Azerbaijan Journal of Mathematics
V. 15, No 1, 2025, January
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2025.1.25

Generalized Hyers-Ulam Stability of n-Dimensional Quadratic Functional Equality in Modular Space and β-Homogeneous Banach Space

A. Baza, M. Rossafi*

Abstract. This paper presents the generalized Hyers-Ulam stability of the n-dimensional quadratic functional equation

$$\phi\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} \phi\left(x_i - x_j\right) = n\sum_{i=1}^{n}\phi(x_i)$$

in modular space, in β-homogeneous Banach space and in fuzzy Banach space.

Key Words and Phrases: modular spaces, Hyers-Ulam stability, Fatou property, Δ_n-condition.

2010 Mathematics Subject Classifications: 39B82, 39B52

1. Introduction and preliminaries

The research on modular and modular spaces as extensions of normed spaces was first done by Nakano [13]. Since the 1950s, numerous eminent mathematicians [3, 16, 20] have worked on it diligently. Orlicz spaces and interpolation theory are two examples of applications for modular and modular spaces in [10, 11, 16]. Now, we present the definition, properties and usual terminologies of the theory of modular spaces.

Definition 1. *Let Y be an arbitrary vector space. A functional $\rho : Y \to [0,\infty)$ is called a modular if for arbitrary $u, v \in Y$:*

1. $\rho(u) = 0$ *if and only if* $u = 0$.

*Corresponding author.

 © 2010 AZJM All rights reserved.

2. *$\rho(\alpha u)=\rho(u)$ for every scalar α with $|\alpha|=1$.*

3. *$\rho(\alpha u+\beta v)\leq \rho(u)+\rho(v)$ if and only if $\alpha+\beta=1$ and $\alpha,\beta\geq 0$.*
   If (3) is replaced by:

4. *$\rho(\alpha u+\beta v)\leq \alpha\rho(u)+\beta\rho(v)$ if and only if $\alpha+\beta=1$ and $\alpha,\beta\geq 0$, then we say that ρ is a convex modular.*

A modular ρ defines a corresponding modular space, i.e., the vector space Y_ρ given by

$$Y_\rho=\{u\in Y:\rho(\lambda u)\to 0 \text{ as } \lambda\to 0\}.$$

A function modular is said to satisfy the Δ_n-condition if there exists $\tau_n>0$ such that $\rho(nu)\leq \tau_n\rho(u)$ for all $u\in Y_\rho$.

Definition 2. *Let $\{u_n\}$ and u be in Y_ρ. Then:*

1. *The sequence $\{u_n\}$, with $u_n\in Y_\rho$, is ρ-convergent to u and we write: $u_n\to u$ if $\rho(u_n-u)\to 0$ as $n\to\infty$.*

2. *The sequence $\{u_n\}$, with $u_n\in Y_\rho$, is called ρ-Cauchy if $\rho(u_n-u_m)\to 0$ as $n:m\to\infty$.*

3. *Y_ρ is called ρ-complete if every ρ-Cauchy sequence in Y_ρ is ρ-convergent.*

Proposition 1. *In modular space,*

- *If $u_n\overset{\rho}{\to}u$ and a is a constant vector, then $u_n+a\overset{\rho}{\to}u+a$.*
- *If $u_n\overset{\rho}{\to}u$ and $v_n\overset{\rho}{\to}v$, then $\alpha u_n+\beta v_n\overset{\rho}{\to}\alpha u+\beta v$, where $\alpha+\beta\leq 1$ and $\alpha,\beta\geq 0$.*

Remark 1. *Note that $\rho(u)$ is an increasing function, for all $u\in X$. Suppose $0<a<b$. Then property (4) of Definition 1 with $v=0$ shows that $\rho(ax)=\rho\left(\frac{a}{b}bu\right)\leq\rho(bu)$ for all $u\in Y$. Moreover, if ρ is a convex modular on Y and $|\alpha|\leq 1$, then $\rho(\alpha u)\leq\alpha\rho(u)$.*

In general, if $\lambda_i\geq 0$, $i=1,\ldots,n$ and $\lambda_1,\lambda_2,\ldots,\lambda_n\leq 1$, then $\rho(\lambda_1u_1+\lambda_2u_2+\cdots+\lambda_nu_n)\leq\lambda_1\rho(u_1)+\lambda_2\rho(u_2)+\cdots+\lambda_n\rho(u_n)$.

If $\{u_n\}$ is ρ-convergent to u, then $\{cu_n\}$ is ρ-convergent to cu, where $|c|\leq 1$. But the ρ-convergence of a sequence $\{u_n\}$ to u does not imply that $\{\alpha u_n\}$ is ρ-convergent to αu for scalars α with $|\alpha|>1$.

If ρ is a convex modular satisfying Δ_n-condition with $0<\tau_n<n$, then $\rho(u)\leq\tau_n\rho(\frac{1}{n}u)\leq\frac{\tau_n}{n}\rho(u)$ for all u. Hence $\rho=0$. Consequently, we must have $\tau_n\geq n$ if ρ is a convex modular.

In many settings, the study of stability problems relies largely on functional equations. Ulam was the first to raise concerns about the stability of group homomorphisms, which allowed for the study of stability problems (see [19]). Hyers [7] solved the stability issue by examining Cauchy's functional equation in Banach spaces. Aoki [1] built on Hyers' work by supposing an infinite Cauchy difference. Rassias [17] reported work on additive mapping, and Gavruta [6] gave similar results in more detail. The general solution and Hyers-Ulam-Rassias stability of finite variable functional equations were reported by Nakmahachalasint [14] in 2007 (see also Khodaei and Rassias [9]), Najati and Moghimi [15], Kenary [18], Gordji [5], and the references therein all focused on specific problems of stability with additive functional equations. The notion of generalized Hyers-Ulam stability comes from historical contexts, and this problem might be addressed for various functional equations types. A biadditive symmetric function is connected to the functional equation (see [2, 8]). Naturally, each equation is referred to as a quadratic functional equation

$$\phi(x+y)+\phi(x-y)=2\phi(x)+2\phi(y). \tag{1}$$

In the present paper which is made up of 4 sections, we study the stability of the functional equation

$$\phi\left(\sum_{i=1}^{n} x_i\right)+\sum_{1\le i<j\le n}\phi\left(x_i-x_j\right)=n\sum_{i=1}^{n}\phi(x_i) \tag{2}$$

in modular space, with or without Δ_n-condition, in β-homogeneous Banach space and in fuzzy Banach space.

2. Stability of (2) in modular space without Δ_n-condition

Theorem 1. *Let X be a linear space, ρ be a convex modular and Y_ρ be a ρ-complete modular space. Let $\varphi : X^n \longrightarrow [0,\infty)$ be a function such as*

$$\psi\left(x_1,\ldots,x_n\right)=\sum_{j=0}^{\infty}\frac{1}{(n^2)^j}\varphi\left(n^j x_1, n^j x_2,\ldots,n^j x_n\right)<\infty \tag{3}$$

and $\lim_{n\to\infty}\frac{1}{(n^2)^k}\varphi\left(n^k x_1,\cdots,n^k x_n\right)=0$ for all $x_1,\ldots,x_n\in X$ and n be a fixed nonnegative integer with $n\ge 2$.

Let $\phi : X\to Y_\rho$ be a mapping such that

$$\rho\left(\phi\left(\sum_{i=1}^{n}x_i\right)+\sum_{1\le i<j\le n}\phi\left(x_i-x_j\right)-n\sum_{i=1}^{n}\phi\left(x_i\right)\right)\le\varphi\left(x_1,\ldots,x_n\right) \tag{4}$$

for all $x_1, \ldots, x_n \in X$. *Then there exists a unique mapping* $h : X \to Y_\rho$ *satisfying* (2) *and*

$$\rho\left(\phi(x) - \frac{n}{2(n+1)}\phi(0) - h(x)\right) \le \frac{1}{n^2}\psi(x, \ldots, x). \tag{5}$$

Proof. Letting $x_1 = x_2 = \ldots = x$ in (4), we get:

$$\rho\left(\phi(nx) + \frac{n(n-1)}{2}\phi(0) - n^2\phi(x)\right) \le \varphi(x, \ldots, x).$$

Hence

$$\rho\big(\hat{\phi}(nx) - n^2\hat{\phi}(x)\big) \le \varphi(x, \cdots, x) \tag{6}$$

where $\hat{\phi}(x) = \phi(x) - \frac{n}{2(n+1)}\phi(0)$.
We can write without using the Δ_n condition, and remarking that $\sum_{j=0}^{m-1} \frac{1}{(n^2)^{j+1}} \le 1$

$$\rho\left(\frac{1}{(n^2)^m}\hat{\phi}\left(n^m x\right) - \hat{\phi}(x)\right) = \rho\left(\sum_{j=0}^{m-1} \frac{n^2\hat{\phi}(n^j x) - \hat{\phi}(n^{j+1}x)}{(n^2)^{j+1}}\right)$$

$$\le \sum_{j=0}^{m-1} \frac{1}{(n^2)^{j+1}}\rho\left(n^2\hat{\phi}(n^j x) - \hat{\phi}(n^{j+1}x)\right) \le \frac{1}{n^2}\sum_{j=0}^{m-1} \frac{1}{(n^2)^j}\varphi(n^j x, \cdots, n^j x)$$

for all $x \in X$, and all positive integers m.
Let m and p be positive integers with $m > p$. We have

$$\rho\left(\frac{\hat{\phi}\left(n^m x\right)}{(n^2)^m} - \frac{\hat{\phi}\left(n^p x\right)}{(n^2)^p}\right) = \rho\left(\frac{1}{(n^2)^p}\left(\frac{\hat{\phi}\left(n^{m-p}\cdot n^p x\right)}{(n^2)^{m-p}} - \hat{\phi}\left(n^p x\right)\right)\right)$$

$$\le \frac{1}{n^2}\sum_{j=0}^{m-p-1} \frac{1}{(n^2)^{p+j}}\varphi\left(n^{p+j}x, \ldots, n^{p+j}x\right) = \frac{1}{n^2}\sum_{k=p}^{m-1} \frac{1}{(n^2)^k}\varphi\left(n^k x, \ldots, n^k x\right). \tag{7}$$

Then, by (7) and (3), we conclude that $\left\{\frac{\hat{\phi}\left(n^m x\right)}{(n^2)^m}\right\}_m$ is a ρ-Cauchy sequence in Y_ρ which is ρ-complete, then the sequence $\left\{\frac{\hat{\phi}(n^m x)}{(n^2)^m}\right\}_m$ is ρ-convergent to $h(x)$. Hence

$h(x) = \rho - \lim_{m\to\infty} \frac{\hat{\phi}(n^m x)}{(n^2)^m}$, i.e. $\lim_{m\longrightarrow\infty}\left(\frac{\hat{\phi}(n^m x)}{(n^2)^m} - h(x)\right) = 0$ for all $x \in X$.

Moreover, by apply the Fatou property, we get:

$$\rho(h(x) - \hat{\phi}(x)) \le \liminf_{n\longrightarrow\infty} \rho\left(\frac{\hat{\phi}(n^m x)}{(n^2)^m} - \hat{\phi}(x)\right)$$

$$\leq \frac{1}{n^2}\sum_{j=0}^{\infty}\frac{1}{(n^2)^j}\varphi(n^jx,\cdots,n^jx)=\frac{1}{n^2}\psi(x,\cdots,x).$$

Therefore, we arrive at (5). Now we prove that h satisfying (2). For all positive integer k, we note that:

$$\rho\left(\frac{2}{3n^2-n+4}\left(h\left(\sum_{i=1}^{n}x_i\right)+\sum_{1\leq i<j\leq n}h(x_i-x_j)-n\sum_{i=1}^{n}h(x_i)\right)\right)$$

$$\leq \frac{2}{3n^2-n+4}\rho\left(h\left(\sum_{i=1}^{n}x_i\right)-\frac{\hat{\phi}(n^k\sum_{i=1}^{n}x_i)}{(n^2)^k}\right)$$

$$+\frac{2}{3n^2-n+4}\sum_{1\leq i<j\leq n}\rho\left(h(x_i-x_j)-\frac{\hat{\phi}(n^k(x_i-x_j)}{(n^2)^k}\right)$$

$$+\frac{2n}{3n^2-n+4}\sum_{i=1}^{n}\rho\left(h(x_i)-\frac{\hat{\phi}(n^kx_i)}{(n^2)^k}\right)$$

$$+\frac{2}{3n^2-n+4}\rho\left(\frac{\hat{\phi}(n^k\sum_{i=1}^{n}x_i)}{(n^2)^k}+\sum_{1\leq i<j\leq n}\frac{\hat{\phi}(n^k(x_i-x_j)}{(n^2)^k}-n\sum_{i=1}^{n}\frac{\hat{\phi}(n^kx_i)}{(n^2)^k}\right)$$

$$\leq \frac{2}{3n^2-n+4}\rho\left(h\left(\sum_{i=1}^{n}x_i\right)-\frac{\hat{\phi}(n^k\sum_{i=1}^{n}x_i)}{(n^2)^k}\right)$$

$$+\frac{2}{3n^2-n+4}\sum_{1\leq i<j\leq n}\rho\left(h(x_i-x_j)-\frac{\hat{\phi}(n^k(x_i-x_j)}{(n^2)^k}\right)$$

$$+\frac{2n}{3n^2-n+4}\sum_{i=1}^{n}\rho\left(h(x_i)-\frac{\hat{\phi}(n^kx_i}{(n^2)^k}\right)$$

$$+\frac{2}{3n^2-n+4}\rho\left(\frac{\phi(n^k\sum_{i=1}^{n}x_i)}{(n^2)^k}+\sum_{1\leq i<j\leq n}\frac{\phi(n^k(x_i-x_j)}{(n^2)^k}-n\sum_{i=1}^{n}\frac{\phi(n^kx_i)}{(n^2)^k}+\frac{n^3+n^2-2n}{4(n+1)(n^2)^k}\phi(0)\right)$$

$$\leq \frac{2}{3n^2-n+4}\rho\left(h\left(\sum_{i=1}^{n}x_i\right)-\frac{\hat{\phi}(n^k\sum_{i=1}^{n}x_i)}{(n^2)^k}\right)$$

$$+\frac{2}{3n^2-n+4}\sum_{1\leq i<j\leq n}\rho\left(h(x_i-x_j)-\frac{\hat{\phi}(n^k(x_i-x_j)}{(n^2)^k}\right)$$

$$+\frac{2n}{3n^2-n+4}\sum_{i=1}^{n}\rho\left(h(x_i)-\frac{\hat{\phi}(n^kx_i)}{(n^2)^k}\right)+\frac{2}{(3n^2-n+4)(n^2)^k}\varphi\left(n^kx_1,\cdots,n^kx_n\right)$$

$$+\frac{2(n^2)^k-2}{(3n^2-n+4)(n^2)^k}\rho\left(\frac{n^3+n^2-2n}{4(n+1)((n^2)^k-1)}\phi(0)\right)\longrightarrow 0 \text{ as } k\to\infty,$$

(because $\phi(0)\in Y_\rho$ and $\lim_{k\longrightarrow\infty}\frac{n^3+n^2-2n}{4(n+1)((n^2)^k-1)}=0$. Then we have:

$$h\left(\sum_{i=1}^n x_i\right)+\sum_{1\le i<j\le n} h\left(x_i-x_j\right)-n\sum_{i=1}^n h\left(x_i\right)=0.$$

Then h is n-dimensional quadratic mapping. Finally, assume that h_1 and h_2 are n-diemensioinal mapping satisfying (5). We have

$$\rho\left(\frac{h(nx)-n^2h(x)}{n^4}\right)=\rho\left(\frac{1}{n^4}\left(h(nx)-\frac{\hat\phi\left(n^{m+1}x\right)}{\left(n^2\right)^m}\right)+\frac{1}{n^2}\left(\frac{\hat\phi\left(n^{m+1}x\right)}{\left(n^2\right)^{m+1}}-h(x)\right)\right)$$

$$\le\frac{1}{n^4}\rho\left(h(nx)-\frac{\hat\phi\left(n^{m+1}x\right)}{\left(n^2\right)^m}\right)+\frac{1}{n^2}\rho\left(\frac{\hat\phi\left(n^{m+1}x\right)}{\left(n^2\right)^{m+1}}-h(x)\right)\longrightarrow 0 \text{ as } m\to\infty.$$

Then $h(nx)=n^2h(x)$, and we have:

$$\rho\left(\frac{h_1(x)-h_2(x)}{2}\right)=\rho\left(\frac{1}{2}\left(\frac{h_1\left(n^kx\right)}{\left(n^2\right)^k}-\frac{\hat\phi\left(n^kx\right)}{\left(n^2\right)^k}\right)+\frac{1}{2}\left(\frac{\hat\phi\left(n^kx\right)}{\left(n^2\right)^k}-\frac{h_2\left(n^kx\right)}{\left(n^2\right)^k}\right)\right)$$

$$\le\frac{1}{2}\rho\left(\frac{h_1\left(n^kx\right)}{\left(n^2\right)^k}-\frac{\hat\phi\left(n^kx\right)}{\left(n^2\right)^k}\right)+\frac{1}{2}\rho\left(\frac{\hat\phi\left(n^kx\right)}{\left(n^2\right)^k}-\frac{h_2\left(n^kx\right)}{\left(n^2\right)^k}\right)\le\frac{1}{2}\cdot\frac{1}{\left(n^2\right)^k}\left\{\rho\left(h_1\left(n^kx\right)-\hat\phi\left(n^kx\right)\right)\right.$$

$$\left.+\rho\left(h_2\left(n^kx\right)-\hat\phi\left(n^kx\right)\right)\right\}\le\frac{1}{n^2}\frac{1}{\left(n^2\right)^k}\psi\left(n^kx,\dots,n^kx\right)=\sum_{j=0}^\infty\frac{1}{\left(n^2\right)^{j+k}}\varphi\left(n^{j+k}x,\dots,n^{j+k}x\right)$$

$$=\sum_{l=k}^\infty\frac{1}{\left(n^2\right)^l}\varphi\left(n^lx,\dots,n^lx\right)\to 0 \text{ as } l\to\infty.$$

This implies that $h_1=h_2$ and this completes the proof. Now, if we put $\varphi=\varepsilon>0$, we have the classical Ulam stability of (2) in modular space without Δ_n-condition. ◀

Corollary 1. *Let X be a linear space, ρ be a convex modular and Y_ρ be a ρ-complete modular space. Let $\phi: X\to Y_\rho$ be a mapping satisfying $\phi(0)=0$ and*

$$\rho\left(\phi\left(\sum_{i=1}^n x_i\right)+\sum_{1\le i<j\le n}\phi\left(x_i-x_j\right)-n\sum_{i=1}^n\phi\left(x_i\right)\right)\le\varepsilon$$

for all $x_1, \ldots, x_n \in X$ and n be a fixed positive integer with $n \geq 2$. Then there exists a unique n-dimensional quadratic mapping $h : X \to Y_\rho$ such that

$$\rho\left(\phi(x) - \frac{n}{2(n+1)}\phi(0) - h(x)\right) \leq \frac{\varepsilon}{n^2-1}\cdot; x \in X.$$

Corollary 2. *Let X be a normed linear space, ρ be a convex modular and Y_ρ be a ρ-complete convex modular space. Let $\theta > 0$ and $0 < p < 2$ be real numbers. Assume that $\phi : X \to Y_\rho$ is a mapping satisfying*

$$\rho\left(\phi\left(\sum_{i=1} nx_i\right) + \sum_{1\leq i<j\leq n} \phi(x_i - x_j) - n\sum_{i=1}^{n}\phi(x_i)\right) \leq \theta\left(\|x_1\|^p + \|x_2\|^p, \ldots, \|x_n\|^p\right)$$

for all $x_1, \ldots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $h : X \longrightarrow Y_\rho$ satisfying

$$\rho\left(\phi(x) - \frac{n}{2(n+1)}\phi(0) - h(x)\right) \leq \frac{n\theta\|x\|^p}{n^2 - n^p}.$$

3. Stability of (2) in modular space with Δ_n-condition

Theorem 2. *Let X be a linear space and Y_ρ be a ρ-complete convex modular space satisfying Δ_n-condition with $k_n \geq n$. Let $\varphi : X^n \to [0, \infty)$ be a function with*

$$\lim_{m\to\infty} k_n^{2m}\varphi\left(\frac{x_1}{n^m}, \frac{x_2}{n^m}, \cdots, \frac{x_n}{n^m}\right) = 0 \tag{8}$$

$$\sum_{i=1}^{\infty}\left(\frac{k_n^3}{n}\right)^i \varphi\left(\frac{x}{n^i}, \frac{x}{n^i}, \cdots, \frac{x}{n^i}\right) < \infty \tag{9}$$

for all $x_1, \ldots, x_n \in X$.
Let $\phi : X \to Y_\rho$ be a mapping satisfying $\phi(0) = 0$ and

$$\rho\left(\phi\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\leq i<j\leq n} \phi(x_i - x_j) - n\sum_{i=1}^{n}\phi(x_i)\right) \leq \varphi(x_n, \ldots, x_n). \tag{10}$$

Then there exists a unique n-dimensional mapping $B : X \to Y_\rho$ such that

$$\rho(\phi(x) - B(x)) \leq \frac{k_2}{2k_n^2}\sum_{i=1}^{n}\left(\frac{k_n^3}{n}\right)^j \varphi\left(\frac{x}{n^i}\cdots\frac{x}{n^i}\right), \qquad x \in X. \tag{11}$$

Proof. Letting $x_1 = x_2 = \cdots = x_n = x$ in we get: $\rho(\phi(nx) - n^2\phi(x)) \leq (x, x, \cdots, x)$, for all $x \in X$, and then it follows from the Δ_n condition and the convexity of the modular ρ that

$$\rho\left(\phi(x) - (n^2)^k\phi\left(\frac{x}{n^k}\right)\right) = \rho\left(\sum_{i=1}^{k}\frac{1}{n^i}\left(n^{3i-2}\phi\left(\frac{x}{n^{i-1}}\right)\right) - n^{3i}\phi\left(\frac{x}{n^i}\right)\right)$$
$$\leq \frac{1}{k_n^2}\sum_{i=1}^{n}\left(\frac{k_n^3}{n}\right)^i\varphi\left(\frac{x}{n^i}, \frac{x}{n^i}, \cdots, \frac{x}{n^i}\right)$$

for all $x \in X$. So for all $n, m \in \mathbb{N}$,with $n \geq m$, we have:

$$\rho\left((n^2)^m\phi\left(\frac{x}{n^m}\right) - (n^2)^{m+s}\phi\left(\frac{x}{n^{m+s}}\right)\right) \leq k_n^{2m}\rho\left(\phi\left(\frac{x}{n^m}\right) - (n^2)^s\phi\left(\frac{x}{n^{m+s}}\right)\right)$$

$$\leq k_n^{2m-2}\sum_{i=1}^{n}\left(\frac{k_n^3}{n}\right)^i\varphi\left(\frac{x}{n^{i+m}}, \cdots, \frac{x}{n^{i+m}}\right) \leq k_n^{2m-2}\sum_{j=m+1}^{m+n}\left(\frac{k_n^3}{n}\right)^{j-m}\varphi\left(\frac{x}{n^j}, \cdots, \frac{x}{n^j}\right)$$

$$\leq \frac{n^m}{k^{m+2}}\sum_{j=m+1}^{n+m}\left(\frac{k_n^3}{n}\right)^j\varphi\left(\frac{x}{n^j}, \cdots, \frac{x}{n^j}\right) \to 0 \text{ as } m \to \infty$$

because $\dfrac{n}{k_n} \leq 1$.

Thus the sequence $\left\{(n^2)^m\phi\left(\frac{x}{n^m}\right)\right\}_m$ is ρ-Cauchy sequence in Y_ρ which is ρ complete and so it is ρ-convergent to a mapping $B : X \to Y_\rho$. Then we write:

$$B(x) = \rho - \lim_{m\to\infty}(n^2)^m\phi\left(\frac{x}{n^m}\right)$$

According to the Δ_n condition, we obtain the following inequality:

$$\rho(\phi(x) - B(x)) \leq \frac{1}{2}\rho\left(2\phi(x) - 2\cdot(n^2)^m\phi\left(\frac{x}{n^m}\right)\right) + \frac{1}{2}\rho\left(2\cdot(n^2)^m\phi\left(\frac{x}{n^m}\right) - 2B(x)\right)$$
$$\leq \frac{k_2}{2}\rho\left(\phi(x) - (n^2)^m\phi\left(\frac{x}{n^m}\right)\right) + \frac{k_2}{2}\rho\left((n^2)^m\phi\left(\frac{x}{n^m}\right) - B(x)\right)$$
$$\leq \frac{k_2}{2k_n^2}\sum_{i=1}^{m}\left(\frac{k_n^3}{n}\right)^j\varphi\left(\frac{x}{n^i}\cdots\frac{x}{n^i}\right) + \frac{k_2}{2}\rho\left((n^2)^m\phi\left(\frac{x}{n^m}\right) - B(x)\right)$$

for all $x \in X$. Taking $m \to \infty$ we obtain the estimation (11). Now, we claim that the mapping B is n-dimensional quadratic. Using the Δ_n-condition we have:

$$\rho\left((n^2)^m\phi\left(\frac{\sum_{i=1}^n x_i}{n^m}\right) + (n^2)^m\sum_{1\leq i<j\leq n}\phi\left(\frac{x_i - x_j}{n^m}\right) - n(n^2)^m\sum_{i=1}^{n}\phi\left(\frac{x_i}{n^m}\right)\right)$$

$$\leq k^{2m}\varphi\left(\frac{x_1}{n^m}, \frac{x_2}{n^m}, \cdots, \frac{x_n}{n^m}\right) \to 0 \text{ as } m \to \infty$$

for all $x_1, x_2, ..., x_n \in X$. And we have:

$$\rho\left(\frac{2}{3n^2-n+4}B\left(\sum_{i=1}^{n} x_i\right) + \frac{2}{3n^2-n+4}\sum_{1\leq i<i\leq n} B\left(x_i - x_j\right) - \frac{2n}{3n^2-n+4}\sum_{i=1}^{n} B(x_i)\right)$$
$$\leq \frac{2}{3n^2-n+4}\rho\left(B\left(\sum_{i=1}^{n} x_i\right) - (n^2)^n\phi\left(\frac{\sum_{i=1}^{n} x_i}{n^m}\right)\right)$$
$$+ \frac{2}{3n^2-n+4}\sum_{1\leq i<j\leq n} \rho\left(B\left(x_i - x_j\right) - (n^2)^m\phi\left(\frac{x_i - x_j}{n^m}\right)\right)$$
$$+ \frac{2n}{3n^2-n+4}\sum_{i=1}^{n} \rho\left(B\left(x_i\right) - (n^2)^n\phi\left(\frac{x^i}{n^m}\right)\right)$$
$$+ \frac{2}{3n^2-n+4}\rho\left((n^2)^m\phi\left(\frac{\sum_{i=1}^{n} x_i}{n^m}\right) + (n^2)^m\sum_{1\leq i<j\leq n} \phi\left(\frac{x_i - x_j}{n^m}\right) - n(n^2)^m\sum_{i=1}^{n}\phi\left(\frac{x_i}{n^m}\right)\right)$$

for all $x_1, x_2, ..., x_n \in X$. Taking the limit as $m \to \infty$, one sees that B is n dimensional quadratic. To show the uniqueness of B, we assume that there exists an other n-dimensional quadratic mapping $B' : X \to Y_\rho$ which satisfies the inequality:
$\rho(\phi(x) - B'(x)) \leq \frac{k_2}{2k_n^2}\sum_{i=1}^{n}\left(\frac{k_n^3}{n}\right)^j \varphi\left(\frac{x}{n^i}\cdots\frac{x}{n^i}\right)$, $x \in X$.

We have:

$$\rho\left(B(x) - B'(x)\right) \leq \frac{1}{2}\rho\left(2\cdot(n^2)^m B\left(\frac{x}{n^m}\right) - 2(n^2)^m\phi\left(\frac{x}{n^m}\right)\right)$$
$$+ \frac{1}{2}\rho\left(2(n^2)^m\phi\left(\frac{x}{n^m}\right) - 2(n^2)^m B'\left(\frac{x}{n^m}\right)\right)$$
$$\leq \frac{k_2k_n^{2m}}{2}\rho\left(B\left(\frac{x}{n^m}\right) - \phi\left(\frac{x}{n^m}\right)\right) + \frac{k_2k_n^{2m}}{2}\rho\left(\phi\left(\frac{x}{n^m}\right) - B'\left(\frac{x}{n^m}\right)\right)$$
$$\leq \frac{k_2^2k_n^{2m}}{2k_n^2}\sum_{i=1}^{n}\left(\frac{k_n^3}{n}\right)^i \varphi\left(\frac{x}{n^{i+m}}, \frac{x}{n^{i+m}}, \cdots, \frac{x}{n^{i+m}}\right)$$
$$\leq \frac{k_2^2k_n^{2m}}{2k_n^2}\sum_{j=m+1}^{m+n}\left(\frac{k_n^3}{n}\right)^{j-m} \varphi\left(\frac{x}{n^j}, \frac{x}{n^j}, \cdots, \frac{x}{n^j}\right)$$
$$\leq \frac{k_2^2}{2}\frac{n^m}{k_n^m}\sum_{j=m+1}^{m+n}\left(\frac{k_n^3}{n}\right)^j \varphi\left(\frac{x}{n^j}, \frac{x}{n^j}, \cdots, \frac{x}{n^j}\right) \to 0 \text{ as } m \to 0.$$

Then

$$B(x) = B'(x)$$

and this complete the proof. ◄

Corollary 3. *Suppose X is a normed space with norm $\|\cdot\|$ ρ be a convex modular satisfying and Y_ρ be a ρ complete convex modular space satisfying Δ_n-condition with $k_n \geq n$. For given real numbers $\theta > 0$ and $p > \log_n \left(\frac{k_n^3}{n}\right)$. If $\varphi : X \to Y_\rho$ is a mapping such that:*

$$\rho\left(\phi\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\leq i<j\leq n} \phi\left(x_i - x_j\right) - n\sum_{i=1}^{n}\phi\left(x_i\right)\right) \leq \theta\left(\|x_1\|^P + \cdots + \|x_n\|^P\right)$$

for all $x_1, x_2, ..., x_n \in X$, then there exists a unique n-dimensional quadratic mapping $B : X \to Y_\rho$ such that

$$\rho(\phi(x) - B(x)) \leq \frac{n\theta k_n k_2}{2\left(n^{P+1} - k_n^3\right)}\|x\|^P \quad \text{for all} \;\; x \in X.$$

4. Stability of (2) in β-homogeneous spaces

Definition 3. *Let X be a linear space over $\mathbb{C}$. The application $\|\cdot\| : X \to [0, \infty)$ is an F-norm if*

1. *$\|x\| = 0$ if and only if $u = 0$,*
2. *$\|\alpha u\| = \|u\|$ for every $u \in X$ and every α with $|\alpha| = 1$,*
3. *$\|u + v\| \leq \|u\| + \|v\|$ for every $u, v \in X$,*
4. *$\|\alpha_n u\| \longrightarrow 0$, implies $\alpha_n \to 0$,*
5. *$\|\alpha u_n\| \longrightarrow 0$ implies $u_n \to 0$.*

Let $d(u, v) = \|u - v\|$. Then (X, d) is a metric space which is called F-space if d is complete.
If $\|\alpha u\| = |\alpha|^\beta\|u\|$ for all $u \in X$ and $\alpha \in \mathbb{C}$, then $\|\cdot\|$ is called β-homogeneous $(\beta > 0)$.
A β-homogeneous F-space is called a β-homogenous Banach space.

Remark 2. *If ρ is a convex modular, then*

$$\|u\|_\rho = \inf\left\{\lambda^k > 0/\rho\left(\frac{u}{\lambda}\right) \le 1\right\}, u \in Y_\rho$$

is an F-norm on Y_ρ that satisfies $\|\alpha u\|_\rho = |\alpha|^k \|u\|_\rho$. Hence, $\|\cdot\|_\rho$ is k-homogeneous. In the case $k=1$, this norm is called the Luxembourg norm.

Theorem 3. *Let X be a linear space, Y be a β-homogeneous complex Banach space $(0 < \beta \le 1)$ and $\varphi : X^n \to [0,\infty)$ be a function such that*

$$\psi(x_1,\ldots,x_n) = \sum_{j=1}^{\infty} \frac{1}{(n^{2\beta})^j}\varphi\left(n^{j-1}x_1,\ldots,n^{j-1}x_n\right) \tag{12}$$

for all $x_1,\ldots,x_n \in X$. Let $\phi : X \longrightarrow Y$ be a mapping satisfying $\phi(0)=0$ and

$$\left\|\phi\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n}\phi(x_i - x_j) - n\sum_{i=1}^{n}\phi(x_i)\right\| \le \varphi(x_1,\ldots,x_n) \tag{13}$$

for all $x_1,\ldots,x_n \in X$ and n be a fixed positive integer with $n \ge 2$. Then there exists a unique mapping $h : X \longrightarrow Y$ satisfying (2) and

$$\|\rho(\phi(x) - h(x))\| \le \psi(x,\ldots,x). \tag{14}$$

Proof. Letting $x_1 = x_2 = \cdots = x_n = x$ in (13), we get

$$\left\|\phi(nx) - n^2\phi(x)\right\| \le \varphi(x,\ldots,x).$$

Then

$$\left\|\frac{1}{n^2}\phi(nx) - \phi(x)\right\| \le \frac{1}{n^{2\beta}}\varphi(x,\ldots,x). \tag{15}$$

By induction on $k \in \mathbb{N}$, we get

$$\left\|\frac{1}{(n^2)^k}\phi\left(n^k x\right) - \phi(x)\right\| \le \sum_{j=1}^{k}\frac{1}{(n^{2\beta})^j}\varphi\left(n^{j-1}x,\ldots,n^{j-1}x\right) \tag{16}$$

for all $x \in X$ and all positive integers k. For $k=1$, we obtain (15). Assume that (16) holds for $k \in \mathbb{N}$. Then, we have

$$\left\|\frac{1}{(n^2)^{k+1}}\phi(n^{k+1}x) - \phi(x)\right\| = \|\frac{1}{n^2}\left(\frac{\phi(n^k.nx)}{(n^2)^k} - \phi(nx)\right)$$

$$+\frac{1}{n^2}\phi(nx)-\phi(x)\| \leq \frac{1}{n^{2\beta}}\| \left(\frac{\phi(n^k.nx)}{(n^2)^k}-\phi(nx)\right)\|$$

$$+\|\frac{1}{n^2}\phi(nx)-\phi(x)\| \leq \sum_{j=1}^{k}\frac{1}{(n^{2\beta})^{j+1}}\varphi\left(n^j x,\ldots,n^j x\right)$$

$$+\frac{1}{n^{2\beta}}\varphi(x,\ldots,x)=\sum_{j=1}^{k+1}\frac{1}{(n^{2\beta})^j}\varphi\left(n^{j-1}x,\ldots,n^{j-1}x\right).$$

Hence (16) holds for every $k \in \mathbb{N}$.
Let m and l be nonnegative integers with $m > l$. We have

$$\left\|\frac{\phi\left(n^m x\right)}{(n^2)^m}-\frac{\phi\left(n^l x\right)}{(n^2)^l}\right\| = \|\frac{1}{(n^2)^l}\left(\frac{\phi\left(n^{m-l}\cdot n^l x\right)}{(n^2)^{m-l}}-\phi\left(n^l x\right)\|\right)$$

$$\leq \frac{1}{n^{2l\beta}}\sum_{j=1}^{m-l}\frac{1}{n^{2\beta j}}\varphi\left(n^{j+l-1}x,\ldots,n^{j+l-1}x\right)=\sum_{k=l+1}^{m}\frac{1}{(n^2\beta)^k}\varphi\left(n^{k-1}x_1\ldots,n^{k-1}x\right). \tag{17}$$

Then, by (12) and (17), we conclude that the sequence $\left\{\frac{\phi\left(n^m x\right)}{(n^2)^m}\right\}_m$ is a Cauchy sequence in Y, which is complete. So, there exists a mapping $h : X \to Y$ defined by

$$h(x)=\lim_{m\to\infty}\frac{\phi\left(n^m x\right)}{(n^2)^m};x\in X.$$

Letting $l = 0$ and passing to the limit as $n \to \infty$ in (17), we get (14). Now, we show that h satisfies (2). We write

$$\|h\left(\sum_{i=1}^{n}x_i\right)+\sum_{1\leq i<j\leq n}h\left(x_i-x_j\right)-n\sum_{i=1}^{n}h\left(x_i\right)\| \leq \left\|h\left(\sum_{i=1}^{n}x_i\right)-\frac{\phi(n^m\sum_{i=1}^{n}x_i)}{(n^2)^m}\right\|$$

$$+\sum_{1\leq i<j\leq n}\left\|h(x_i-x_j)-\frac{\phi(n^m(x_i-x_j)}{(n^2)^m}\right\|+n^\beta\sum_{i=1}^{n}\|h(x_i)-\frac{\phi(n^m x_i)}{(n^2)^m}\|$$

$$+\|\frac{\phi(n^m\sum_{i=1}^{n}x_i)}{(n^2)^m}+\sum_{1\leq i<j\leq n}\frac{\phi(n^m(x_i-x_j))}{(n^2)^m}-n\sum_{i=1}^{n}\frac{\phi(n^m x_i)}{(n^2)^m}\| \to 0 \text{ as } m\to\infty.$$

Then, we get

$$h(\sum_{i=1}^{n}x_i)+\sum_{1\leq i<j\leq n}h(x_i-x_j)-n\sum_{i=1}^{n}h(x_i)=0$$

for all $x_1, \ldots, x_n \in X$.
Next, let h_1 and h_2 be mappings satisfying (14). We have

$$\begin{aligned}\|h_1(x) - h_2(x)\| &\leq \left\| \frac{h_1(n^m x) - \phi(n^m x)}{(n^2)^m} \right\| + \left\| \frac{h_2(n^m x) - \phi(n^m x)}{(n^2)^m} \right\| \\ &\leq \frac{2}{(n^{2\beta})^m} \sum_{j=1}^{\infty} \frac{1}{(n^{2\beta})^j} \varphi\left(n^{m+j-1}x, \ldots, n^{m+j-1}x\right) \\ &= 2 \sum_{k=m+1}^{\infty} \frac{1}{(n^{2\beta})^k} \varphi\left(n^{k-1}x, \ldots, n^{k-1}x\right) \to 0 \text{ as } m \to \infty\end{aligned}$$

for all $x \in X$. From which it follows that $h_1 = h_2$. ◄

Now, we obtain a result on classical Ulam stability of n-dimensional quadratic functional equation by putting $\varphi = \varepsilon > 0$.

Corollary 4. *Let X be a linear space, Y be a β-homogeneous complex Banach space with $0 < \beta \leq 1$. Let $\phi : X \to Y$ be a mapping satisfying $\phi(0) = 0$ and*

$$\left\| \phi\left(\sum_{i=1}^{n} x_i\right) + \sum_{1 \leq i < j \leq n} \phi(x_i - x_j) - n \sum_{i=1}^{n} \phi(x_i) \right\| \leq \varepsilon$$

for all $x_1, \ldots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $h : X \to Y$ such that

$$\|\phi(x) - h(x)\| \leq \frac{\varepsilon}{n^{2\beta} - 1}; x \in X.$$

5. Fuzz stability of (2) in fuzzy Banach space

The following theorem is a fundamental result in fixed point theory.

Theorem 4 ([4]). *Let (X, d) be a complex generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $u \in X$, either $d\left(J^n u, J^{n+1} u\right) = \infty$, for all non-negative integers n or there exists a positive integer n_0 such that*

1. *$d\left(J^n u, J^{n+1} u\right) < \infty$ for all $n \geq n_0$;*
2. *The sequence $\{J^n u\}$ converges to a fixed point v^* of J;*
3. *v^* is the unique fixed point of J in the set $Y = \{v \in X / d(J^{n_0} u, v) < \infty\}$;*
4. *$d(v, v^*) < \frac{1}{1-L} d(v, Jv)$ for all $v \in Y$.*

We use the definition of fuzzy normed spaces.

Definition 4. *Let X be a real space. A function $N : X \times R \to [0,1]$ is called a fuzzy norm on X if for all $u, v \in X$ and all $s, t \in \mathbb{R}$,*

1. $N(u,t) = 0$ *for* $t \leq 0$;
2. $u = 0$ *if and only if* $N(u,t) = 1$ *for all* $t > 0$;
3. $N(cu,t) = N\left(u, \frac{t}{|c|}\right)$ *if* $c \neq 0$;
4. $N(u+v, s+c) \geq \min\{N(u,s), N(v,c)\}$;
5. $N(u,\cdot)$ *is a non-decreasing function of* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(u,t) = 1$;
6. *for* $u \neq 0, N(u,.)$ *is continuous on* $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed space.

Example 1. *Let $(X, ||.||)$ be a normed linear space Then*

$$N(u,t) = \begin{cases} \frac{t}{t+\|u\|}; & u \in X; t > 0 \\ 0 & ; u \in X; t \leq 0 \end{cases}$$

is a fuzzy norm on X.

Definition 5. *Let (X,N) be a fuzzy normed vector space. A sequence $\{u_n\}$ in X is said to be convergent to $u \in X$ if $\lim_{n\to\infty} N(u_n - u, t) = 1$ for all $t > 0$ and we denote it by $N - \lim_{m\to\infty} u_n = u$. A sequence $\{u_n\}$ in X is called a Cauchy sequence if $\lim_{n,m\to\infty} N(u_n - u_m, t) = 1$ for all $t > 0$. The fuzzy norm is said to be complete if each Cauchy sequence is convergent, and the fuzzy normed vector space is called a fuzzy Banach space.*

Theorem 5. *Let X be a real vector space, and (Y,N) be a fuzzy Banach space. Let $\psi : X^n \to [0,\infty)$ be a function with $\psi(0,\ldots,0) = 0$ and there exist $0 < L < 1$ such that*

$$\psi(x_1,\ldots,x_n) \leq \frac{L}{n^2}\psi(nx_1,\ldots,nx_n) \tag{18}$$

for all $x_1,\ldots,x_n \in X$, where n is a fixed positive integer with $n \geq 2$.
Let $\phi : X \longrightarrow Y$ be a mapping that satisfies

$$N\left(\phi(\sum_{i=1}^{n} x_i) + \sum_{1\leq i<j\leq n} \phi(x_i - x_j) - n\sum_{i=1}^{n}\phi(x_i), t\right) \geq \frac{t}{t+\psi(x_1,\ldots,x_n)} \tag{19}$$

for all $x_1, \ldots, x_n \in X$ *and all* $t > 0$. *Then there exists a unique mapping* $h : X \to Y$ *satisfying* (2) *and*

$$N(\phi(x) - h(x)) \geq \frac{n^2(1-L)t}{n^2(1-L)t + L\psi(x, \ldots, x)}, \qquad x \in X, t > 0. \tag{20}$$

The mapping h *is defined by* $h(x) = N - \lim_{m\to\infty} \left(n^2\right)^m \phi\left(\frac{x}{n^m}\right)$.

Proof. Letting $x_1 = x_2 = \ldots = x_n = 0$ in (19), we get

$$N\left(\left(\frac{n^2+n-2}{2}\right)\phi(0), t\right) \geq 1, \qquad t > 0.$$

So $\phi(0) = 0$. Replacing $(x_1, x_2, \ldots, x_n)$ with $(x, \ldots, x)$ in (19), we get

$$N\left(\phi(nx) - n^2\phi(x), t\right) \geq \frac{t}{t + \psi(x, \ldots, x)}. \tag{21}$$

So $N\left(\phi(x) - n^2\phi(\frac{x}{n}), t\right) \geq \frac{t}{t+\psi(\frac{x}{n},\ldots,\frac{x}{n})}$ for all $x \in X$.
Consider the set $S = \{p : X \to Y\}$ and introduce the generalized metric on S:

$$d(u,v) = \inf\left\{\mu \in \mathbb{R}^+ / N(p(x) - q(x), \mu t) \geq \frac{t}{t + \psi(x, \cdots, x)}, x \in X, t > 0\right\}.$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see Lemma 2.1 in [12]).

Now we consider the linear mapping : $J : S \longrightarrow S$ such that

$$Jp(x) = n^2 p\left(\frac{x}{n}\right)$$

for all $x \in X$.

Let $p, q \in S$ be given such that $d(p,q) = \varepsilon$. Then $N\left(p(x) - q(x), \varepsilon t\right) \geq \frac{t}{t + \psi(x, \cdots, x)}$ for all $x \in X$ and all $t > 0$. Hence

$$N\left(Jp(x) - Jq(x), L\varepsilon t\right) = N\left(n^2 p\left(\frac{x}{n}\right) - n^2 q\left(\frac{x}{n}\right), L\varepsilon t\right)$$

$$= N\left(p\left(\frac{x}{n}\right) - q\left(\frac{x}{n}\right), \frac{L\varepsilon t}{n^2}\right) \geq \frac{\frac{Lt}{n^2}}{\frac{Lt}{n^2} + \psi\left(\frac{x}{n}, \cdots, \frac{x}{n}\right)} \geq \frac{t}{t + \psi(x, \cdots, x)}$$

for all $x \in X$ and all $t > 0$. So $d(p,q) = \varepsilon$ implies that $d\left(Jp, Jq\right) \leq L\varepsilon$. This means that $d(Jp, Jq) \leq Ld(p,q)$ for all $p, q \in S$.
It follows from (21) that

$$N\left(\phi(x) - n^2\phi\left(\frac{x}{n}\right), \frac{Lt}{n^2}\right) \geq \frac{\frac{Lt}{n^2}}{\frac{Lt}{n^2} + \psi\left(\frac{x}{n}, \cdots, \frac{x}{n}\right)}$$

$$\geq \frac{\frac{Lt}{n^2}}{\frac{Lt}{n^2} + \frac{L}{n^2}\psi(x, \cdots, x)} = \frac{t}{t + \psi(x, \cdots, x)}$$

for all $x \in X$ and all $t > 0$. So $d(\phi, J\phi) \leq \frac{L}{n^2}$.
By Theorem 4, there exists a mapping $h : X \longrightarrow Y$ satisfying the following:

1. h is a fixed point of J, i.e.

$$h\left(\frac{x}{n}\right) = \frac{1}{n^2}h(x), \text{ for all } x \in X.$$

The mapping h is a unique fixed point of J in the set

$$M = \{p \in S; d(\phi, p) < \infty\}$$

this implies that h is a unique mapping satisfying (5) such that there exists a $\mu \in (0, \infty)$ satisfying $N(\phi(x) - h(x), \mu t) \geq \dfrac{t}{t + \psi(x, \cdots, x)}$ for all $x \in X$.

2. $d(J^m\phi, h) \longrightarrow 0$ as $m \longrightarrow \infty$. This implies the equality

$$N - lim_{m\longrightarrow\infty}(n^2)^m\phi\left(\frac{x}{n^m}\right) = h(x) \text{ for all } x \in X.$$

3. $d(\phi, h) \leq \dfrac{1}{1-L}d(\phi, J\phi)$, which implies the inequality $d(\phi, h) \leq \dfrac{L}{n^2(1-L)}$.

This implies that the inequality (20) holds. By (19) we have

$$N\left((n^2)^m\phi\left(\frac{\sum_{i=1}^n x_i}{n^m}\right) + \left(n^2\right)^m \sum_{1\leq i<j\leq n} \phi\left(\frac{x_i - x_j}{n^m}\right) \right.$$
$$\left. -n\left(n^2\right)^m \sum_{i=1}^n \phi\left(\frac{x_i}{n^m}\right), \left(n^2\right)^m t\right) \geq \frac{t}{t + \psi\left(\frac{x_1}{n^m}, \ldots, \frac{x_n}{n^m}\right)}$$

and so

$$N\left((n^2)^m\phi\left(\frac{\sum_{i=1}^n x_i}{n^m}\right) + \left(n^2\right)^m \sum_{1\leq i<j\leq n} \phi\left(\frac{x_i - x_j}{n^m}\right) \right.$$
$$\left. -n\left(n^2\right)^m \sum_{i=1}^n \phi\left(\frac{x_i}{n^m}\right), t\right) \geq \frac{\frac{t}{(n^2)^m}}{\frac{t}{(n^2)^m} + \left(\frac{L}{n^2}\right)^m \psi\left(x_1, \ldots, x_n\right)}$$

Since

$$\lim_{m\to\infty} \frac{\frac{t}{(n^2)^m}}{\frac{t}{(n^2)^m} + \left(\frac{L}{n^2}\right)^m \psi(x_1, \ldots, x_n)} = 1,$$

it follows that

$$h\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} h(x_i - x_j) - n\sum_{i=1}^{n} h(x_i) = 0.$$

Finally, to prove the uniqueness of h, we assume that h_1 and h_2 are two mappings satisfying (20). Then

$$N(h_2(x) - h_1(x), 2t) = N\left((n^2)^m \left(h_2\left(\frac{x}{n^m}x\right) - h_1\left(\frac{x}{n^m}\right)\right), 2t\right)$$

$$\ge \min\left\{N\left(\left(n^2\right)^m\left(\phi\left(\frac{x}{n^m}\right) - h_1\left(\frac{x}{n^m}\right)\right), t\right), N\left(\left(n^2\right)^m\left(\phi\left(\frac{x}{n^m}\right) - h_2\left(\frac{x}{n^m}\right)\right), t\right)\right\}$$

$$\ge \frac{n^2(1-L)\frac{t}{(n^2)^m}}{n^2(1-L)\frac{t}{(n^2)^m} + L\psi\left(\frac{x}{n^m}, \ldots, \frac{x}{n^m}\right)} \ge \frac{n^2(1-L)\frac{t}{(n^2)^m}}{n^2(1-L)\frac{t}{(n^2)^m} + \left(\frac{L}{(n^2)^m}\right)^m \psi(x, \ldots, x)}$$

$$\longrightarrow 1 \text{ as } m \longrightarrow \infty.$$

This yields $h_1 = h_2$. ◄

Corollary 5. *Let X be a real normed space, and (Y, N) be a fuzzy Banach space. Let $\theta > 0$ and $r > 2$ be a real number. Let $\phi : X \longrightarrow Y$ be a mapping satisfying*

$$N\left(\phi\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} \phi(x_i - x_j) - n\sum_{i=1}^{n} \phi(x_i)\right) \ge \frac{t}{t + \theta\left(\|x_1\|^r + \cdots + \|x_n\|^r\right)}$$

for all $x_1, \ldots, x_n \in X$ and $t > 0$. Then there exists a unique mapping $h : X \longrightarrow Y$ satisfying (2) and

$$N(\phi(x) - h(x), t) \ge \frac{\left(n^r - n^2\right)t}{\left(n^r - n^2\right)t + \theta n\|x\|^r}, \quad x \in X \text{ and } t > 0.$$

Proof. the proof follows from Theorem 5 by taking $\psi(x_1, \cdots, x_2) = \theta\left(\|x_1\|^p, \cdots, \|x_n\|^p\right)$ for all $x_1, \cdots, x_n \in X$, and $L = n^{2-r}$. ◄

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Ipn., **2**, 1950, 64-66.

[2] J. Aczél, J. Dhombres, *Functional Equations in Secceral Variables; Encyclopedia of Mathematics and its Applications*, v.31, Cambridge University Press: Cambridge, UK, 1989.

[3] L. Amemiya, *On the representation of complemented modular lattices*, J. Math. Soc. Jpn., **9(2)**, 1957, 263-279.

[4] J. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., **74**, 1968, 305–309.

[5] M. Eshaghi Gordji, *Stability of a functional equation deriving from quartic and additive functions*, Bull. Korean Math. Soc., **47**, 2010, 491-502.

[6] P.A. Gavruţa, *Generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. AnaL. Appl., **184**, 1994, 431-436.

[7] D.H. Hyers, *On the stability of the linear functional equation*, Prac. Nat. Acad. Sci. USA, **27**, 1941, 222-224.

[8] P. Kannappan, *Quadratic functional equation and inner product spaces*, Results Math., **27**, 1995, 368-372.

[9] H. Khodaei, T.M. Rassias, *Approximately generalized additive functions in several variables*, Int. J. Nonlineur Anal. Appl., **1**, 2010, 22-41.

[10] M. Krbec, *Modular interpolation spaces*, I. Z. Anal. Anwen., **1**, 1982, 25-40.

[11] L. Maligranda, *Orlicz Spaces and Interpolation*, Seminários de Matemática, 5, Universidade Estadual de Campinas, Departamento de Matemática: Campinas, Brazil, 1989.

[12] D. Mihet, V. Radu, *On the Stability of the Additive Cauchy Functional Equation in Random Normed Spaces*, Journal of Mathematical Analysis and Applications, **343**, 2008, 567-572.

[13] H. Nakano, *Modulared Semi-Ordered Linear Spaces*, Maruzen Co., Ltd.: Tokyo, Japan, 1950.

[14] P. Nakmahachalasint, *On the Hyers-Ulam-Rassias stability of an n-dimensional additive functional equation*, Thai J. Math., **5**, 2007, 81-86.

[15] A. Najati, M.B. Moghimi, *Stability of a functional equation deriving from quadratic and additive functions in quasi-Banach spaces*, I. Math. Anal. Appl., **337**, 2008, 399-415.

[16] W. Orlicz, *Collected Papers*, Part I, II; PWN-Polish Scientific Publishers: Warsaw, Poland, 1988.

[17] T.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Am. Math. Soc., **72**, 1978, 297-300.

[18] D.Y. Shin, H.A. Kenary, N. Sahami, *Hyers-Ulam-Rassias stability of functional equation in NBS-spaces*, Int. J. Pure Appl. Math., **95**, 2014, 1-11.

[19] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, No. 8; Interscience Publishers: New York, NY, USA, 1960.

[20] S. Yamamuro, *On conjugate spaces of Nakano spaces*, Trams. Am. Math. Soc., **90**, 1959, 291-311.

Abderrahman Baza
Laboratory Partial Differential Equations, Spectral Algebra and Geometry, Departement of Mathematics, Ibn Tofail University, Kenitra, Morocco
E-mail: abderrahmane.baza@gmail.com; abderrahman.baza@uit.ac.ma

Mohamed Rossafi
Laboratory Partial Differential Equations, Spectral Algebra and Geometry, Higher School of Education and Training, Ibn Tofail University, Kenitra, Morocco
E-mail: rossafimohamed@gmail.com; mohamed.rossafi@uit.ac.ma; mohamed.rossafi1@uit.ac.ma

Received 14 September 2023
Accepted 16 February 2024