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Some Results About the Analytic Representation of Functions of the Space S_0

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Abstract. In this paper we consider the convolution of functions which are the elements of the space $S_0(\mathbb{R})$ and we generalize the results in $\mathbb{R}^n$. We deal with the functions from the spaces $S_0(\mathbb{R})$ and L^1, their convolution is an element of $S_0(\mathbb{R})$. Also, we give analytic representation for the functions of the space $S_0(\mathbb{R})$ and conclude that the convolution of the sequence of functions from $S_0(\mathbb{R})$ with another function belongs to the same space.

Key Words and Phrases: convolution, distribution, space S_0.

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1. Introduction

In the introduction part we will use general notations found in [2,4,5,6]. By $S(\mathbb{R})$ we denote the space of all rapidly decreasing functions $\varphi \in C^{\infty}(\mathbb{R})$ for which

$$\rho^1_{k,n}(\varphi) = \sup_{x\in\mathbb{R}} \left| x^k \varphi^{(n)}(x) \right| < \infty, \ \ \forall k,\, n \in \mathbb{N}_0.$$

The dual space of $S(\mathbb{R})$ is the space of tempered distributions, denoted by $S'(\mathbb{R})$.

L. Schwartz has considered the Fourier transform F of distributions in $S'(\mathbb{R})$. The space $S'(\mathbb{R})$ has the important property that the Fourier transform of distribution in $S'(\mathbb{R})$ is also a distribution in $S'(\mathbb{R})$.

If $\varphi \in S$, then the Fourier transform is

$$F\left(\varphi, z\right) = \int_{\mathbb{R}} \varphi(t) e^{itz} dt,$$

and it is an element of S.

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Also, for $\psi \in S$, the inverse Fourier transform is

$$F^{-1}(\psi, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(t) e^{-itz} dt,$$

and it is an element of the space S.

For $T \in S'$, the Fourier transform and the inverse Fourier transform are defined by $\langle F(T), \varphi \rangle = \langle T_t, F(\varphi, t) \rangle$ and $\langle F^{-1}(T), \varphi \rangle = \langle T_t F^{-1}(\varphi, t) \rangle, \varphi \in S$, respectively [3,7,8].

The function $\varphi \in L^2(\mathbb{R})$ is called a progressive (regressive) function if and only if $\operatorname{supp} \overset{\wedge}{\varphi} \subseteq (0, \infty]$ ($\operatorname{supp} \overset{\wedge}{\varphi} \subseteq [-\infty, 0)$), where $\overset{\wedge}{\varphi}(z) = F(\varphi, -2\pi z)$.

Lemma 1. *[6,7] Let $\varphi \in L^2(\mathbb{R})$ be a progressive function. Then the following conditions are equivalent:*

1. $\sup\limits_{x \in \mathbb{R}} (1 + |x|^2)^{p/2} |\varphi(x)| + \sup\limits_{w \geq 0} \frac{(1+w)^{2p+1}}{w^p} |\widehat{\varphi}(w)| < \infty, \ \forall p > 0;$

2. $\sup\limits_{x \in \mathbb{R}} (1 + |x|^2)^{p/2} |\varphi(x)| + \sup\limits_{w \geq 0} (1 + w^2)^{p/2} |\widehat{\varphi}(w)| < \infty, \ \forall p > 0.$

Definition 1. *i) Let $\varphi \in L^2(\mathbb{R})$ be a progressive function. Then $\varphi \in S_+(\mathbb{R})$ if and only if condition 1) or condition 2) from Lemma 1 is true.*

ii) $\varphi \in S_-(\mathbb{R}) \Leftrightarrow \varphi(-x) \in S_+(\mathbb{R})$.

iii) $S_0(\mathbb{R}) = S_+(\mathbb{R}) \otimes S_-(\mathbb{R})$.

The space $S_0(\mathbb{R})$ may be defined as the space of all functions of $S(\mathbb{R})$ with all their moments zero, i.e. $\varphi \in S_0(\mathbb{R})$ if and only if $\int_{\mathbb{R}} x^m \varphi(x) dx = 0, \forall m \in \mathbb{N}_0$, or $\widehat{\varphi}^{(n)}(0) = 0, \ \forall n \in \mathbb{N}_0$.

It is true that $S_0(\mathbb{R}) \subset S(\mathbb{R})$ is dense and $S_0'(\mathbb{R}) \simeq S'(\mathbb{R})/P(\mathbb{R})$, where $P(\mathbb{R})$ is the space of polynomials, is the space of Lizorkin distributions.

For $\alpha \in \mathbb{Z}^+ \cup \{0\}$, the functions $x_+^\alpha = \begin{cases} x^\alpha, & x > 0 \\ 0, & x \leq 0 \end{cases}$ and $x_-^\alpha = \begin{cases} (-x)^\alpha, & x < 0 \\ 0, & x \geq 0 \end{cases}$ define Lizorkin distributions $x_+^\alpha : \varphi \to \int_0^\infty x^\alpha \varphi(x) dx$ and $x_-^\alpha : \varphi \to \int_{-\infty}^0 (-x)^\alpha \varphi(x) dx$, $\varphi(x) \in S(\mathbb{R})$, i.e. $\langle x_+^\alpha, \varphi \rangle = \int_0^\infty x^\alpha \varphi(x) dx$and $\langle x_-^\alpha, \varphi \rangle = \int_{-\infty}^0 (-x)^\alpha \varphi(x) dx$, $\varphi(x) \in S(\mathbb{R})$.

Theorem 1. *[1, 7, 8] Let $f \in S$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}/\{0\}$. Then*

1) $F(f^{(n)}, \omega) = (-i\omega)^n F(f(\omega))$;

2) $F(f(t-a), \omega) = e^{awi} F(f(\omega))$;

3) $F(f(at),\omega) = \frac{1}{|a|}F(f(\frac{\omega}{a}))$.

Theorem 2. *[1] Let* $T \in S'$*. Then*

1) $F(T^{(n)}) = (-it)^n F(T)$;

2) $F(T) = S, \quad S^{(n)} = F((i\omega)^n T)$.

Definition 2. *Let* $f, g \in S_0$*. We define* $(f * g)(x) = \int_{\mathbb{R}^n} f(t)g(x-t)dt$.

Lemma 2. *[1, 4] Let* $f \in L^1$*. Then* $F(f,w) = \int_{\mathbb{R}^n} f(t)e^{i<t,w>}dt$ *exists, is continuous and is uniformly bounded on* $\mathbb{R}^n$.

Theorem 3. *[1, 3](Parseval's formula). Let* $f, g \in L^2$*. Then*

$$\int_{\mathbb{R}} F(f,t)g(t)dt = \int_{\mathbb{R}} f(w)F(g,w)dw,$$

$$\int_{\mathbb{R}} F^{-1}(f,t)g(t)dt = \int_{\mathbb{R}} f(w)F^{-1}(g,w)dw.$$

2. Main results

Theorem 4. *Let* $f,\ g \in S_0$*. Then* $f * g \in S_0$.

Proof. While $f \in S_0$, $\int_{\mathbb{R}} x^m f(x)dx = 0\ \forall m \in \mathbb{N}_0$ and as $g \in S_0$, $\int_{\mathbb{R}} x^m g(x)dx = 0$, $\forall m \in \mathbb{N}_0$. We have $\int_{\mathbb{R}}(f * g)(x)\, x^m\, dx = \int_{\mathbb{R}}\int_{\mathbb{R}} f(t)g(x-t)dt\, x^m dx$.
By Fubini's theorem we get

$$\int_{\mathbb{R}}(f * g)(x)\, x^m\, dx = \int_{\mathbb{R}}\int_{\mathbb{R}} f(t)g(x-t)dt\, x^m dx = \int_{\mathbb{R}} f(t)dt \int_{\mathbb{R}} x^m g(x-t)dx.$$

While

$$\int_{\mathbb{R}} x^m g(x-t)dx = \int_{\mathbb{R}} (x+t)^m g(x)dx$$

$$= \int_{\mathbb{R}} (\left(\begin{array}{c} m \\ 0 \end{array}\right) x^m + \left(\begin{array}{c} m \\ 1 \end{array}\right) x^{m-1}t + ... \left(\begin{array}{c} m \\ m \end{array}\right) x^0 t^m)g(x)dx.$$

We get $\int_{\mathbb{R}} x^m g(x-t)dx = 0 + 0 ... + 0 = 0$. ◄

Theorem 5. *Let* $f \in L^1, g \in \mathrm{S}_0$ *and* $h = f * g$*. Then* $h \in S_0$ *and the Cauchy representation* $\hat{h}(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{h(t)}{t-z}dt$, $z = x + iy$, $Imz \neq 0$, *is valid.*

Proof. For $\varphi \in D$

$$\lim_{y\to 0^+} \int_{\mathbb{R}} (\hat{h}(x+iy) - \hat{h}(x-iy))\varphi(x)dx = \lim_{y\to 0^+} \int_{\mathbb{R}} \frac{1}{2\pi i}(\int_{\mathbb{R}}(\frac{h(t)}{t-z} - \frac{h(t)}{t-\bar{z}})dt)\varphi(x)dx =$$

$$= \lim_{y\to 0^+} \int_{\mathbb{R}} \frac{1}{2\pi i}(\int_{\mathbb{R}}\int_{\mathbb{R}}(\frac{f(u)g(t-u)du}{t-z} - \frac{h(u)g(t-u)du}{t-\bar{z}})dt)\varphi(x)dx.$$

Using Holder inequality and Fubini's theorem we get

$$\lim_{y\to 0^+} \int_{\mathbb{R}} (\hat{h}(x+iy) - \hat{h}(x-iy))\varphi(x)dx =$$

$$= \lim_{y\to 0^+} \frac{1}{2\pi i} \int_{\mathbb{R}} (\frac{\varphi(x)}{t-z} - \frac{\varphi(x)}{t-\bar{z}})dx \int_{\mathbb{R}} f(u)du \int_{\mathbb{R}} g(t-u)dt$$

$$= \lim_{y\to 0^+} \frac{y}{\pi} \int_{\mathbb{R}} \frac{\varphi(x)}{|t-z|^2} dx \int_{\mathbb{R}} f(u)du \int_{\mathbb{R}} g(t-u).$$

While $\frac{y}{\pi}\int_{\mathbb{R}} \frac{\varphi(x)dx}{|t-z|^2} = \overset{\wedge}{\varphi}(t+iy)$, we have $\int_{\mathbb{R}} f(u)du \int_{\mathbb{R}} g(t-u)\overset{\wedge}{\varphi}(t+iy)dt \to \int_{\mathbb{R}} f(u)du \int_{\mathbb{R}} g(t-u)\varphi(t)dt$ as $y \to 0^+$.

Finally,

$$\lim_{y\to 0^+} \int_{\mathbb{R}} [\hat{h}(x+iy) - \hat{h}(x-iy]\varphi(x)dx =$$
$$= \int_{\mathbb{R}} f(u)g(t-u)du \int_{\mathbb{R}} \varphi(t)dt = \int_{\mathbb{R}} (f*g)(t)\varphi(t)dt = < f*g, \varphi > .$$

◀

Theorem 6. *Suppose that the sequence $\{f_n\}$ converges to $f \in S_0$ and $g \in S_0$. Then the sequence $\{l_n\} = \{f_n * g\}$ converges to $l = f * g \in S_0$.*

Proof. We have

$$\left| \int_{\mathbb{R}} x^m l_n(x)dx - \int_{\mathbb{R}} x^m l(x)dx \right| = \left| \int_{\mathbb{R}} x^m (f_n * g)(x)dx - \int_{\mathbb{R}} x^m (f*g)(x)dx \right|$$
$$= \left| \int_{\mathbb{R}} \int_{\mathbb{R}} [f_n(t)g(x-t)dt - f(t)g(x-t)dt] x^m dx \right|$$
$$= \left| \int_{\mathbb{R}} \int_{\mathbb{R}} [f_n(t) - f(t)]g(x-t)x^m dtdx \right|.$$

Using Fubini's theorem, in the last integral we may change the order of integration. Then

$$\left| \int_{\mathbb{R}} x^m l_n(x)dx - \int_{\mathbb{R}} x^m l(x)dx \right| \le \int_{\mathbb{R}} |f_n(t) - f(t)|\, dt \int_{\mathbb{R}} g(x-t)x^m dx =$$

$$= \int_{\mathbb{R}} |f_n(t) - f(t)|\, dt \cdot 0 = 0.$$

So, $f_n * g$ converges to $f * g \in S_0$. ◀

Theorem 7. *Suppose that $f \in S_0$ and consider the sequence $\{g_n\}$, $g_n \in S_0$. Then the sequence $\{l_n\}$, $l_n = f * g_n$ converges to $l = f * g$ in S_0.*

Proof. Let $l_n = f * g_n$ and $l = f * g$. Then

$$\left|\int_{\mathbb{R}} x^m\, l_n(x)dx - \int_{\mathbb{R}} x^m\, l(x)dx\right| = \left|\int_{\mathbb{R}}\int_{\mathbb{R}} x^m\, [f(t)g_n(x-t)dt - f(t)g(x-t)]dtdx\right|$$
$$\leq \int_{\mathbb{R}} \left|\int_{\mathbb{R}} x^m [f(t)g_n(x-t) - f(t)g(x-t)]dt\right| dx.$$

Applying Fubini's theorem in the last integral, we have

$$\left|\int_{\mathbb{R}} x^m\, l_n(x)dx - \int_{\mathbb{R}} x^m\, l(x)dx\right| \leq \int_{\mathbb{R}} |f(t)|\, dt \int_{\mathbb{R}} x^m\, |g_n(x-t) - g(x-t)|\, dx =$$

$$= \int_{\mathbb{R}} |f(t)|\, dt \int_{\mathbb{R}} (x+t)^m\, |g_n(x) - g(x)|\, dx \leq \int_{\mathbb{R}} |f(t)|\, dt \int_{\mathbb{R}} (x+t)^m\, |g_n(x)|\, dx$$

$$+ \int_{\mathbb{R}} |f(t)|\, dt \int_{\mathbb{R}} (x+t)^m\, |g(x)|\, dx = \int_{\mathbb{R}} |f(t)|\, (0+0) = 0.$$

So, $l_n = f * g_n$ converges to $l = f * g$ in S_0. ◀

Theorem 8. *Let $\{f_n\}$ be a sequence of functions in S_0 that converges to f in S_0 as $n \to \infty$ and let*

$$\hat{f}(z) = \frac{1}{2\pi i} \int \frac{f(t)}{t-z} dt,\ z = x + iy,\ Imz \neq 0.$$

Then $\hat{f}(z)$ is an analytic representation of $f(t)$.

Proof. Let $z = x + iy$ be a complex number such that $Imz \neq 0$. For any $\varphi \in D$ and $n \in \mathbb{N}$

$$\int_{\mathbb{R}} [\hat{f}(x+iy) - \hat{f}(x-iy)]\varphi(x)dx = \int_{\mathbb{R}} \frac{1}{2\pi i}\Big(\int_{\mathbb{R}} [\frac{\hat{f}(t)}{t-z} - \frac{\hat{f}(t)}{t-\bar{z}}]dt\Big)\varphi(x)dx.$$

On the other hand

$$\tfrac{1}{2\pi i(t-z)} = F^{-1}(H(w)e^{iwz}, t)\ ,\ \ y > 0,$$

$$\tfrac{1}{2\pi i(t-\bar{z})} = F^{-1}(H(-w)e^{iwz}, t)\ ,\ \ y < 0.$$

From the Parseval's formula, we get

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{1}{2\pi i}\Big(\int_{\mathbb{R}}\Big[\frac{f(t)}{t-z} - \frac{f(t)}{t-\bar{z}}\Big]dt\Big)\varphi(x)dx = \\
&= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(t)F^{-1}(H(w)e^{iwz}, t)dt + \int_{\mathbb{R}} f(t)F^{-1}(H(-w)e^{iw\bar{z}}, t)dt\Big)\varphi(x)dx = \\
&= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} H(w)e^{iwz}F^{-1}(f,w)dw + \int_{\mathbb{R}} H(-w)e^{iw\bar{z}}F^{-1}(f,w)dw\Big)\varphi(x)dx = \\
&= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} H(w)e^{iwx}e^{-wy}F^{-1}(f,w)dw + \int_{\mathbb{R}} H(-w)e^{iwx}e^{wy}F^{-1}(f,w)dw\Big)\varphi(x)dx = \\
&= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} e^{iwx}F^{-1}(f,w)(H(w)e^{-wy} + H(-w)e^{wy})dw\Big)\varphi(x).
\end{aligned}$$

Using Fubini's theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}} [\hat{f}(x+iy) - \hat{f}(x-iy)]\varphi(x)dx = \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x)e^{iwx}\,dx\,F^{-1}(f,w)(H(w)e^{-wy} + H(-w)e^{wy})dw = \\
&= \int_{\mathbb{R}} F(\varphi,w)F^{-1}(f,w)(H(w)e^{-wy} + H(-w)e^{wy})dw.
\end{aligned}$$

By the Lebesgue dominated convergence theorem and Parseval's formula (while $f \in S_0 \subset S \subset L^2$) we obtain

$$\begin{aligned}
&\lim_{y\to 0^+}\int_{\mathbb{R}} [\hat{f}(x+iy) - \hat{f}(x-iy)]\varphi(x)dx = \int_{\mathbb{R}} F(\varphi,w)F^{-1}(f,w)dw = \\
&= \int_{\mathbb{R}} f(t)F^{-1}(F(\varphi,t))dt = \int_{\mathbb{R}} f(t)\varphi(t)dt.
\end{aligned}$$

◀

Note 1. *Let* $< t, w >$ *denote* $t_1w_1 + t_2w_2 + ... + t_nw_n,\;\; w = (w_1, w_2, ..., w_n)$.

Theorem 9. *Let* $f, g \in S_0$. *Then* $f * g \in L^1$.

Proof. While $S_0 \subset S \subset L^1$, we have $\int_{\mathbb{R}^n} |f(x)|\,dx = C_1,\; \int_{\mathbb{R}^n} |g(x-t)|\,dx = C_2$,

$$\int_{\mathbb{R}^n} |f(t)| \int_{\mathbb{R}^n} g(x-t)dxdt = C_1 \int_{\mathbb{R}^n} |f(t)|\,dt = C_1\,C_2.$$

So, $\exists \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(t)g(x-t)|\,dx\,dt$, which implies $f * g \in L^1$. ◀

Theorem 10. *Let* $f, g \in S_0$. *Then* $F(f * g, w) = F(f, w)\,F(g, w)$.

Proof. Since $f, g \in S_0$, we have $f, g \in L^1$. Then

$$F(f * g, w) = \int_{\mathbb{R}^n} (f * g)(t)e^{i<t,w>}dt = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(m)g(t-m)dm\,e^{i<t,w>}dt.$$

Since $|f(m)g(m-t)|$ is integrable in $\mathbb{R}^n$, by Fubini's theorem we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(m)g(t-m)dm\, e^{i<t,w>}dt = \int_{\mathbb{R}^n} f(m)\int_{\mathbb{R}^n} g(t-m)e^{i<t,w>}dm\, dt =$$
$$= \int_{\mathbb{R}^n} f(m)\int_{\mathbb{R}^n} g(t)e^{i<t+m,w>}dm\, dt = \int_{\mathbb{R}^n} f(m)dm\int_{\mathbb{R}^n} g(t)e^{i<t+m,w>}dt =$$
$$= \int_{\mathbb{R}^n} f(m)e^{i<m,w>}dm\int_{\mathbb{R}^n} g(t)e^{i<t,w>}dt.$$

Consequently, $F(f*g,w) = F(f,w)\,F(g,w)$.

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