

Discrete Delayed Perturbation of Mittag-Leffler Function and Its Application to Linear Fractional Delayed Difference System

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Abstract. The linear nonhomogeneous fractional difference system with constant coefficients is introduced. An explicit solution to this system is obtained by proposing a newly discrete retarded perturbation of the nabla Mittag-Leffler-type function containing matrix equations that provide non-permutability. A couple of special cases obtained from our results are discussed.

Key Words and Phrases: discrete delayed perturbation, fractional difference, nabla Mittag-Leffler, linear system, time-delay.

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1. Introduction

In the last two decades, fractional differential equations have become a powerful tool for lots of phenomena in many branches of science such as mathematical physics [1], [2], biology, computed tomography, diffusion, biophysics, signal processing, engineering, electrochemistry, control theory [3]-[8], etc.

Retardation is mostly related to chemical processes, economics, heredity in population, hydraulic and electrical networks. Generally, a hallmark of the corresponding mathematical structure is that the rate of change of these courses depends on past history. Any differential equation consisting of at least one delay is known as a delayed differential equation. Combining a delayed differential equation with a fractional differential equation gives rise to highly realistic models for various systems having memory such as stabilization, automatic steering, control, and so on. Some articles about fractional delayed differential/difference equations can be found in [4],[9]-[16],[22],[26],[36],[41].

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As it is well-known, $z(t) = e^{Mt}z(0)$ is a solution to a linear system $z'(t) = Mz(t)$, $t \geq 0$, where e^{Mt} is the exponential matrix function. Sometimes it is not so easy to find a solution to some linear systems like the following linear delayed system:

$$\begin{cases} z'(t) = Mz(t) + Nz(t-r), & t \in (0, T], \\ z(t) = \phi(t), & t \in [-r, 0], \quad r > 0, \end{cases} \quad (1)$$

where M and N are square matrices. Under the condition that M and N are commutative, Khusainov and Shuklin [14] give a representation of solutions to system (1) by defining delayed exponential matrix function. Li and Wang [15] examine the fractional version of the same system with $M = \Theta$. Mahmudov [16] manages to obtain a solution to the delayed Caputo fractional differential system which is more general version of system (1) having a linear function by proposing delay perturbation of Mittag-Leffler (DPML) matrix function.

We observe the similar process for difference systems. Diblík and Khusainov [17] [18] focus on the linear discrete system having only one retardation and give a representation of solutions by introducing delayed discrete exponential matrix function e_h^{Nk} . Diblík and Morávková [20],[21] find a representation of solutions of linear two-retarded discrete systems by extending the delayed discrete exponential matrix function to the delayed discrete exponential matrix function for two delays. Pospíšil [22] concentrates on the linear discrete multi-delayed system with commutative coefficient matrices and solves this system by using Z -transform. Mahmudov [23] removes the commutativity in the same linear discrete multi-delayed system by proposing the delayed perturbation of discrete exponential matrix function $X_h^{M,N}(k)$ with the same method.

Jia et al. in the work [24] prove that the M-L function $\mathbb{E}_{b,\beta,\beta-1}(k, \rho(a)) = \sum_{i=0}^{\infty} b^i H_{i\beta+\beta-1}(k, \rho(a))$ is the unique solution to the following nabla Riemann-Liouville fractional difference equation:

$$\begin{cases} \nabla_{\rho(a)}^{\beta} z(k) = bz(k), & \beta \in (0, 1), \quad k \in \mathbb{N}_{a+1}, \\ z(a) = \frac{1}{1-b} > 0. \end{cases} \quad (2)$$

Jia et al. in [25] prove that the M-L function $\mathbb{E}_{b,\beta,0}(k, a) = \sum_{i=0}^{\infty} b^i H_{i\beta}(k, a)$ is the unique solution to the following nabla Caputo fractional difference equation:

$$\begin{cases} {}^C \nabla_a^{\beta} z(k) = bz(k), & \beta \in (0, 1), \quad k \in \mathbb{N}_{a+1}, \\ z(a) = 1. \end{cases} \quad (3)$$

Du and Lu in [26] consider the nonhomogeneous delayed Riemann-Liouville fractional difference system

$$\begin{cases} \nabla_{-r}^{\alpha} z(k) = Nz(k-r) + \mathfrak{T}(k), & k \in \mathbb{N}_1, \\ z(k) = \phi(k), & k \in \mathbb{N}_{1-r}^0, \quad r > 0, \end{cases} \quad (4)$$

present representation of solutions to system (4) by proposing the discrete delayed M-L type matrix function $\mathbb{F}_r^{Nk^\alpha}$, and examine its finite-time stability.

As stated in [27], the Riemann-Liouville fractional derivative has a widespread usage in the real-world problems. In the recent times, the Riemann-Liouville fractional derivative presents an excellent tool to express anomalous diffusion, Levy flights, and so forth. The set of all functions which satisfy the definition of Riemann-Liouville fractional derivative is bigger than that of ones which fulfill the definition of the Caputo fractional derivative. Moreover, the RL fractional derivative is closer to the classical one than the Caputo one in terms of their features and similarities.

Motivated by the above-cited works and the aforementioned superiorities of both the fractional-order derivative and the one in the sense of Riemann-Liouville, this paper is dedicated to the exploration of the following linear retarded Riemann-Liouville fractional difference system:

$$\begin{cases} \nabla_{-r}^\alpha z(k) = Mz(k) + Nz(k-r) + \mathfrak{T}(k), & k \in \mathbb{N}_1, \\ z(k) = \phi(k), & k \in \mathbb{N}_{1-r}^0, \quad r > 0, \end{cases} \quad (5)$$

where ∇_{-r}^α is the Riemann-Liouville fractional difference of order $0 < \alpha < 1$, $z : \mathbb{N}_1 \rightarrow \mathbb{R}^n$, $\mathfrak{T} : \mathbb{N}_1 \rightarrow \mathbb{R}^n$ is a continuous function, $r \in \mathbb{N}_2$ is a retardation, $M, N \in \mathbb{R}^{n \times n}$ are constant coefficient matrices, $\phi : \mathbb{N}_{1-r}^0 \rightarrow \mathbb{R}^n$ is an initial conditional function.

The major contributions are highlighted below:

- we introduce the nabla Riemann-Liouville fractional delayed difference system with noncommutative coefficient matrices (see Section 2);
- we newly define the discrete delay perturbation of the nabla M-L type matrix function, investigate some of its features, establish a couple of its relations with the available ones in the literature;
- we search for a representation of solutions to the nabla Riemann-Liouville fractional delayed difference system with noncommutative coefficient matrices in the homogeneous and nonhomogeneous forms (see Section 4);
- we share some valuable results (see Section 5);
- we state some open problems (see Section 6).

2. Preliminaries

In this section, we present the available tools in the literature.

$\mathbb{N}^a = \{\dots, a-2, a-1, a\}$, $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$, $\mathbb{N}_a^b = \{a, a+1, a+2, \dots, b\}$ where $a, b \in \mathbb{R}$ with $b-a \in \mathbb{N}_1$.

Definition 1. [28, Definition 3.4] The generalized rising function is defined by $k^{\bar{r}} = \frac{\Gamma(k+r)}{\Gamma(k)}$, whenever the right-hand side of this equation is sensible for those values of k and r .

Definition 2. [28, Definition 3.56] Assume that $\alpha \notin \mathbb{N}^{-1}$. The α -th order (nabla) fractional Taylor monomial $H_\alpha(k, a)$ is defined by $H_\alpha(k, a) = \frac{(k-a)^\alpha}{\Gamma(\alpha+1)}$, where the right-hand side of the above equation is sensible.

In the following definitions, we present the (nabla) Leibniz formula, the (nabla) fractional sum with respect to the (nabla) fractional Taylor monomial and the (nabla) Riemann-Liouville fractional difference, respectively.

Theorem 1. [28, Theorem 3.41] Assume $z : \mathbb{N}_a \times \mathbb{N}_{a+1} \rightarrow \mathbb{R}$. Then

$$\nabla \left(\int_a^k z(k, s) \nabla s \right) = \int_a^k \nabla_k z(k, s) \nabla s + z(\rho(k), k), \quad k \in \mathbb{N}_{a+1}.$$

Definition 3. [28, Definition 3.58] Let $z : \mathbb{N}_{a+1} \rightarrow \mathbb{R}$ and $\alpha > 0$. For $k \in \mathbb{N}_a$, and $\rho(s) = s-1$, the (nabla) fractional sum is given by

$$\nabla_a^{-\alpha} z(k) = \int_a^k H_{\alpha-1}(k, \rho(s)) z(s) \nabla s = \sum_{s=a+1}^k H_{\alpha-1}(k, \rho(s)) z(s).$$

Definition 4. [28, Definition 3.61, Theorem 3.62] Let $z : \mathbb{N}_{a+1} \rightarrow \mathbb{R}$. For $k \in \mathbb{N}_a$, the (nabla) Riemann-Liouville fractional difference of order $0 < \alpha < 1$ is given by

$$\nabla_a^\alpha z(k) = \nabla \nabla_a^{-(1-\alpha)} z(k) = \int_a^k H_{-\alpha-1}(k, \rho(s)) z(s) \nabla s = \sum_{s=a+1}^k H_{-\alpha-1}(k, \rho(s)) z(s).$$

The composition of two different (nabla) fractional sums is given in the below theorem.

Theorem 2. [28, Theorem 3.107] Let $z : \mathbb{N}_{a+1} \rightarrow \mathbb{R}$ and $\alpha, \beta > 0$. Then

$$\nabla_a^{-\alpha} \nabla_a^{-\beta} z(k) = \nabla_a^{-\alpha-\beta} z(k), \quad k \in \mathbb{N}_a.$$

Lemma 1. [28] Let $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $\beta-1$ and $\alpha+\beta-1$ are nonnegative integers. Then

$$\nabla_a^{-\alpha} H_{\beta-1}(k, a) = H_{\alpha+\beta-1}(k, a), \quad k \in \mathbb{N}_a,$$

and

$$\nabla_a^\alpha H_{\beta-1}(k, a) = H_{\alpha-\beta-1}(k, a), \quad k \in \mathbb{N}_a,$$

and for $0 < \beta < 1$ and $t \in \mathbb{N}_{a+1}$,

$$\nabla_a^{-\beta} \nabla_{\rho(a)}^\beta z(k) = z(k) - H_{\beta-1}(k, \rho(a))z(a).$$

3. Discrete DPML function

In this section, the discrete DPML function is presented to obtain the exact analytical solution formulas for the homogeneous and nonhomogeneous delayed Riemann-Liouville fractional difference systems. Some of its properties to be used in the forthcoming proofs are discussed.

For noncommutative coefficient constant matrices, the perturbational matrix equation embedded in the discrete DPML matrix function is used in the following definition.

Definition 5. The discrete DPML matrix function $\mathbb{D}_{\alpha, \beta, r}^{M, N}$ generated by M, N is defined as follows:

$$\mathbb{D}_{\alpha, \beta, r}^{M, N}(k) := \begin{cases} \Theta, & k \in \mathbb{N}^{-r-1}, \\ I, & k = -r, \\ \sum_{i=0}^{\infty} A^i \frac{(k+r)^{\overline{i\alpha+\beta-1}}}{\Gamma(i\alpha+\beta)}, & k \in \mathbb{N}_{1-r}^0, \\ \sum_{i=0}^{\infty} \sum_{j=0}^p Q(i+1, j) \frac{(k-(j-1)r)^{\overline{i\alpha+\beta-1}}}{\Gamma(i\alpha+\beta)}, & k \in \mathbb{N}_{(p-1)r+1}^{pr}, \end{cases} \quad (6)$$

where Θ and I are the zero and identity matrices, respectively, and the matrix equation $Q(i, j)$ has the following recursive form:

$$Q(i+1, j) = MQ(i, j) + NQ(i, j-1), \quad (7)$$

and

$$Q(0, j) = Q(i, -1) = \Theta, \quad Q(1, 0) = I, \quad (8)$$

for $i, j \in \mathbb{N}_0$.

Remark 1. Note that $Q(i, j)$ was used in [19] and [29] to define delayed perturbation of discrete matrix exponential and DPML matrix function, respectively. By using the above recursive equation, one can easily obtain the explicit form in the below table:

	$j = 0$	$j = 1$	$j = 2$	$j = 3$	\dots	$j = p$
$Q(0, j)$	I	Θ	Θ	Θ	\dots	Θ
$Q(1, j)$	M	N	Θ	Θ	\dots	
$Q(2, j)$	M^2	$MN + NM$	N^2	Θ	\dots	Θ
$Q(3, j)$	M^3	$M(MN + NM) + NM^2$	$MN^2 + N(MN + NM)$	N^3	\dots	
\dots	\dots	\dots	\dots		\dots	Θ
$Q(p, j)$	M^p	Θ	Θ	Θ	\dots	N^p

We can reexpress the discrete DPML matrix function $\mathbb{D}_{\alpha,\beta,r}^{M,N}$ in terms of the (nabla) fractional Taylor monomial as follows:

$$\mathbb{D}_{\alpha,\beta,r}^{M,N}(k) := \begin{cases} \Theta, & k \in \mathbb{N}^{-r-1}, \\ I, & k = -r, \\ \sum_{i=0}^{\infty} M^i H_{i\alpha+\beta-1}(k, -r), & k \in \mathbb{N}_{1-r}^0, \\ \sum_{i=0}^{\infty} \sum_{j=0}^p Q(i+1, j) H_{i\alpha+\beta-1}(k, (j-1)r), & k \in \mathbb{N}_{(p-1)r+1}^{pr}. \end{cases} \quad (9)$$

For commutative constant coefficient matrices, the following remark provides an explicit form for the matrix entries $Q(i, j)$.

Remark 2. *Under the commutativity of the constant coefficient matrices M and N , we have*

$$Q(i, j) = \binom{i}{j} M^{i-j} N^j, \quad i, j \in \mathbb{N}_0.$$

In the following remark, it is seen that the discrete DPML matrix function we define reduces to the available famous functions in the literature depending on the special choice of parameters.

Remark 3. *Let $\mathbb{D}_{\alpha,\beta,r}^{M,N}$ be defined by (6). Then the following assertions hold true:*

1. *if $\alpha = \beta = 1$, $r = h + 1$, then $\mathbb{D}_{\alpha,\beta,h}^{M,N}(k) = X_h^{M,N}(k)$,*
2. *if $M = \Theta$, $r = h + 1$, and $\alpha = \beta = 1$, then $\mathbb{D}_{\alpha,\beta,h}^{M,N}(k) = e_h^{Nk}$,*
3. *if $M = \Theta$ and $\alpha = \beta$, then $\mathbb{D}_{\alpha,\beta,r}^{M,N}(k) = \mathbb{F}_r^{Nk\bar{\alpha}}$,*
4. *if $N = \Theta$, then $\mathbb{D}_{\alpha,\beta,r}^{M,N}(k) = \mathbb{E}_{M,\alpha,\beta-1}(k, -r)$,*
5. *if $\alpha = \beta = 1$, $r = h + 1$, and $MN = NM$, then $\mathbb{D}_{\alpha,\beta,h}^{M,N}(k) = e^{Mk} e_h^{N_1(k-h)}$, where $N_1 = e^{-Mh} N$.*

This remark can be easily confirmed by keeping aforesaid definitions in mind.

Remark 4. *Comparison of the discrete delay perturbation of (nabla) M - L type matrix function $\mathbb{D}_{\alpha,\beta,r}^{M,N}(k)$, the (nabla) M - L matrix function $\mathbb{E}_{M,\alpha,\beta}(k, r)$, and discrete delayed M - L -type matrix function $\mathbb{F}_r^{Nk\bar{\alpha}}$ is presented in Figure 1.*

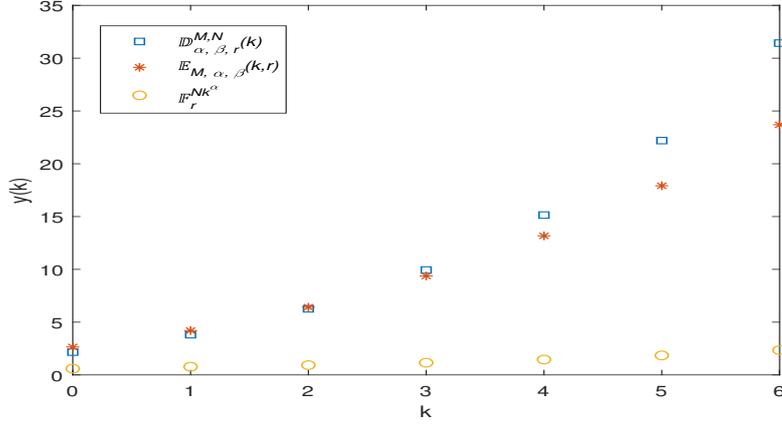


Figure 1: Comparison of functions $\mathbb{D}_{\alpha, \beta, r}^{M, N}(k)$, $\mathbb{E}_{M, \alpha, \beta}(k, r)$, and $\mathbb{F}_r^{Nk^\alpha}$ for $\alpha = 0.9$, $\beta = 0.6$, $M = 5$, $N = 3$, $r = 2$.

4. The explicit solution of RL fractional retarded difference system

In this section, divided into two subsections, we investigate an explicit solution to the linear Riemann-Liouville fractional retarded difference system.

We share a couple of main theorems to achieve our objective. When it comes to most of their proofs related to natural numbers, we use the method of mathematical induction, which is the most useful and powerful proof technique in the field of natural numbers.

4.1. Homogeneous case

In this subsection, we firstly consider the following homogeneous Riemann-Liouville fractional delayed difference system:

$$\begin{cases} \nabla_{-r}^\alpha z(k) = Mz(k) + Nz(k-r), & k \in \mathbb{N}_1, \\ z(k) = \mathbb{E}_{M, \alpha, \alpha-1}(k, -r), & k \in \mathbb{N}_{1-r}^0, \quad r > 0, \end{cases} \quad (10)$$

where ∇_{-r}^α is the Riemann-Liouville fractional difference of order $0 < \alpha < 1$, $z : \mathbb{N}_1 \rightarrow \mathbb{R}^n$, $r \in \mathbb{N}_2$ is a retardation, $M, N \in \mathbb{R}^{n \times n}$ are constant coefficient matrices.

Theorem 3. *The discrete DPML matrix function $\mathbb{D}_{\alpha, \beta, r}^{M, N}$ satisfies homogeneous system (10) in the case $\alpha = \beta$.*

Proof. Firstly we show that it verifies the initial condition $z(k) = \mathbb{E}_{A, \alpha, \alpha-1}(k, -r)$, for $k \in \mathbb{N}_{1-r}^0$. From the formulation given in equation (9), which is derived from

Definition 5 using Taylor monomials, one can easily see that

$$\mathbb{D}_{\alpha,\alpha,r}^{M,N}(k) = \sum_{i=0}^{\infty} M^i H_{i\alpha+\alpha-1}(k, -r) = \mathbb{E}_{M,\alpha,\alpha-1}(k, -r).$$

Now, let's show for $k \in \mathbb{N}_1$ that

$$\nabla_{-r}^{\alpha} \mathbb{D}_{\alpha,\alpha,r}^{M,N}(k) = M \mathbb{D}_{\alpha,\alpha,r}^{M,N}(k) + N \mathbb{D}_{\alpha,\alpha,r}^{M,N}(k-r). \quad (11)$$

For $k \in \mathbb{N}_1$, there exists $p \in \mathbb{N}_1$ such that $k \in \mathbb{N}_{(p-1)r+1}^{pr}$. We apply the mathematical induction on $p \in \mathbb{N}_1$ to demonstrate its validity. For $p = 1$, we have

$$\mathbb{D}_{\alpha,\alpha,r}^{M,N}(k) = \sum_{i=0}^{\infty} \sum_{j=0}^1 Q(i+1, j) H_{i\alpha+\alpha-1}(k, (j-1)r). \quad (12)$$

Taking the Riemann-Liouville fractional difference ∇_{-r}^{α} of (12), we obtain

$$\nabla_{-r}^{\alpha} \mathbb{D}_{\alpha,\alpha,r}^{M,N}(k) = \sum_{i=0}^{\infty} \sum_{j=0}^1 Q(i+1, j) \nabla_{-r}^{\alpha} H_{i\alpha+\alpha-1}(k, (j-1)r).$$

Taking the subintervals and Lemma 1 into consideration, we have

$$\begin{aligned} \nabla_{-r}^{\alpha} \mathbb{D}_{\alpha,\alpha,r}^{M,N}(k) &= \sum_{i=0}^{\infty} \sum_{j=0}^1 Q(i+1, j) \nabla_{(j-1)r}^{\alpha} H_{i\alpha+\alpha-1}(k, (j-1)r) \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^1 Q(i+1, j) H_{i\alpha-1}(k, (j-1)r). \end{aligned}$$

Using (7) and (8), we obtain

$$\begin{aligned} &\nabla_{-r}^{\alpha} \mathbb{D}_{\alpha,\alpha,r}^{M,N}(k) \\ &= M \sum_{i=1}^{\infty} \sum_{j=0}^1 Q(i, j) H_{i\alpha-1}(k, (j-1)r) + N \sum_{i=1}^{\infty} \sum_{j=1}^1 Q(i, j-1) H_{i\alpha-1}(k, (j-1)r) \\ &= M \sum_{i=0}^{\infty} \sum_{j=0}^1 Q(i+1, j) H_{i\alpha+\alpha-1}(k, (j-1)r) + N \sum_{i=0}^{\infty} \sum_{j=0}^0 Q(i+1, j) H_{i\alpha+\alpha-1}(k, jr) \\ &= M \mathbb{D}_{\alpha,\alpha,r}^{M,N}(k) + N \mathbb{D}_{\alpha,\alpha,r}^{M,N}(k-r). \end{aligned}$$

Now, let us assume the validity of (11) for $p = n$, that is

$$\mathbb{D}_{\alpha,\alpha,r}^{M,N}(k) = \sum_{i=0}^{\infty} \sum_{j=0}^n Q(i+1, j) H_{i\alpha+\alpha-1}(k, (j-1)r)$$

satisfies (11). For $p = n + 1$, by similar calculations we have

$$\begin{aligned}
\nabla_{-r}^{\alpha} \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k) &= \sum_{i=0}^{\infty} \sum_{j=0}^{n+1} Q(i+1, j) \nabla_{(j-1)r}^{\alpha} H_{i\alpha+\alpha-1}(k, (j-1)r) \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{n+1} Q(i+1, j) H_{i\alpha-1}(k, (j-1)r) \\
&= M \sum_{i=1}^{\infty} \sum_{j=0}^{n+1} Q(i, j) H_{i\alpha-1}(k, (j-1)r) \\
&\quad + N \sum_{i=1}^{\infty} \sum_{j=1}^{n+1} Q(i, j-1) H_{i\alpha-1}(k, (j-1)r) \\
&= M \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k) + N \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k-r),
\end{aligned}$$

which is the desired result. ◀

Now let's consider the below homogeneous Riemann-Liouville fractional delayed difference system

$$\begin{cases} \nabla_{-r}^{\alpha} z(k) = Mz(k) + Nz(k-r), & k \in \mathbb{N}_1, \\ z(k) = \phi(k), & k \in \mathbb{N}_{1-r}^0, \quad r > 0, \end{cases} \quad (13)$$

where $\phi : \mathbb{N}_{1-r}^0 \rightarrow \mathbb{R}^n$ is an initial condition function, and the rest of details are the same as before.

Theorem 4. *The function*

$$z(k) = \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k) \phi(1-r) + \int_{1-r}^0 \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k-r-\rho(s)) (\nabla_{-r}^{\alpha} \phi(s) - M\phi(s)) \nabla s$$

satisfies the homogeneous system (13).

Proof. In the light of equation (9), which is derived from Definition 5, and taking into account $\mathbb{D}_{\alpha, \alpha, r}^{M, N}(k-r-\rho(s)) = 0$, for $k \in \mathbb{N}_{1-r}^0$ we easily obtain

$$\begin{aligned}
z(k) &= \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k) \phi(1-r) + \int_{1-r}^k \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k-r-\rho(s)) (\nabla_{-r}^{\alpha} \phi(s) - M\phi(s)) \nabla s \\
&= \sum_{i=0}^{\infty} M^i H_{i\alpha+\alpha-1}(k, -r) \phi(1-r) \\
&\quad + \sum_{i=0}^{\infty} M^i \int_{1-r}^k H_{i\alpha+\alpha-1}(k, \rho(s)) (\nabla_{-r}^{\alpha} \phi(s) - M\phi(s)) \nabla s
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{\infty} M^i H_{i\alpha+\alpha-1}(k, -r) \phi(1-r) \\
&+ \sum_{i=0}^{\infty} M^i \left(\nabla_{1-r}^{-(i\alpha+\alpha)} \nabla_{-r}^{\alpha} \phi(k) - M \nabla_{1-r}^{-(i\alpha+\alpha)} \phi(k) \right).
\end{aligned}$$

It is time to apply Theorem 2 and Lemma 1 to the above equation, so we get

$$\begin{aligned}
z(k) &= \sum_{i=0}^{\infty} M^i H_{i\alpha+\alpha-1}(k, -r) \phi(1-r) \\
&= \sum_{i=0}^{\infty} M^i H_{i\alpha+\alpha-1}(k, -r) \phi(1-r) - M \sum_{i=0}^{\infty} M^i \nabla_{1-r}^{-(i\alpha+\alpha)} \phi(k) \\
&+ \sum_{i=0}^{\infty} M^i \left(\nabla_{1-r}^{-i\alpha} (\phi(k) - H_{i\alpha+\alpha-1}(k, -r) \phi(1-r)) \right) \\
&= \sum_{i=1}^{\infty} M^i \nabla_{1-r}^{-i\alpha} \phi(k) + \phi(k) - M \sum_{i=0}^{\infty} M^i \nabla_{1-r}^{-(i\alpha+\alpha)} \phi(k) = \phi(k),
\end{aligned}$$

which means that $z(k)$ satisfies the initial condition for $k \in \mathbb{N}_{1-r}^0$. In order to demonstrate that $z(k)$ fulfills homogeneous system (10) for $k \in \mathbb{N}_1$, we use the Riemann-Liouville fractional difference, the well-known Fubini theorem, and Theorem 3 to obtain

$$\begin{aligned}
\nabla_{-r}^{\alpha} z(k) &= \nabla_{-r}^{\alpha} \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k) \phi(1-r) \\
&+ \int_{1-r}^0 \nabla_{-r}^{\alpha} \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k-r-\rho(s)) (\nabla_{-r}^{\alpha} \phi(s) - M \phi(s)) \nabla s \\
&= M \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k) \phi(1-r) + N \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k-r) \phi(1-r) \\
&+ M \int_{1-r}^0 \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k-r-\rho(s)) (\nabla_{-r}^{\alpha} \phi(s) - M \phi(s)) \nabla s \\
&+ N \int_{1-r}^0 \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k-2r-\rho(s)) (\nabla_{-r}^{\alpha} \phi(s) - M \phi(s)) \nabla s \\
&= Mz(k) + Nz(k-r),
\end{aligned}$$

which completes the proof. ◀

Remark 5. *Since the solution of the homogeneous part is known from Theorem 3, the variation of constants' technique also can be used to prove the last theorem.*

4.2. Nonhomogeneous case

In this subsection, we consider the following nonhomogeneous Riemann-Liouville fractional delayed difference system:

$$\begin{cases} \nabla_{-r}^{\alpha} z(k) = Mz(k) + Nz(k-r) + \mathfrak{T}(k), & k \in \mathbb{N}_1, \\ z(k) = 0, & k \in \mathbb{N}_{1-r}^0, \quad r > 0, \end{cases} \quad (14)$$

where ∇_{-r}^{α} is the Riemann-Liouville fractional difference of order $0 < \alpha < 1$, $z : \mathbb{N}_1 \rightarrow \mathbb{R}^n$, $\mathfrak{T} : \mathbb{N}_1 \rightarrow \mathbb{R}^n$ is a continuous function, $r \in \mathbb{N}_2$ is a retardation, $M, N \in \mathbb{R}^{n \times n}$ are constant coefficient matrices.

Before giving one of main theorems, we prove some useful equalities in the following lemma.

Lemma 2. *The below equalities hold true:*

- $\int_{-r}^0 H_{-\alpha}(k, \rho(s)) \int_0^s \mathbb{D}_{\alpha, \alpha, r}^{M, N}(s-r-\rho(r)) \mathfrak{T}(r) \nabla \rho(r) \nabla s = 0,$
- $\nabla_{\rho(k)}^{\alpha-1} \mathfrak{T}(k) = \mathfrak{T}(k).$

Proof. For the first item, take $s = 0$. Then the whole term should be zero as the limits of the inner integral are equal to zero. If $0 \leq \rho(r) \leq s$ and $-r \leq s < 0$, then $s-r-\rho(r) \leq -r-1$, and so $\mathbb{D}_{\alpha, \alpha, r}^{M, N}(s-r-\rho(r)) = \Theta$ because of Definition 9. The whole term again should be zero. For the second item, we need a simple calculation as follows:

$$\begin{aligned} \nabla_{\rho(k)}^{\alpha-1} \mathfrak{T}(k) &= \sum_{s=\rho(k)+1}^k H_{\alpha-1}(k, \rho(s)) \mathfrak{T}(s) = \sum_{s=k}^k H_{\alpha-1}(k, \rho(s)) \mathfrak{T}(s) \\ &= H_{\alpha-1}(k, \rho(k)) \mathfrak{T}(k) = \mathfrak{T}(k), \end{aligned}$$

where $H_{\alpha-1}(k, \rho(k)) = 1$. ◀

Theorem 5. *The integral term*

$$z(k) = \int_0^k \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k-r-\rho(s)) \mathfrak{T}(s) \nabla s$$

satisfies the nonhomogeneous system (14).

Proof. We will firstly show, for $k \in \mathbb{N}_{1-r}^0$, that

$$z(k) = \int_0^k \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k-r-\rho(s)) \mathfrak{T}(s) \nabla s = 0.$$

Due to the property of integral, $z(k) = \int_0^0 \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k - r - \rho(s)) \Upsilon(s) \nabla s = 0$ for $k = 0$. From [30, Theorem 8.48] it follows that

$$\begin{aligned} z(k) &= \int_0^k \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k - r - \rho(s)) \Upsilon(s) \nabla s = - \int_k^0 \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k - r - \rho(s)) \Upsilon(s) \nabla s \\ &= - \sum_{s=k+1}^0 \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k - r - \rho(s)) \Upsilon(s). \end{aligned}$$

If $k < s \leq 0$ and $1 - r \leq k \leq -1$, due to a property of integer arithmetic, we have $k - r - \rho(s) \in \mathbb{N}^{-1-r}$, and so $\mathbb{D}_{\alpha, \alpha, r}^{M, N}(k - r - \rho(s)) = \Theta$. Thus, $z(k) = 0$ for \mathbb{N}_{1-r}^{-1} . To sum up, $z(k) = 0$ for \mathbb{N}_{1-r}^0 , that is, $z(k)$ fulfills the initial condition.

In the rest of this proof, we use the technique of mathematical induction on $p \in \mathbb{N}_1$. For $k \in \mathbb{N}_1$, there exists $p \in \mathbb{N}_1$ such that $k \in \mathbb{N}_{(p-1)r+1}^{pr}$. Suppose that it is true for $p = 1$ such that $k \in \mathbb{N}_1^r$.

It is clear, from the initial condition, that $z(k - r) = 0$ for $k \in \mathbb{N}_1^r$, and so

$$Mz(k) + Nz(k - r) + \Upsilon(k) = Mz(r) + \Upsilon(r).$$

On the other hand, we have for $k \in \mathbb{N}_1^r$

$$\begin{aligned} \nabla_{-r}^\alpha z(k) &= \nabla_{-r}^\alpha \left[\int_0^k \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k - r - \rho(s)) \Upsilon(s) \nabla s \right] \\ &= \nabla \nabla_{-r}^{\alpha-1} \left[\int_0^k \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k - r - \rho(s)) \Upsilon(s) \nabla s \right] \\ &= \nabla \int_{-r}^k H_{-\alpha}(k, \rho(s)) \left(\int_0^s \mathbb{D}_{\alpha, \alpha, r}^{M, N}(s - r - \rho(t)) \Upsilon(t) \nabla t \right) \nabla s \\ &= \nabla \int_{-r}^k H_{-\alpha}(k, \rho(s)) \left(\int_0^s \mathbb{D}_{\alpha, \alpha, r}^{M, N}(s - r - \rho(t)) \Upsilon(t) \nabla \rho(t) \right) \nabla s. \end{aligned}$$

It is time to use Lemma 2 and Fubini theorem, so we get

$$\begin{aligned} \nabla_{-r}^\alpha z(k) &= \nabla \int_0^k \int_{\rho(t)}^k H_{-\alpha}(k, \rho(s)) \mathbb{D}_{\alpha, \alpha, r}^{M, N}(s - r - \rho(t)) \Upsilon(t) \nabla s \nabla t \\ &= \nabla \int_0^k \nabla_{\rho(t)}^{\alpha-1} \mathbb{D}_{\alpha, \alpha, r}^{M, N}(s - r - \rho(t)) \Upsilon(t) \nabla t. \end{aligned}$$

By implementing Lemma 1, Lemma 2, and equation (9), which is based on Definition 5, to the above equality, we obtain

$$\nabla_{-r}^\alpha z(k) = \int_0^k \nabla_{\rho(t)}^\alpha \mathbb{D}_{\alpha, \alpha, r}^{M, N}(s - r - \rho(t)) \Upsilon(t) \nabla t + \Upsilon(k)$$

$$\begin{aligned}
 &= M \int_0^k \mathbb{D}_{\alpha,\alpha,r}^{M,N}(s-r-\rho(t)) \mathfrak{T}(t) \nabla t + \mathfrak{T}(k) + N \int_0^k \mathbb{D}_{\alpha,\alpha,r}^{M,N}(s-2r-\rho(t)) \mathfrak{T}(t) \nabla t \\
 &= Mz(k) + \mathfrak{T}(k).
 \end{aligned}$$

On comparing the obtained results for $k \in \mathbb{N}_1^r$, we have

$$\nabla_{-r}^\alpha z(k) = Mz(r) + \mathfrak{T}(r) = Mz(k) + Nz(k-r) + \mathfrak{T}(k).$$

Suppose that it is valid for $p = n$ such that $k \in \mathbb{N}_{(n-1)r+1}^{nr}$, that is

$$\mathbb{D}_{\alpha,\alpha,r}^{M,N}(k) = \sum_{i=0}^{\infty} \sum_{j=0}^n Q(i+1, j) H_{i\alpha+\alpha-1}(k, (j-1)r).$$

In the same manner as in the first case, we obtain for $k \in \mathbb{N}_{nr+1}^{(n+1)r}$

$$\begin{aligned}
 \nabla_{-r}^\alpha z(k) &= \int_0^k \nabla_{\rho(t)}^\alpha \mathbb{D}_{\alpha,\alpha,r}^{M,N}(s-r-\rho(t)) \mathfrak{T}(t) \nabla t + \mathfrak{T}(k) \\
 &= M \int_0^k \mathbb{D}_{\alpha,\alpha,r}^{M,N}(s-r-\rho(t)) \mathfrak{T}(t) \nabla t + \mathfrak{T}(k) + N \int_0^k \mathbb{D}_{\alpha,\alpha,r}^{M,N}(s-2r-\rho(t)) \mathfrak{T}(t) \nabla t \\
 &= M \int_0^k \mathbb{D}_{\alpha,\alpha,r}^{M,N}(s-r-\rho(t)) \mathfrak{T}(t) \nabla t + \mathfrak{T}(k) + N \int_0^{k-r} \mathbb{D}_{\alpha,\alpha,r}^{M,N}(s-2r-\rho(t)) \mathfrak{T}(t) \nabla t \\
 &= Mz(k) + Nz(k-r) + \mathfrak{T}(k),
 \end{aligned}$$

which is what we wanted to prove. ◀

Remark 6. *Based on the homogeneous solution from Theorem 3, the variation of constants' technique also can be used to prove the last theorem.*

Now, we will examine the nonhomogeneous retarded Riemann-Liouville fractional difference system

$$\begin{cases} \nabla_{-r}^\alpha z(k) = Mz(k) + Nz(k-r) + \mathfrak{T}(k), & k \in \mathbb{N}_1, \\ z(k) = \phi(k), & k \in \mathbb{N}_{1-r}^0, \quad r > 0, \end{cases} \quad (15)$$

where ∇_{-r}^α is the Riemann-Liouville fractional difference of order $0 < \alpha < 1$, $z : \mathbb{N}_1 \rightarrow \mathbb{R}^n$, $\mathfrak{T} : \mathbb{N}_1 \rightarrow \mathbb{R}^n$ is a continuous function, $r \in \mathbb{N}_2$ is a retardation, $M, N \in \mathbb{R}^{n \times n}$ are constant coefficient matrices, $\phi : \mathbb{N}_{1-r}^0 \rightarrow \mathbb{R}^n$ is an initial condition function.

Theorem 6. *The function*

$$\begin{aligned} z(k) &= \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k) \phi(1-r) + \int_{1-r}^0 \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k-r-\rho(s)) (\nabla_{-r}^{\alpha} \phi(s) - M\phi(s)) \nabla s \\ &+ \int_0^k \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k-r-\rho(s)) \Upsilon(s) \nabla s, \end{aligned} \quad (16)$$

satisfies the nonhomogeneous system (15).

Proof. It is clear that $z(k) = z_1(k) + z_2(k)$ in (17), where $z_1(k)$ and $z_2(k)$ are the solutions of the systems (13) and (5) given by Theorem 4 and 5 respectively, which form the system (15). By implementing the principle of superposition technique, $z(k)$ fulfills the system (15). Indeed, for $k \in \mathbb{N}_1$ we obtain

$$\begin{aligned} \nabla_{-r}^{\alpha} z(k) &= \nabla_{-r}^{\alpha} [z_1(k) + z_2(k)] \\ &= Mz_1(k) + Nz_1(k-r) + Mz_2(k) + Nz_2(k-r) + \Upsilon(k) \\ &= M[z_1(k) + z_2(k)] + N[z_2(k-r) + z_2(k-r)] + \Upsilon(k) \\ &= Mz(k) + Nz(k-r) + \Upsilon(k), \end{aligned}$$

and for $k \in \mathbb{N}_{1-r}^0$,

$$\nabla_{-r}^{\alpha} z(k) = \nabla_{-r}^{\alpha} [z_1(k) + z_2(k)] = \phi(k) + 0 = \phi(k),$$

which is the desired result. ◀

Remark 7. *Theorem 6 corresponds to [26, Theorem 4], for $M = \Theta$ and $\alpha = \beta$,*

5. Special Cases

In this section, we consider a couple of special cases.

Example 1. *Let us reconsider the matrix nabla fractional delayed difference system (5) when $MN = NM$. In this context, Theorem 6 may be reformulated as follows.*

Proposition 1. *The exact analytical solution of nonhomogeneous version of system (5) has the following explicit form:*

$$\begin{aligned} z(k) &= \sum_{i=0}^{\infty} \sum_{j=0}^p \binom{i+1}{j} M^{i+1-j} N^j H_{(i+1)\alpha-1}(k, (j-1)r) \phi(1-r) \\ &+ \sum_{i=0}^{\infty} \sum_{j=0}^p \binom{i+1}{j} M^{i+1-j} N^j \int_{1-r}^0 H_{(i+1)\alpha-1}(k-r-\rho(s), (j-1)r) \nabla_{-r}^{\alpha} \phi(s) \nabla s \end{aligned}$$

$$\begin{aligned}
& - \sum_{i=0}^{\infty} \sum_{j=0}^p \binom{i+1}{j} M^{i+2-j} N^j \int_{1-r}^0 H_{(i+1)\alpha-1}(k-r-\rho(s), (j-1)r) \phi(s) \nabla s \\
& + \sum_{i=0}^{\infty} \sum_{j=0}^p \binom{i+1}{j} M^{i+1-j} N^j \int_0^k H_{(i+1)\alpha-1}(k-r-\rho(s), (j-1)r) \mathfrak{T}(s, z(s)) \nabla s.
\end{aligned}$$

Remark 8. Proposition 1 is new for matrix nabla RL fractional delayed difference system with commutative coefficient matrices.

Example 2. Let us reconsider delta analogue of the nabla fractional delayed difference system (5) via the relation $\nabla z(k) = \Delta z(k-1)$. In this context, Theorem 6 may be reformulated as follows.

Proposition 2. The exact analytical solution of the delta fractional delayed difference system has the following form:

$$\begin{aligned}
z(k) &= \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k-1) \phi(1-r) + \int_{-r}^{-1} \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k-1-r-s) N \phi(s) \Delta s \\
&+ \int_1^k \mathbb{D}_{\alpha, \alpha, r}^{M, N}(k-r-s) \mathfrak{T}(s-1) \Delta s.
\end{aligned} \tag{17}$$

Remark 9. Here are some of our findings:

1. Proposition 2 is new for the delta fractional delayed difference system with commutative coefficient matrices.
2. Theorem 6 matches up with [19, Theorem 3.6], for $\alpha = \beta = 1$, $r = h + 1$,
3. Theorem 6 coincides with [18, Theorem 3.5], for $\alpha = \beta = 1$, $r = h + 1$, and $MN = NM$.

6. Conclusion

In this paper, we introduce the nabla Riemann-Liouville fractional retarded difference system with the noncommutative coefficients. In the sequel, we launch out the discrete delayed perturbation of the nabla M-L type matrix function in a new way, investigate its properties, determine some relations between it and the available ones related to it, and graphically compare it with some of the present ones. We investigate solutions of homogeneous and nonhomogeneous versions of the RL fractional delayed difference system brick by brick and present their solutions based on the discrete retarded perturbation of the M-L function, and lastly discuss some special cases.

A number of advanced topics, many of which are represented in the following references, remain open for investigation in the context of the current system. These include, for instance, studying whether the RL fractional retarded system in (4) exhibits relative controllability [31, 32, 33], iterative learning controllability [34, 35], exponential stability [36, 37] asymptotic stability [38, 39, 40], and finite-time stability [41]. Furthermore, the proposed system can be extended to include higher-order discrete versions [42, 43], multi-retarded structures [44], and systems with variable coefficients. Each of these extensions invites further exploration regarding the above-mentioned control and stability properties.

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