of two integrals. The countable system of integral equations and functional integral equations are solved by the method of successive approximations and contraction mapping. Absolute and uniform convergence of the obtained Fourier series is proved.

Key Words and Phrases: pseudoparabolic differential equation, nonlinear optimal control, degeneration, necessary conditions of optimality, minimization of the functional, Gerasimov-Caputo fractional operator.

2010 Mathematics Subject Classifications: 35B50, 35D30, 35K61, 35Q93, 35R30, 35L82

1. Introduction. Problem statement

Solving some problems of mathematical modelling of thermal processes often leads to the nonlocal inverse problems for parabolic and pseudo-parabolic equations. Inverse problems are among the most important directions of the theory of differential equations of mathematical physics. Nonlocal problems with final value conditions occur in mathematical modeling, when the initial data of the

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process flow domain is inaccessible for direct measurements. As an example, we can mention some problems of diffusion of particles in a turbulent plasma and heat transfer processes. In the process of aluminum production, the raw material passes through fire before the start of the production cycle, and the state of the raw material by the beginning of the production cycle is unknown.

The theory of optimal control for systems with distributed parameters is widely used in solving different practical problems. Many interesting methods for solving optimal control problems have been developed (see, for example, [1-15]).

When solving many problems in mathematical physics, mechanics and geometry, mixed problems for differential and integro-differential equations of parabolic and pseudo-parabolic types often occur. Therefore, a lot of research has been done in this field (see, for example [16-39]).

Interesting results in the theory and applications of fractional calculus have been obtained by many authors (see, for example, [40-45]). Different boundary value and inverse problems for fractional differential and integro-differential equations have been considered in [46-70].

In this paper, we consider the optimal control problem for a fractional order pseudoparabolic differential equation with a quadratic optimality criterion. The necessary optimality conditions are stated by using the maximum principle. The control function and state function are calculated.

In the domain $\Omega = \left\{(t,x,y)\,|\,0<t<T,\ 0<x,y<l\right\}$, we consider a partial differential equation

$$\left[{}_C D_{0t}^{\alpha} - {}_C D_{0t}^{\alpha}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - t^{\beta}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right] U(t,x,y) =$$

$$= a(t)\,U(t,x,y) + f(t,x,y,p(t)) \tag{1}$$

with a final point integral condition

$$U(T,x,y) = \varphi(x,y) + \omega \int\limits_0^T \int\limits_0^l \int\limits_0^l R(s,\eta,\xi) U(s,\eta,\xi) d\eta d\xi ds, \ \ 0 \le x,y \le l, \tag{2}$$

where $\beta,\ T$ and l are the given positive real numbers, ω is a positive real parameter, $p(t) \in C(\Omega_T)$ is a control function, for $0<\alpha\le 1$ the integral

$${}_C D_{0t}^{\alpha}\eta(t) = I_{0t}^{1-\alpha}\eta'(t) = \frac{1}{\Gamma(1-\alpha)}\int\limits_0^t \frac{\eta'(s)}{(t-s)^{\alpha}}ds, \ \ {}_C D_{0t}^{1}\eta(t) = \eta'(t), \ \ t\in(0,T)$$

is a Gerasimov-Caputo type fractional operator, the integral

$$I_{0t}^{\alpha}\eta(t)=\frac{1}{\Gamma(\alpha)}\int\limits_{0}^{t}\frac{\eta(s)ds}{(t-s)^{1-\alpha}},\quad t\in(0,T)$$

is a Riemann-Liouville integral operator, $a(t)\in C(\Omega_T)$ is a known function, $\varphi(x,y)$ is a given function, $\Omega_T\equiv[0,T],\ \Omega_l\equiv[0,l]$.

The mixed optimal control problem involves a pair of unknown functions:

$$\left\{U(t,x,y)\in C\left(\bar{\Omega}\right),\ \ p(t)\in C(\Omega_T)\right\},$$

where $\overline{\Omega}=\left\{(t,x,y)\,|\,0\leq t\leq T,\ 0\leq x,y\leq l\right\}$.

To solve the equation (1), we use the following boundary value conditions:

$$U(t,0,y)=U(t,l,y)+U(t,x,0)=U(t,x,l)=0. \tag{3}$$

We assume that the given functions satisfy the following boundary conditions:

$$\varphi(0,y)=\varphi(l,y)=\varphi(x,0)=\varphi(x,l)=0,$$

$$f(t,0,y,p)=f(t,l,y,p)=f(t,x,0,p)=f(t,x,l,p)=0.$$

Nontrivial solutions of the problem (1)-(3) are sought as a Fourier series

$$U(t,x,y)=\sum_{n,m=1}^{\infty}u_{n,m}(t)\,\vartheta_{n,m}(x,y) \tag{4}$$

and we suppose that

$$f(t,x,y,p(t))=\sum_{n,m=1}^{\infty}f_{n,m}(t,p)\,\vartheta_{n,m}(x,y), \tag{5}$$

where

$$\left[\begin{array}{l} u_{n,m}(t)=\int\limits_0^l\int\limits_0^l U(t,\eta,\xi)\,\vartheta_{n,m}(\eta,\xi)d\eta d\xi, \\ f_{n,m}(t)=\int\limits_0^l\int\limits_0^l f(t,\eta,\xi)\,\vartheta_{n,m}(\eta,\xi)d\eta d\xi, \end{array}\right. \tag{6}$$

$$\vartheta_{n,m}(x,y)=\frac{2}{l}\sin\frac{\pi\,n}{l}x\,\sin\frac{\pi\,m}{l}y,\ n,m=1,\,2,\,\ldots\,.$$

Problem. Find the control function

$$p(t)\in\left\{p:\,|\,p(t)\,|\leq M^{*},\ 0<M^{*}=\text{const},\ t\in\Omega_T\right\}$$

and the corresponding state function $U(t,x,y)$, which deliver a minimum to the functional

$$J[p]=\int_0^l\int_0^l\left[U(T,\eta,\xi)-\psi(\eta,\xi)\right]^2 d\eta d\xi+b\int_0^T p^2(t)dt, \tag{7}$$

where $\psi(x,y)$ is a given continuous function such that

$$\psi(x,y)=\sum_{n,m=1}^{\infty}\psi_{n,m}\,\vartheta_n(x,y),\ \psi_{n,m}=\int_0^l\int_0^l\psi(\eta,\xi)\vartheta_{n,m}(\eta,\xi)d\eta d\xi,$$

$$\psi(0,y)=\psi(l,y)=\psi(x,0)=\psi(x,l)=0,\ \sum_{n,m=1}^{\infty}|\,\psi_{n,m}\,|<\infty,\ 0<b=\text{const}.$$

Substituting Fourier series (4) and (5) into partial differential equation (1), we obtain a countable system of ordinary fractional differential equations of order α $(0<\alpha<1)$ with degeneration

$${}_C D_{0t}^{\alpha}u_{n,m}(t)+\mu_{n,n}^2 t^{\beta}u_{n,m}(t)=\frac{1}{1+\lambda_{n,m}^2}\left[a(t)\,u_{n,m}(t)+f_{n,m}(t,p)\right], \tag{8}$$

where

$$\mu_{n,m}^2=\frac{\lambda_{n,m}^2}{1+\lambda_{n,m}^2},\ \ \lambda_{n,m}^2=\left(\frac{\pi}{l}\right)^2(n^2+m^2).$$

We also suppose that the following Fourier expansion holds:

$$\varphi(x,y)=\sum_{n,m=1}^{\infty}\varphi_{n,m}\,\vartheta_{n,m}(x,y),$$

where

$$\varphi_{n,m}(t)=\int_0^l\int_0^l\varphi(\eta,\xi)\,\vartheta_{n,m}(\eta,\xi)d\eta d\xi. \tag{9}$$

Using the Fourier coefficients (7) and (9), we rewrite the final value integral condition (2) as follows:

$$u_{n,m}(T)=\varphi_{n,m}+\omega\int_0^T\int_0^l\int_0^l R(s,\eta,\xi)u_{n,m}(s)d\eta d\xi ds. \tag{10}$$

We use the well known Kilbas-Saigo type function, which is a generalization of the following two-parameter Mittag-Leffler function:

$$E_{\alpha,\beta}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha\, m+\beta)},\quad z,\alpha,\beta\in\mathbb{C},\quad \mathrm{Re}\,(\alpha)>0.$$

Kilbas-Saigo function is defined for real $\alpha, m, l\in\mathbb{R}$ and complex $l\in\mathbb{C}$ as follows:

$$E_{\alpha,m,l}(z)=\sum_{k=0}^{\infty}c_k z^k,\ \ c_0<1,\ \ c_k=\prod_{j=1}^{k-1}\frac{\Gamma(\alpha[jm+l]+1)}{\Gamma(\alpha[jm+l+1]+1)},\quad k=1,2,\ldots.$$

This function belongs to the class of entire functions in the complex plane.

We consider a countable system of ordinary differential equations of fractional order with degeneration

$$ {}_C D_{0t}^{\alpha} u_{n,m}(t)=-\mu_{n,m}^2\, t^{\beta} u_{n,m}(t)+g_{n,m}(t),\ \ u_{n,m}(0)=\varsigma_{n,m}, \tag{11}$$

where $\beta, \varsigma_{n,m}\in\mathbb{R},\ 0<\mu_{n,m}^2<1$,

$$g_{n,m}(t)=\frac{1}{1+\lambda_{n,m}^2}\big[a(t)\, u_{n,m}(t)+f_{n,m}(t,p)\big].$$

Let $\gamma\in[0,1)$. Then we consider the class of following functions:

$$C_{\gamma}(\Omega_T)=\big\{g_{n,m}(t):\ t^{\gamma} g_{n,m}(t)\in C(\Omega_T)\big\},$$

$$C_{\gamma}^{\alpha}(\Omega_T)=\big\{g_{n,m}(t)\in C(\Omega_T):\ {}_C D_{0t}^{\alpha}\, g_{n,m}(t)\in C_{\gamma}(\Omega_T)\big\}.$$

Lemma 1. *Let $\gamma\in[0,\alpha]$, $\beta\geq 0$. Then for all $g_{n,m}(t)\in C_{\gamma}(\Omega_T)$ there exists a unique solution $u_{n,m}(t)\in C_{\gamma}^{\alpha}(\Omega_T)$ of the Cauchy problem* (11)*. This solution has the following form:*

$$u_{n,m}(t)=\varsigma_{n,m} E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\mu_{n,m}^2\, t^{\alpha+\beta}\right)+\int_0^t K(t,\tau) g_{n,m}(\tau)d\tau, \tag{12}$$

where

$$K(t,\tau)=\sum_{i=1}^{n}K_i(t,\tau), \tag{13}$$

$$K_0(t,\tau)=\frac{1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1},\ \ K_i(t,\tau)=\frac{\mu_{n,m}^2}{\Gamma(\alpha)}\int_{\tau}^{t}s^{\beta}(t-s)^{\alpha-1}K_{i-1}(s,\tau)ds, \tag{14}$$

$i = 1, 2, \ldots,$ *and* $E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(\mu_{n,m}^2 t^{\alpha+\beta}\right)$ *is a Kilbas-Saigo function.*

Moreover, for the kernel (13) *in the case of* $\gamma \in [0, \alpha]$, $\beta \geq 0$, *the following estimate holds:*

$$|\, K(t,\tau)\,| \leq (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\mu_{n,m}^2 t^{\beta}(t-\tau)^{\alpha}\right) \leq (t-\tau)^{\alpha-1} M_3, \tag{15}$$

where $M_3 = \text{const}$.

Indeed, from (13) and (14) we obtain the validity of the following estimates:

$$|K(t,\tau)| \leq |K_0(t,\tau)| + |K_1(t,\tau)| + |K_2(t,\tau)| + \ldots + |K_k(t,\tau)| + \ldots, \tag{16}$$

$$|\, K_0(t,\tau)\,| \leq \frac{1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}, \tag{17}$$

$$|\, K_1(t,\tau)\,| \leq \frac{\mu_{n,m}^2}{\Gamma(\alpha)} \int\limits_{\tau}^{t} s^{\beta}(t-s)^{\alpha-1} \,|\, K_0(s,\tau)\,|\, ds \leq$$

$$\leq \frac{\mu_{n,m}^2}{\Gamma^2(\alpha)} \int\limits_{\tau}^{t} s^{\beta}(t-s)^{\alpha-1}(s-\tau)^{\alpha-1} ds \leq \frac{\mu_{n,m}^2 t^{\beta}}{\Gamma^2(\alpha)} \int\limits_{\tau}^{t} (t-s)^{\alpha-1}(s-\tau)^{\alpha-1} ds. \tag{18}$$

If we use the following known formula

$$\int\limits_{\tau}^{t} (t-s)^{\rho-1}(s-\tau)^{\sigma-1} ds = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}(t-\tau)^{\rho+\sigma-1} \tag{19}$$

and put $\rho = \sigma = \alpha$, then from the estimate (18) we derive

$$|\, K_1(t,\tau)\,| \leq \frac{\mu_{n,m}^2 t^{\beta}}{\Gamma^2(\alpha)} \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}(t-\tau)^{2\alpha-1} = \frac{\mu_{n,m}^2}{\Gamma(2\alpha)} t^{\beta}(t-\tau)^{2\alpha-1}. \tag{20}$$

Similarly, we estimate the next kernel $K_2(t,\tau)$:

$$|\, K_2(t,\tau)\,| \leq \frac{\mu_n^2}{\Gamma(\alpha)} \int\limits_{\tau}^{t} s^{\beta}(t-s)^{\alpha-1} \,|\, K_1(s,\tau)\,|\, ds \leq$$

$$\leq \frac{\mu_{n,m}^2}{\Gamma^2(\alpha)} \frac{\mu_{n,m}^2}{\Gamma(2\alpha)} \int\limits_{\tau}^{t} s^{2\beta}(t-s)^{\alpha-1}(s-\tau)^{2\alpha-1} ds \leq \frac{\mu_{n,m}^4 t^{2\beta}}{\Gamma(\alpha)\,\Gamma(2\alpha)} \int\limits_{\tau}^{t} (t-s)^{\alpha-1}(s-\tau)^{2\alpha-1} ds.$$

Hence, taking the estimate (20) into account, by the aid of formula (19), for $\rho=\alpha,\ \sigma=2\alpha$ we derive

$$|\,K_2(t,\tau)\,| \le \frac{\mu_{n,m}^4}{\Gamma(3\alpha)} t^{2\beta}(t-\tau)^{3\alpha-1}. \tag{21}$$

By the induction method we obtain

$$|\,K_k(t,\tau)\,| \le \frac{\mu_{n,m}^{2k}}{\Gamma((k+1)\alpha)} t^{k\beta}(t-\tau)^{(k+1)\alpha-1}. \tag{22}$$

Taking the estimates (17)-(22) into account, for (16) we have

$$|\,K(t,\tau)\,| \le \frac{1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1} + \frac{\mu_{n,m}^2}{\Gamma(2\alpha)} t^{\beta}(t-\tau)^{2\alpha-1} + \frac{\mu_{n,m}^4}{\Gamma(3\alpha)} t^{2\beta}(t-\tau)^{3\alpha-1} +$$

$$+ ... + \frac{\mu_{n,m}^{2k}}{\Gamma((k\alpha+\alpha)} t^{k\beta}(t-\tau)^{(k+1)\alpha-1} + ... =$$

$$= (t-\tau)^{\alpha-1}\left[\frac{1}{\Gamma(\alpha)} + \frac{\mu_{n,m}^2}{\Gamma(2\alpha)} t^{\beta}(t-\tau)^{\alpha} + \frac{\mu_{n,m}^4}{\Gamma(3\alpha)} t^{2\beta}(t-\tau)^{2\alpha} +\right.$$

$$\left. + ... + \frac{\mu_{n,m}^{2k}}{\Gamma((k\alpha+\alpha)} t^{k\beta}(t-\tau)^{k\alpha} + ...\right] \le (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\mu_{n,m}^2 t^{\beta}(t-\tau)^{\alpha}\right). \tag{23}$$

Further, we use the known fact that for $|\arg z| \le \sigma_0$ and $|z| \ge 0$ the following estimate is true

$$|E_{\alpha,\mu}(z)| \le \sigma_1(1+|z|)^{\frac{1-\varepsilon}{\alpha}} e^{\operatorname{Re} z^{\frac{1}{\alpha}}} + \frac{\sigma_2}{1+|z|}, \tag{24}$$

where σ_1 and σ_2 are constants, not depending on z; $\alpha < 2,\ z \in \mathbb{C},\ \varepsilon$ is a real constant and σ_0 is a fixed number from the interval $\left(\frac{\pi\alpha}{2}, \min\{\pi, \pi\alpha\}\right)$. If we put $z = \mu_{n,m}^2 t^{\beta}(t-\tau)^{\alpha},\ \varepsilon = \alpha$, then from (24) we have

$$\left|\, E_{\alpha,\alpha}\left(\mu_{n,m}^2 t^{\beta}(t-\tau)^{\alpha}\right) \right| \le \sigma_1 \left[1 + \mu_{n,m}^2 t^{\beta}(t-\tau)^{\alpha}\right]^{\frac{1-\alpha}{\alpha}} e^{\left(\mu_{n,m}^2 t^{\beta}(t-\tau)^{\alpha}\right)^{\frac{1}{\alpha}}} +$$

$$+ \frac{\sigma_2}{1+\mu_{n,m}^2 t^{\beta}(t-\tau)^{\alpha}} \le \sigma_3. \tag{25}$$

By virtue of (25), from (23) we obtain the estimate (15).

The general solution of the countable system of ordinary differential equations (12) can be written as

$$u_{n,m}(t) = C_{n,m} E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}\left(-\mu_{n,m}^2\, t^{\alpha+\beta}\right) +$$

$$+\frac{1}{1+\lambda_{n,m}^2}\int\limits_0^t K(t,s)\big[a(s)\,u_{n,m}(s)+f_{n,m}(s,p)\big]ds, \tag{26}$$

where $C_{n,m}$ is an arbitrary constant, the kernel $K(t,s)$ is defined by the formulas (13), (14) and the estimate (15) holds.

To find the unknown coefficients $C_{n,m}$ in (26), we use the condition (10). So, substituting the equation (26) into (10), we derive

$$C_{n,m}=\varphi_{n,m}Q_T^{-1}+\frac{\omega Q_T^{-1}}{1+\lambda_{n,m}^2}\times$$

$$\times\int\limits_0^T\left\{\int\limits_0^l\int\limits_0^l R(s,\eta,\xi)\int\limits_0^s K(s,\theta)f_{n,m}\left(\theta,p(\theta)\right)d\theta d\eta d\xi-K(T,s)f_{n,m}\left(s,p(s)\right)\right\}ds+$$

$$+\frac{\omega Q_T^{-1}}{1+\lambda_{n,m}^2}\int\limits_0^T\left\{\int\limits_0^l\int\limits_0^l R(s,\eta,\xi)\int\limits_0^s K(s,\theta)a(\theta)\,u_{n,m}(\theta)d\theta d\eta d\xi-\right.$$

$$\left.-K(T,s)a(s)\,u_{n,m}(s)\right\}ds, \tag{27}$$

where

$$Q_T=E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\mu_{n,m}^2T^{\alpha+\beta}\right)-$$

$$-\omega\int\limits_0^T\int\limits_0^l\int\limits_0^l R(s,\eta,\xi)E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\mu_{n,m}^2s^{\alpha+\beta}\right)d\eta d\xi ds\neq 0. \tag{28}$$

We consider the regular values of parameter ω, for which the condition (28) is fulfilled. We denote the set of these values of parameter ω by Λ.

Further, substituting (27) into equation (26), for $\omega\in\Lambda$ we derive a countable system of linear integral equations (CSLIE)

$$u_{n,m}(t)=\varphi_{n,m}Q_T^{-1}E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\mu_{n,m}^2t^{\alpha+\beta}\right)+$$

$$+\frac{1}{1+\lambda_{n,m}^2}\int\limits_0^t K(t,s)f_{n,m}(s,p)ds-\frac{\omega Q_T^{-1}}{1+\lambda_{n,m}^2}\int\limits_0^T K(T,s)f_{n,m}(s,p)ds+$$

$$+\frac{\omega Q_T^{-1}}{1+\lambda_{n,m}^2}\int\limits_0^l\int\limits_0^l\int\limits_0^T R(s,\eta,\xi)\int\limits_0^s K(s,\theta)f_{n,m}(\theta,p)d\theta ds d\eta d\xi+$$

$$+\frac{1}{1+\lambda_{n,m}^2}\int\limits_0^t K(t,s)a(s)\,u_{n,m}(s)ds-\frac{\omega Q_T^{-1}}{1+\lambda_{n,m}^2}\int\limits_0^T K(T,s)a(s)u_{n,m}(s)ds+$$

$$+\frac{\omega Q_T^{-1}}{1+\lambda_{n,m}^2}\int\limits_0^l\int\limits_0^l\int\limits_0^T R(s,\eta,\xi)\int\limits_0^s K(s,\theta)a(\theta)\,u_{n,m}(\theta)d\theta ds d\eta d\xi.$$

We take into account that

$$\int\limits_0^l\int\limits_0^l\int\limits_0^T R(s,\eta,\xi)\int\limits_0^s K(s,\theta)f_{n,m}(\theta,p)d\theta ds d\eta d\xi=$$

$$=\int\limits_0^l\int\limits_0^l\int\limits_0^T f_{n,m}(\theta,p)\int\limits_\theta^T R(s,\eta,\xi)K(s,\theta)ds d\theta d\eta d\xi=$$

$$=\int\limits_0^T f_{n,m}(s,p)\int\limits_s^T\int\limits_0^l\int\limits_0^l R(\theta,\eta,\xi)K(\theta,s)d\theta ds d\eta d\xi.$$

Then, introducing new notation, for $\omega\in\Lambda$ we have

$$u_{n,m}(t)=J_1(t;u_{n,m},p)\equiv\varphi_{n,m}Q_T^{-1}E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\mu_{n,m}^2t^{\alpha+\beta}\right)+$$

$$+\frac{1}{1+\lambda_{n,m}^2}\int\limits_0^T\bar{K}(t,s)f_{n,m}(s,p)ds+\frac{1}{1+\lambda_{n,m}^2}\int\limits_0^T\bar{K}(t,s)a(s)u_{n,m}(s)ds,\qquad(29)$$

where

$$\bar{K}(t,s)=\begin{cases}-\omega Q_T^{-1}\left[K(T,s)-\int\limits_s^T\int\limits_0^l\int\limits_0^l R(\theta,\eta,\xi)K(\theta,s)d\theta d\eta d\xi\right],\ t<s\le T,\\ K(t,s)-\omega Q_T^{-1}\left[K(T,s)-\int\limits_s^T\int\limits_0^l\int\limits_0^l R(\theta,\eta,\xi)K(\theta,s)d\theta d\eta d\xi\right],\ 0\le s<t.\end{cases}$$

Substituting the equation (29) into Fourier series (4), for $\omega\in\Lambda$ we obtain

$$U(t,x,y)=\sum_{n,m=1}^{\infty}\vartheta_{n,m}(x,y)\varphi_{n,m}Q_T^{-1}E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\mu_{n,m}^2t^{\alpha+\beta}\right)+$$

$$+\sum_{n,m=1}^{\infty}\frac{\vartheta_{n,m}(x,y)}{1+\lambda_{n,m}^2}\int_0^T \bar{K}(t,s)f_{n,m}(s,p)ds+$$

$$+\sum_{n,m=1}^{\infty}\frac{\vartheta_{n,m}(x,y)}{1+\lambda_{n,m}^2}\int_0^T \bar{K}(t,s)a(s)\,u_{n,m}(s)ds. \tag{30}$$

2. Control function

Let $p^*(t)$ be an optimal control function:

$$\Delta\,J\big[p^*(t)\big]=J\big[p^*(t)+\Delta\,p^*(t)\big]-J\big[p^*(t)\big]\geq 0,$$

where $p^*(t)+\Delta\,p^*(t)\in \bar{H}(\Omega_T)$.

With functional (7) and Fourier series (30) in mind, we consider the Pontryagin function

$$V(t,x,y)\left\{\sum_{n,m=1}^{\infty}\vartheta_{n,m}(x,y)\varphi_{n,m}Q_T^{-1}E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\mu_{n,m}^2t^{\alpha+\beta}\right)+\right.$$

$$+\sum_{n,m=1}^{\infty}\frac{\vartheta_{n,m}(x,y)}{1+\lambda_{n,m}^2}\int_0^T \bar{K}(t,s)f_{n,m}(s,p^*(s))ds+$$

$$\left.+\sum_{n,m=1}^{\infty}\frac{\vartheta_{n,m}(x,y)}{1+\lambda_{n,m}^2}\int_0^T \bar{K}(t,s)a(s)u_{n,m}(s)ds\right\}=b\,[p^*(t)]^2, \tag{31}$$

where $V(t,x,y)$ is defined by solving the following mixed problem:

$$\left[{}_CD_{0t}^{\alpha}+{}_CD_{0t}^{\alpha}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+t^{\beta}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]V(t,x,y)=0, \tag{32}$$

$$V(T,x,y)=-2\left[U(T,x,y)-\psi(x,y)\right], \tag{33}$$

$$V(t,0,y)=V(t,l,y)=V(t,x,0)=V(t,x,l)=0, \tag{34}$$

which is conjugated to problem (1)-(3).

Let us rewrite the equation (31) in convenient form:

$$V(t,x,y)\left[\Phi(t,x,y,u)+\int_0^T R_1(t,s,x,y)*f\left(s,p^*(s)\right)ds\right]=b\,[p^*(t)]^2, \tag{35}$$

where

$$\Phi(t,x,y,u)=Q_T^{-1}\sum_{n,m=1}^{\infty}\vartheta_{n,m}(x,y)\varphi_{n,m}E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\mu_{n,m}^2t^{\alpha+\beta}\right)+$$

$$+\sum_{n,m=1}^{\infty}\frac{\vartheta_{n,m}(x,y)}{1+\lambda_{n,m}^2}\int_0^T\bar{K}(t,s)a(s)\,u_{n,m}(s)ds,$$

$$\int_0^T R_1(t,s,x,y)*f\left(s,p^*(s)\right)ds=\sum_{n,m=1}^{\infty}\frac{\vartheta_{n,m}(x,y)}{1+\lambda_{n,m}^2}\int_0^T\bar{K}(t,s)f_{n,m}(s,p^*)ds.$$

According to the maximum principle, we calculate derivatives in (35) with respect to the control function and we obtain the following necessary condition for optimality

$$V(t,x,y)\int_0^T R_1(t,s,x,y)*f_p\left(s,p^*(s)\right)ds=2b\,p^*(t). \tag{36}$$

Calculating derivatives in (36) with respect to the control function $p^*(t)$, for $\omega\in\Lambda$ we obtain another necessary condition for optimality

$$V(t,x,y)\int_0^T R_1(t,s,x,y)*f_{pp}\left(s,p^*(s)\right)ds-2b<0.$$

We solve the conjugate differential equation (32) in the same way as we solved the equation (1). According to the condition (34), we find the nonzero solution of the equation (32) from the CS of fractional differential equations

$${}_CD_{0t}^{\alpha}q_{n,m}(t)=\mu_{n,m}^2t^{\beta}q_{n,m}(t), \tag{37}$$

where

$$q_{n,m}=\int_0^l\int_0^l V(\eta,\xi)\,\vartheta_{n,m}(\eta,\xi)d\eta d\xi.$$

To solve the CS of differential equations (37), we use the condition (33) in the following form

$$q_{n,m}(T)=-2\int_0^l\int_0^l\left[U(T,\eta,\xi)-\psi(\eta,\xi)\right]\vartheta_{n,m}(\eta,\xi)d\eta d\xi=2\psi_{n,m}-2u_{n,m}(T),$$

or, using the equation (29), we obtain

$$q_{n,m}(T)=2\psi_{n,m}-2\varphi_{n,m}Q_T^{-1}E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\mu_{n,m}^2T^{\alpha+\beta}\right)-$$

$$-\frac{2}{1+\lambda_{n,m}^2}\int\limits_0^T\bar{K}(t,s)f_{n,m}(s,p(s))ds-$$

$$-\frac{2}{1+\lambda_{n,m}^2}\int\limits_0^T\bar{K}(t,s)a(s)u_{n,m}(s)ds. \tag{38}$$

The general solution of the CS of homogeneous equations (32) has a form

$$q_{n,m}(t)=B_{n,m}E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(\mu_{n,m}^2t^{\alpha+\beta}\right), \tag{39}$$

where $B_{n,m}$ is an arbitrary constant. To find this constant, by virtue of the condition (33), from (38) and (39) we have

$$B_{n,m}=E^{-1}_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(\mu_{n,m}^2T^{\alpha+\beta}\right)\Bigg[2\psi_{n,m}-2\varphi_{n,m}Q_T^{-1}E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\mu_{n,m}^2T^{\alpha+\beta}\right)-$$

$$-\frac{2}{1+\lambda_{n,m}^2}\int\limits_0^T\bar{K}(t,s)f_{n,m}(s,p(s))ds-\frac{2}{1+\lambda_{n,m}^2}\int\limits_0^T\bar{K}(t,s)a(s)u_{n,m}(s)ds\Bigg]. \tag{40}$$

Substituting (40) into the solution (39), we obtain the CS

$$q_{n,m}(t)=\chi_{n,m}(t)\Bigg[2\psi_{n,m}-2\varphi_{n,m}Q_T^{-1}E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\mu_{n,m}^2T^{\alpha+\beta}\right)-$$

$$-\frac{2}{1+\lambda_{n,m}^2}\int\limits_0^T\bar{K}(t,s)f_{n,m}(s,p(s))ds-\frac{2}{1+\lambda_{n,m}^2}\int\limits_0^T\bar{K}(t,s)a(s)u_{n,m}(s)ds\Bigg], \tag{41}$$

where

$$\chi_{n,m}(t)=E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(\mu_{n,m}^2t^{\alpha+\beta}\right)E^{-1}_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(\mu_{n,m}^2T^{\alpha+\beta}\right).$$

Hence, from (41) we obtain a desired function

$$V(t,x,y)=\sum_{n,m=1}^{\infty}\vartheta_{n,m}(x,y)\chi_{n,m}(t)\Bigg[2\psi_{n,m}-2\varphi_{n,m}Q_T^{-1}E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\big(-\mu_{n,m}^2T^{\alpha+\beta}\big)-$$

$$-\frac{2}{1+\lambda_{n,m}^2}\int\limits_0^T \bar{K}(t,s)f_{n,m}(s,p(s))ds-\frac{2}{1+\lambda_{n,m}^2}\int\limits_0^T \bar{K}(t,s)a(s)u_{n,m}(s)ds\Bigg]. \quad (42)$$

We rewrite (42) in a more compact form

$$V(t,x,y)=\Psi(t,x,y,u)+\int\limits_0^T R_2(t,s,x,y)*f\left(s,p(s)\right)ds, \quad (43)$$

where

$$\Psi(t,x,y,u)=\sum_{n,m=1}^{\infty}\vartheta_{n,m}(x,y)\chi_{n,m}(t)\Bigg[2\psi_{n,m}-2\varphi_{n,m}Q_T^{-1}E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\mu_{n,m}^2T^{\alpha+\beta}\right)-$$

$$-\frac{2}{1+\lambda_{n,m}^2}\int\limits_0^T \bar{K}(t,s)a(s)u_{n,m}(s)ds\Bigg],$$

$$\int\limits_0^T R_2(t,s,x,y)*f\left(s,p(s)\right)ds=$$

$$=\sum_{n,m=1}^{\infty}\vartheta_{n,m}(x,y)\chi_{n,m}(t)\left[\frac{-2}{1+\lambda_{n,m}^2}\int\limits_0^T \bar{K}(t,s)f_{n,m}(s,p(s))ds\right].$$

From the solution (43) of conjugate problem (32)-(34) and the optimality condition (36), we obtain the integral equation, which involves the product of two integrals,

$$p(t)=J_2(t;u_{n,m},p)\equiv\frac{1}{2b}\int\limits_0^T R_1(t,s,x,y)*f_p\left(s,p(s)\right)ds\times$$

$$\times\left[\Psi(t,x,y,u)+\int\limits_0^T R_2(t,s,x,y)*f\left(s,p(s)\right)ds\right]. \quad (44)$$

There are two unknown functions in the integral equation (44): control function $p(t)$ and Fourier coefficients of main unknown function $u_{n,m}(t)$. The equation (29) also includes two unknown functions: Fourier coefficients of main unknown function $u_{n,m}(t)$ and control function $p(t)$. So, we solve these two equations together as a system of two functional integral equations.

Let $\varphi(x,y) \in C^2\left(\Omega_l^2\right)$. Then, we integrate by parts

$$\varphi_{n,m} = \int_0^l \int_0^l \varphi(\eta,\xi)\vartheta_{n,m}(\eta,\xi)d\eta d\xi$$

two times and obtain the following result

$$|\,\varphi_{n,m}\,| \leq \left(\frac{l}{\pi}\right)^4 \frac{\left|\,\varphi_{n,m}^{(IV)}\,\right|}{n^2 m^2}, \tag{45}$$

where

$$\varphi_{n,m}^{(IV)} = \int_0^l \int_0^l \frac{\partial^4 \varphi(\eta,\xi)}{\partial\eta^2 \partial\xi^2}\vartheta_{n,m}(\eta,\xi)d\eta d\xi.$$

We have also the Bessel's inequalities

$$\left\|\,\varphi^{(IV)}\,\right\|_{\ell_2} = \sqrt{\sum_{n,m=1}^{\infty}\left|\,\varphi_{n,m}^{(IV)}\,\right|^2} \leq \left(\frac{2}{l}\right)^2 \left\|\,\frac{\partial^4\varphi(x,y)}{\partial x^2 \partial y^2}\,\right\|_{L_2(\Omega_l^2)}. \tag{46}$$

Theorem 1. *Let the following conditions be satisfied:*

1) $\int_0^T \int_0^l \int_0^l |\,R(s,\eta,\xi)\,|\,d\eta d\xi ds < \infty$; $\varphi(x,y) \in C^2\left(\Omega_l^2\right)$;

2) $0 < \max\left\{\max_t \|\,f\,(t,x,y,p(t))\,\|_{L_2(\Omega_l^2)}\,,\,\max_t \|\,f_p\,(t,x,y,p(t))\,\|_{L_2(\Omega_l^2)}\right\} \leq M_1$, $0 < M_1 = \text{const}$;

3) $|\,f\,(t,x,y,p_1(t)) - f\,(t,x,y,p_2(t))\,| \leq M_2(x,y)\,|\,p_1(t)-p_2(t)\,|$, $0 < \|\,M_2(x,y)\,\|_{L_2(\Omega_l^2)}$;

4) $|\,f_p\,(t,x,y,p_1(t)) - f_p\,(t,x,y,p_2(t))\,| \leq M_3(x,y)\,|\,p_1(t)-p_2(t)\,|$, $0 < \|\,M_3(x,y)\,\|_{L_2(\Omega_l^2)}$;

5) $\rho = \rho_1 + \rho_2 < 1$, *where* ρ_1, ρ_2 *are defined by the formulas* (52) *and* (54), *respectively.*

Then the system of equations (29) *and* (42) *has a unique pair of solutions for* $\omega \in \Lambda$.

Proof. The function $p(t)$ will be sought in the space of continuous functions $C(\Omega_T)$ and the function $u_{n,m}(t)$ will be sought in the Banach space $B_2(\Omega_T)$ with a norm

$$\|\,\vec{u}(t)\,\|_{B_2(\Omega_T)} \leq \sqrt{\sum_{n,m=1}^{\infty}\left[\max_{0\leq t\leq T}|\,u_{n,m}(t)\,|\right]^2}.$$

For this system, we build iterative process:

$$\begin{cases} u_{n,m}^{k+1}(t) = J_1(t; u_{n,m}^k, p^k), \ \ u_{n,m}^0(t) = \varphi_{n,m} Q_T^{-1} E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}} \left(-\mu_{n,m}^2 t^{\alpha+\beta} \right), \\ p^{k+1}(t) = J_2(t; u_{n,m}^k, p^k), \quad p^0(t) = 0. \end{cases} \tag{47}$$

Let us estimate the zero approximation $u_{n,m}^0(t)$. We use the known estimate

$$\frac{1}{1+\Gamma(1-\alpha)x} \le E_{\alpha,m,m-1}(-x) \le \frac{1}{1+\frac{\Gamma(1+\alpha(m-1))}{\Gamma(1+\alpha m)} x},$$

which is true for every $\alpha \in [0,1]$, $m > 0$ and $x \ge 0$. We put $x = -\mu_n^2 t^{\alpha+\beta}$, $m = 1+\frac{\beta}{\alpha}$, $\Gamma(1+\alpha(m-1)) = \Gamma(1+\beta)$, $\Gamma(1+\alpha m) = \Gamma(1+\alpha+\beta)$, then we obtain

$$E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}} \left(-\mu_{n,m}^2 t^{\alpha+\beta} \right) \le \frac{1}{1+\frac{\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} \mu_{n,m}^2 t^{\alpha+\beta}} \le 1.$$

So, for Q_T we have

$$|Q_T| \le 1 + \omega \int_0^T \int_0^l \int_0^l |R(s,\eta,\xi)| \, dy ds = \frac{1}{\delta_0} < \infty, \quad \delta_0 = \text{const}.$$

Therefore, for the first approximation in (47) we have

$$\| \vec{u}^0(t) \|_{B_2(\Omega_T)} \le \sum_{n,m=1}^{\infty} \max_{0 \le t \le T} | u_{n,m}^0(t) | \le$$

$$\le \sum_{n,m=1}^{\infty} \max_{0 \le t \le T} |\varphi_{n,m}| \, |Q_T^{-1}| \, E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}} \left(-\mu_{n,m}^2 t^{\alpha+\beta} \right) \le \delta_0 \left(\frac{l}{\pi} \right)^4 \sum_{n,m}^{\infty} \frac{\left| \varphi_{n,m}^{(IV)} \right|}{n^2 m^2} \le$$

$$\le \delta_0 \left(\frac{l}{\pi} \right)^4 \sqrt{\sum_{n,m=1}^{\infty} \frac{1}{n^4 m^4}} \sqrt{\sum_{n,m=1}^{\infty} \left| \varphi_{n,m}^{(IV)} \right|^2} \le$$

$$\le \delta_0 \left(\frac{l}{\pi} \right)^4 \left(\frac{2}{l} \right)^2 \sqrt{\sum_{n,m=1}^{\infty} \frac{1}{n^4 m^4}} \left\| \frac{\partial^4 \varphi(x,y)}{\partial x^2 \partial y^2} \right\|_{L_2(\Omega_l^2)} = r_0 < \infty. \tag{48}$$

Then, by virtue of estimates (15), (45), (46), (48), applying Cauchy-Schwarz and Bessel inequalities, for the first difference in (47) we obtain

$$\| \vec{u}^1(t) - \vec{u}^0(t) \|_{B_2(\Omega_T)} \le \sqrt{\frac{l}{2}} \sum_{n,m=1}^{\infty} \frac{1}{\lambda_{n,m}^2} \max_{0 \le t \le T} \int_0^T | \bar{K}(t,s) | \, | f_{n,m}(s, p^0(s)) | \, ds +$$

$$+\sqrt{\frac{l}{2}}\sum_{n,m=1}^{\infty}\frac{1}{\lambda_{n,m}^2}\max_{0\le t\le T}\int_0^T \left|\bar{K}(t,s)\right|\left|a(s)\right|\left|u_{n,m}^0(s)\right|ds\le$$

$$\le\sqrt{\frac{l}{2}}\left(\frac{l}{\pi}\right)^4\max_{0\le t\le T}\int\limits_0^T\left|\bar{K}(t,s)\right|\sum_{n,m=1}^{\infty}\frac{1}{n^2m^2}\left|f_{n,m}(s,p^0(s))\right|ds+$$

$$+a_0\sqrt{\frac{l}{2}}\left(\frac{l}{\pi}\right)^4\max_{0\le t\le T}\int\limits_0^T\left|\bar{K}(t,s)\right|\sum_{n,m=1}^{\infty}\frac{1}{n^2m^2}\left|u_{n,m}^0(s)\right|ds\le$$

$$\le\delta_1\sqrt{\frac{l}{2}}\left(\frac{l}{\pi}\right)^4\sqrt{\sum_{n,m=1}^{\infty}\frac{1}{n^4m^4}}\left\{\left\|\vec{f}(t,0)\right\|_{B_2(\Omega_T)}+a_0\left\|\vec{u}^0(t)\right\|_{B_2(\Omega_T)}\right\}\le$$

$$\le\delta_1(M_1+a_0r_0)\sqrt{\frac{l}{2}}\left(\frac{l}{\pi}\right)^4\sqrt{\frac{2}{l}}\sqrt{\sum_{n,m=1}^{\infty}\frac{1}{n^4m^4}}=r_{1,1}<\infty,\qquad(49)$$

where

$$a_0=\max_{0\le t\le T}\left|a(t)\right|,\quad\delta_1=\max_{0\le t\le T}\int\limits_0^T\left|\bar{K}(t,s)\right|ds.$$

Similar to the estimate (49), we have

$$\left\|p^1(t)-p^0(t)\right\|_C\le$$

$$\le\frac{M_4}{2lb}\max_{0\le t\le T}\left\{\left(\frac{l}{\pi}\right)^4\max_{0\le t\le T}\int\limits_0^T\left|\bar{K}(t,s)\right|\sum_{n,m=1}^{\infty}\frac{1}{n^2m^2}\left|\left(f_{n,m}(s,p^0(s))\right)_p'\right|ds\times\right.$$

$$\times\left[\sum_{n,m=1}^{\infty}\left|\psi_{n,m}\right|+\delta_0\left(\frac{l}{\pi}\right)^4\sum_{n,m=1}^{\infty}\frac{\left|\varphi_{n,m}^{(IV)}\right|}{n^2m^2}+\right.$$

$$+a_0\left(\frac{l}{\pi}\right)^4\max_{0\le t\le T}\int\limits_0^T\left|\bar{K}(t,s)\right|\sum_{n,m=1}^{\infty}\frac{1}{n^2m^2}\left|u_{n,m}^0(s)\right|ds+$$

$$\left.\left.+\left(\frac{l}{\pi}\right)^4\max_{0\le t\le T}\int\limits_0^T\left|\bar{K}(t,s)\right|\sum_{n,m=1}^{\infty}\frac{1}{n^2m^2}\left|f_{n,m}(s,p^0(s))\right|ds\right]\right\}\le$$

$$\leq \frac{M_4}{2lb}\delta_1\left(\frac{l}{\pi}\right)^4\sqrt{\sum_{n,m=1}^{\infty}\frac{1}{n^4m^4}}\left\|\vec{f}_p(t,0)\right\|_{B_2(\Omega_T)}\times$$

$$\times\left[\sum_{n,m=1}^{\infty}|\,\psi_{n,m}\,|+\delta_0\left(\frac{l}{\pi}\right)^4\sqrt{\sum_{n,m=1}^{\infty}\frac{1}{n^4m^4}}\sqrt{\sum_{n,m=1}^{\infty}\left|\varphi_{n,m}^{(IV)}\right|^2}+\right.$$

$$+a_0\delta_1\left(\frac{l}{\pi}\right)^4\sqrt{\sum_{n,m=1}^{\infty}\frac{1}{n^4m^4}}\,\|\,\vec{u}^0(t)\,\|_{B_2(\Omega_T)}+$$

$$\left.+\delta_1\left(\frac{l}{\pi}\right)^4\sqrt{\sum_{n,m=1}^{\infty}\frac{1}{n^4m^4}}\left\|\vec{f}(t,0)\right\|_{B_2(\Omega_T)}\right]\leq\frac{M_4}{2lb}\delta_1M_1\ell_1^2\frac{2}{l}\sum_{n,m=1}^{\infty}\frac{1}{n^4m^4}\times$$

$$\times\left[\sum_{n,m=1}^{\infty}|\psi_{n,m}|+\delta_0\left(\frac{2}{l}\right)^2\left\|\frac{\partial^4\varphi(x,y)}{\partial x^2\partial y^2}\right\|_{L_2(\Omega_l^2)}+\right.$$

$$\left.+a_0r_0\delta_1+\delta_1\frac{2}{l}M_1\right]=r_{1,2}<\infty, \tag{50}$$

where

$$M_4=\max_{0\leq t\leq T}|\,\chi_{n,m}(t)\,|,\quad \ell_1=\max\left\{1;\left(\frac{l}{\pi}\right)^4\right\}.$$

Due to the estimate (49), for arbitrary difference we obtain the following estimate

$$\left\|\vec{u}^{k+1}(t)-\vec{u}^k(t)\right\|_{B_2(\Omega_T)}\leq$$

$$\leq\delta_1\left(\frac{l}{\pi}\right)^4\left|p^k(s)-p^{k-1}(s)\right|\sum_{n,m=1}^{\infty}\frac{1}{n^2m^2}\left|\int_0^l\int_0^l M_2(\eta,\xi)\vartheta_{n,m}(\eta,\xi)d\eta d\xi\right|+$$

$$+a_0\delta_1\left(\frac{l}{\pi}\right)^4\sum_{n,m=1}^{\infty}\frac{1}{n^2m^2}\left|\,u_{n,m}^k(s)-u_{n,m}^{k-1}(s)\,\right|\leq$$

$$\leq\rho_1\left[\left\|p^k(t)-p^{k-1}(t)\right\|_C+\left\|\vec{u}^k(t)-\vec{u}^{k-1}(t)\right\|_{B_2(\Omega_T)}\right], \tag{51}$$

where

$$\rho_1=\delta_1\ell_2\left(\frac{l}{\pi}\right)^4\sqrt{\sum_{n,m=1}^{\infty}\frac{1}{n^4m^4}},\quad \ell_2=\max\left\{a_0;\ \frac{2}{l}\left\|M_2(x,y)\right\|_{L_2(\Omega_l^2)}\right\}. \tag{52}$$

Similar to (51), we have

$$\left\| p^{k+1}(t) - p^k(t) \right\|_C \leq \frac{1}{2b} \max_{0 \leq t \leq T} \left| \int_0^T R_1(t,s,x,y) * f_p\left(s, p^k(s)\right) ds \times \right.$$

$$\times \left[\Psi(t,x,y,u^k) + \int_0^T R_2(t,s,x,y) * f\left(s, p^k(s)\right) ds \right] -$$

$$-\frac{1}{2b} \int_0^T R_1(t,s,x,y) * f_p\left(s, p^{k-1}(s)\right) ds \times$$

$$\left. \times \left[\Psi(t,x,y,u^{k-1}) + \int_0^T R_2(t,s,x,y) * f\left(s, p^{k-1}(s)\right) ds \right] \right| \leq$$

$$\leq \frac{1}{2b} \max_{0 \leq t \leq T} \left\{ \int_0^T \left| R_1(t,s,x,y) * \left[f_p\left(s, p^k(s)\right) - f_p\left(s, p^{k-1}(s)\right) \right] \right| ds \times \right.$$

$$\left. \times \left[\left| \Psi(t,x,y,u^k) \right| + \int_0^T \left| R_2(t,s,x,y) * f\left(s, p^k(s)\right) \right| ds \right] \right\} +$$

$$+\frac{1}{2b} \max_{0 \leq t \leq T} \left\{ \int_0^T \left| R_1(t,s,x,y) * f_p\left(s, p^{k-1}(s)\right) \right| ds \times \right.$$

$$\times \left[\left| \Psi(t,x,y,u^k) - \Psi(t,x,y,u^{k-1}) \right| + \right.$$

$$\left. \left. + \int_0^T \left| R_2(t,s,x,y) * \left[f\left(s, p^k(s)\right) - f\left(s, p^{k-1}(s)\right) \right] \right| ds \right] \right\}.$$

We continue this estimating process to obtain

$$\left\| p^{k+1}(t) - p^k(t) \right\|_C \leq$$

$$\leq \frac{2}{l^2 b} \left(\frac{l}{\pi} \right)^8 \max_{0 \leq t \leq T} \sum_{n,m=1}^{\infty} \frac{1}{n^2 m^2} \int_0^T \left| \bar{K}(t,s) \right| \left| p^k(s) - p^{k-1}(s) \right| \times$$

$$\times \left| \int_0^l \int_0^l M_3(\eta,\xi)\vartheta_{n,m}(\eta,\xi)d\eta d\xi \right| ds \left[a_0 \int_0^T \left|\bar{K}(t,s)\right| \sum_{n,m=1}^{\infty} \frac{1}{n^2m^2} \left| u_{n,m}^k(s) \right| ds + \right.$$

$$\left. + \int_0^T \left| \bar{K}(t,s) \right| \sum_{n,m=1}^{\infty} \frac{1}{n^2m^2} \left| f_{n,m}(s,p^k(s)) \right| ds \right] +$$

$$+ \frac{2}{l^2 b}\left(\frac{l}{\pi}\right)^8 \max_{0\le t\le T} \left\{ \sum_{n,m=1}^{\infty} \frac{1}{n^2m^2} \int_0^T \left| \bar{K}(t,s) \right| \left| \left(f_{n,m}(s,p^0(s))\right)_p' \right| ds \right\} \times$$

$$\times \max_{0\le t\le T} \left[a_0 \int_0^T \left| \bar{K}(t,s) \right| \sum_{n,m=1}^{\infty} \frac{1}{n^2m^2} \left| u_{n,m}^k(s) - u_{n,m}^{k-1}(s) \right| ds + \right.$$

$$\left. + \int_0^T \left|\bar{K}(t,s)\right| \left| p^k(s) - p^{k-1}(s) \right| \sum_{n,m=1}^{\infty} \frac{1}{n^2m^2} \left| \int_0^l \int_0^l M_2(\eta,\xi)\vartheta_{n,m}(\eta,\xi)d\eta d\xi \right| ds \right].$$

At the end of this process, we arrive at the following estimate:

$$\left\| p^{k+1}(t) - p^k(t) \right\|_C \le$$

$$\le \delta_1^2 \frac{2}{l^2 b}\left(\frac{l}{\pi}\right)^8 \sum_{n,m=1}^{\infty} \frac{1}{n^4m^4} \| M_2(x,y) \|_{L_2(\Omega_l^2)} \left\| p^k(t) - p^{k-1}(t) \right\|_C \times$$

$$\times \left(a_0 \left\| \vec{u}^k(t) \right\|_{B_2(\Omega_T)} + \frac{2}{l} \max_t \left\| f\left(t,x,y,p^k(t)\right) \right\|_{L_2\left(\Omega_l^2\right)} \right) +$$

$$+ \delta_1^2 \frac{4}{l^3 b}\left(\frac{l}{\pi}\right)^8 \sum_{n,m=1}^{\infty} \frac{1}{n^4m^4} \max_t \left\| f_p\left(t,x,y,p^k(t)\right) \right\|_{L_2\left(\Omega_l^2\right)} \times$$

$$\times \left(a_0 \left\| \vec{u}^k(t) - \vec{u}^{k-1}(t) \right\|_{B_2(\Omega_T)} + \frac{2}{l} \|M_2(x,y)\|_{L_2(\Omega_l^2)} \left\| p^k(t) - p^{k-1}(t) \right\|_C \right) \le$$

$$\le \delta_1^2 \frac{2}{l^2 b} \frac{l^8}{\pi^8} \sum_{n,m=1}^{\infty} \frac{1}{n^4m^4} \|M_2(x,y)\|_{L_2(\Omega_l^2)} \left(a_0 r_{2,k} + \frac{2}{l} M_1 \right) \left\| p^k(t) - p^{k-1}(t) \right\|_C +$$

$$+ \delta_1^2 \ell_2 M_1 \frac{4}{l^3 b} \frac{l^8}{\pi^8} \sum_{n,m=1}^{\infty} \frac{1}{n^4m^4} \left(\left\| \vec{u}^k(t) - \vec{u}^{k-1}(t) \right\|_{B_2(\Omega_T)} + \left\| p^k(t) - p^{k-1}(t) \right\|_C \right) \le$$

$$\le \rho_2 \left[\left\| \vec{u}^k(t) - \vec{u}^{k-1}(t) \right\|_{B_2(\Omega_T)} + \left\| p^k(t) - p^{k-1}(t) \right\|_C \right], \qquad (53)$$

where

$$\rho_2=\delta_1^2\frac{l^6}{\pi^8}\frac{2}{b}\sum_{n,m=1}^{\infty}\frac{1}{n^4m^4}\left[\| M_2(x,y)\|_{L_2(\Omega_l^2)}\left(a_0r_{2,k}+\frac{2}{l}M_1\right)+\ell_2M_1\frac{2}{l}\right], \quad (54)$$

$r_{2,k}=\text{const}$.

From the estimates (51) and (53) we obtain

$$\left\| \vec{u}^{k+1}(t)-\vec{u}^{k}(t)\right\|_{B_2(\Omega_T)}+\left\| p^{k+1}(t)-p^{k}(t)\right\|_{C}\leq$$

$$\leq\rho\cdot\left[\left\| \vec{u}^{k}(t)-\vec{u}^{k-1}(t)\right\|_{B_2(\Omega_T)}+\left\| p^{k}(t)-p^{k-1}(t)\right\|_{C}\right], \quad (55)$$

where $\rho=\rho_1+\rho_2$.

Since $\rho<1$, the estimates (48)-(51), (53) and (55) imply the existence and uniqueness of the pair of solutions of the system (29), (42): $p(t)\in C(\Omega_T)$, $\vec{u}(t)\in B_2(\Omega_T)$. Theorem 1 is proved. ◄

We consider the Fourier series (30) as a formal solution of the problem (1)-(3).

Theorem 2. *Let the conditions of Theorem 1 be satisfied. If* $p(t)\in C(\Omega_T)$, $\vec{u}(t)\in B_2(\Omega_T)$ *is a unique pair of solutions of the system of equations* (29) *and* (42), *then the series* (30), *for* $\omega\in\Lambda$, *is a unique solution to the mixed problem* (1)-(3).

The proof of convergence of the series (30) is the same as the way of obtaining the estimates (48)-(51).

Conclusion

In the domain $\Omega=\{(t,x,y)\,|\,0<t<T,\,0<x,y<l\}$, the unique solvability of mixed optimal control problem (1)-(3) with minimization of quality functional for a partial differential equation is considered in the case of α-order Gerasimov-Caputo type fractional operator, $0<\alpha\leq 1$. The solution of the optimal control problem is studied in the class of regular functions. The considered equation depends on three independent arguments. First argument is a time argument, and with respect to this argument the equation is a fractional Gerasimov-Caputo type ordinary differential equation. Final point integral condition is used. Second and third arguments are spatial and the equations with respect to these arguments are second order differential equations. The Fourier series method is used and a countable system of differential equations is obtained.

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