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Problem of the Calculus of Variations with a Quadratic Functional and Higher-Order Derivatives

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Abstract. This paper presents a complete study of the problem of the calculus of variations with higher-order derivatives and a quadratic functional. The main result is the theorem about quadratic functionals. We answer the following question: when does the feasible extrema provide an absolute minimum in the given problem with a quadratic functional? This problem was previously investigated by V. Alekseev, E. Galeev, V. Tikhomirov in [1]. Points *c*) and *d*) of the above theorem were firstly formulated in [1] without proof. In this article we prove these points and add two more cases *a*) and *b*). Why are these two new points significant? What does happen if necessary condition is satisfied (the Jacobi condition), but sufficient condition is not (the strong Jacobi condition)? Usually, in such cases we use definition of absolute minimum and then we prove that feasible extrema provide absolute minimum or not by using sequences of feasible functions. But now, using the result of this paper we can answer this question just by checking two conditions.
To prove the above theorem, we modified and applied methods used by E. Galeev in [2] to prove similar result in a case of the classic problem of the calculus of variations.
Key Words and Phrases: problem of the calculus of cariations with higher-order derivatives, Jacobi condition, Legendre condition, quadratic functional.
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1. Introduction

We shall consider a problem with higher-order derivatives. The extrema is sought in the function space $x(\cdot) \in C^n([t_0;t_1],\mathbb{R})$ with the given conditions:

$$J(x(\cdot)) = \int_{t_0}^{t_1} L(t,x,\dot{x},\ddot{x},\ldots,x^{(n)})dt \to \min;$$

$$x^{(k)}(t_0) = x_{k0},\ x^{(k)}(t_1) = x_{k1},\ k = 0,\ldots,n-1. \quad (P)$$

Definition 1. *We shall say that the functional $J(x)$ has a weak local minimum for $\widehat{x}(\cdot) \in C^n([t_0;\, t_1])$ ($\widehat{x}(\cdot) \in \text{wlocmin}\, P$) if there exists $\delta > 0$ such that $J(x(\cdot)) \geq J(\widehat{x}(\cdot))$ for any feasible function $x(\cdot)$ such that $\|x(\cdot) - \widehat{x}(\cdot)\|_{C^n([t_0;\, t_1])} < \delta$.*

Definition 2. *We shall say that the functional $J(x)$ has a strong local minimum for $\widehat{x}(\cdot) \in PC^n([t_0;\, t_1])$ ($\widehat{x}(\cdot) \in \text{strlocmin}\, P$) if there exists $\delta > 0$ such that $J(x(\cdot)) \geq J(\widehat{x}(\cdot))$ for any feasible function $x(\cdot)$ such that $\|x(\cdot) - \widehat{x}(\cdot)\|_{C^{n-1}([t_0;\, t_1])} < \delta$.*

2. Definitions of Legendre and Jacobi conditions

We shall suppose that the integrand L is $2n$ times continuously differentiable in a neighborhood of $\Gamma_{\hat{x}\dot{\hat{x}}\ldots\hat{x}^{(n)}} = \{(t, x(t), \widehat{x}(t), \ldots, \widehat{x}^n(t)) : t \in [t_0; t_1]\}$. Let $\widehat{x}(\cdot) \in C^{2n}([t_0;\, t_1], \mathbb{R})$ be feasible extrema of (P), i.e. let the Euler-Poisson equation

$$\sum_{k=0}^{n} (-1)^k \frac{d^k}{dt^k} \widehat{L}_{x^{(k)}} = 0$$

be satisfied. The functional $J(x(\cdot))$ has the second variation at the point $x = \widehat{x}$ on the space of functions $h \in C^n([t_0;\, t_1])$:

$$J''(\hat{x})[h, h] = \int_{t_0}^{t_1} \left(\sum_{i,j=0}^{n} \hat{L}_{x^{(i)}x^{(j)}} h^{(i)} h^{(j)} \right).$$

Definition 3. *We say that the minimum problem on the extrema $\widehat{x}$ satisfies the Legendre condition if $\widehat{L}_{x^{(n)}x^{(n)}}(t) \geq 0, \forall t \in [t_0;\, t_1]$ and the strong Legendre condition if $\widehat{L}_{x^{(n)}x^{(n)}}(t) > 0$, $\forall t \in [t_0;\, t_1]$.*

Let's suppose that extrema $\widehat{x}$ satisfy the strong Legendre condition. Consider the Euler-Poisson equation for the integrand $\widetilde{L} = \sum\limits_{i,j=0}^{n} \widehat{L}_{x^{(i)}x^{(j)}} h^{(i)} h^{(j)}$:

$$\sum_{k=0}^{n} (-1)^k \frac{d^k}{dt^k} \widetilde{L}_{h^{(k)}} = 0$$

This equation is *the Jacobi equation.* If we plug in definition of $\widetilde{L}$, then we get the following equation:

$$\sum_{k=0}^{n} (-1)^k \frac{d^k}{dt^k} \left(\sum_{j=0}^{n} \widehat{L}_{x^{(k)}x^{(j)}} h^{(j)} \right) = 0.$$

Definition 4. *A point τ is called conjugate to a point t_0 if there exists a non-trivial solution $h(\cdot)$ of the Jacobi equation such that $h^{(k)}(t_0) = h^{(k)}(\tau) = 0$, $k = 0, 1, \dots n-1$.*

Definition 5. *The Jacobi condition is satisfied on the extrema $\widehat{x}$ if there are no points conjugate to t_0 in the interval $(t_0; t_1)$, and the strong Jacobi condition is satisfied if there are no points conjugate to t_0 in the semi open interval $(t_0; t_1]$.*

We shall consider analytical approach to find conjugate points. The Jacobi equation is a linear differential equation of order $2n$. Let $h_1, \dots, h_n$ be a fundamental system of solutions of the Jacobi equation with initial conditions $H(t_0) = 0, H^{(n)}(t_0) = I$, where I is a unit matrix:

$$H(t) = \begin{pmatrix} h_1(t) & \dots & h_n(t) \\ \dot{h}_1(t) & \dots & \dot{h}_n(t) \\ \dots & \dots & \dots \\ h_1^{(n-1)}(t) & \dots & h_n^{(n-1)}(t) \end{pmatrix},$$

$$H^{(n)}(t) = \begin{pmatrix} h_1^{(n)}(t) & \dots & h_n^{(n)}(t) \\ \dots & \dots & \dots \\ h_1^{(2n-1)}(t) & \dots & h_n^{(2n-1)}(t) \end{pmatrix}.$$

We'll show that τ is conjugate to t_0 if and only if $\det H(\tau) = 0$. The determinant of the matrix $H(t)$ is equal to zero if and only if the columns of this matrix are linearly dependent with non-zero coefficients $a_1, \dots, a_n$. Then the function $h = \sum\limits_{i=0}^{n} a_i h_i$ satisfies the following conditions at $t = \tau$:

$$h^{(k)}(\tau) = 0, k = 0, \dots n-1.$$

The condition $H(t_0) = 0$ is equivalent to $h_i^{(k)}(t_0) = 0$, $k = 0, \dots n-1$, $i = 1, \dots n$. Thus, $h^{(k)}(t_0) = 0, k = 0, \dots n-1$, i.e. τ is a conjugate point.

3. Quadratic functional

Now we shall consider the problem with quadratic functional in "diagonal" form:

$$J(x(\cdot)) = \int\limits_{t_0}^{t_1} \left(\sum_{k=0}^{n} A_k(t)(x^{(k)}(t))^2 dt \right) \to \min;$$

$$x^{(k)}(t_0) = x_{k0},\ x^{(k)}(t_1) = x_{k1},\ k = 0, \dots, n-1. \quad (P)$$

The main result of this paper is the following theorem about quadratic functional. Parts *c*) and *d*) were formulated in [1] with no proof. The essential results are parts *a*) and *b*), where necessary condition for extrema is satisfied (the Jacobi condition), but sufficient condition for extrema is not satisfied (the strong Jacobi condition).

Theorem 1. *Let the functions* $A_k \in C^k([t_0;t_1])$, $k = 0, \dots, n$ *be given and the strong Legendre condition for a minimum be satisfied* $(\hat{L}_{x^{(n)}x^{(n)}}(t) = 2A_n(t) > 0)$. *Then*

a) *if the Jacobi condition is satisfied, but the strong Jacobi condition is not satisfied and feasible extrema exist, then* $\hat{x} \in \operatorname{absmin} P$;

b) *if the Jacobi condition is satisfied, but the strong Jacobi condition is not satisfied and the feasible extrema do not exist and*

$$G = \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^{j+1} \frac{d^{j-1}}{dt^{j-1}} (A_k(t) h^{(k)}) \tilde{x}^{(k-j)} \Big|_{t_0}^{t_1} \neq 0,$$

then $S_{\text{absmin}} = -\infty$;

c) *if the strong Jacobi condition is satisfied, then the feasible extrema exists, is unique and* $\hat{x} \in \operatorname{absmin} P$;

d) *if the Jacobi condition is not satisfied, then* $S_{\text{absmin}} = -\infty$.

Proof. a) Let the Jacobi condition be satisfied, but the strong Jacobi condition isn't satisfied and a feasible extrema $\widehat{x}$ exist. The extrema $\hat{x}(t)$ can be embedded in a central field of extremals that covers $t_0 \le t \le t_1 - \varepsilon$. Let x_ε be an arbitrary function such that $x_\varepsilon^{(k)}(t_0) = x_{k0}$, $x_\varepsilon^{(k)}(t_1 - \varepsilon) = \hat{x}^{(k)}(t_1 - \varepsilon), k = 0, \dots, n-1$ and $x_\varepsilon \in C^n[t_0; t_1 - \varepsilon]$. According to the formula for the Weierstrass E-Function in quadratic case

$$J(x_\varepsilon) - J(\hat{x}) = \int_{t_0}^{t_1-\varepsilon} E(t, x_\varepsilon, ..., x_\varepsilon^{(n-1)}, u, x_\varepsilon^{(n)}) dt = \int_{t_0}^{t_1-\varepsilon} A_n (x_\varepsilon^{(n)} - u)^2 dt \ge 0,$$

where E is a Weierstrass E-Function. Taking limit as $\varepsilon \to 0$, we obtain $J(x) - J(\hat{x}) \ge 0$. Thus, $\hat{x} \in \operatorname{absmin} P$.

b) Assume that the Jacobi condition is satisfied, but the strong Jacobi condition is not satisfied and feasible extrema doesn't exist. Let's suppose that the feasible space is non-empty. Then we can take any feasible function $\tilde{x}$ such that $\tilde{x}^{(k)}(t_0) = x_{k0}$, $\tilde{x}^k(t_1) = x_{k1}$, where $k = 0, ..., n-1$. Using Taylor's theorem we obtain

$$J(\tilde{x} + h) = J(\tilde{x}) + J'(\tilde{x})[h] + \frac{1}{2} J''[h, h]. \tag{1}$$

Here h is the solution of the Jacobi equation such that $h^{(k)}(t_0) = h^{(k)}(t_1) = 0$ $k = 0, \ldots, n-1$. We can find solution that complies with these boundary conditions as the Jacobi condition is satisfied, but the strong Jacobi condition is not satisfied. Now let's calculate the second variation of the functional $J(\cdot)$ at the point h:

$$\frac{1}{2}J''[h,h] = \int_{t_0}^{t_1} \left(\sum_{k=0}^{n} A_k(t)(h^{(k)}(t))^2 dt \right)$$

Applying integration by parts k times to each term and using the fact that $h^{(k)}(t_0) = h^{(k)}(t_1) = 0$, $k = 0, \ldots, n-1$, we obtain

$$\int_{t_0}^{t_1} A_k(t)(h^{(k)})^2 dt = \int_{t_0}^{t_1} A_k(t) h^{(k)} dh^{(k-1)} =$$

$$= A_k h^{(k)} h^{(k-1)} \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt}(A_k(t)h^{(k)})h^{(k-1)} dt =$$

$$= \int_{t_0}^{t_1} -\frac{d}{dt}(A_k(t)h^{(k)})h^{(k-1)} dt = \ldots =$$

$$= \int_{t_0}^{t_1} h \left((-1)^k \frac{d^k}{dt^k}(A_k h_k) \right) dt.$$

Then

$$\frac{1}{2}J''[h,h] = \int_{t_0}^{t_1} \sum_{k=0}^{n} h \left((-1)^k \frac{d^k}{dt^k}(A_k h_k) \right) dt = 0,$$

where the expression under the integral sign is the Jacobi equation and h is its solution.

$$J(\tilde{x} + h) - J(\tilde{x}) = J'(\tilde{x})[h] =$$

$$= 2 \int_{t_0}^{t_1} \sum_{k=0}^{n} A_k(t) \tilde{x}^{(k)}(t) h^{(k)}(t) dt =$$

$$= 2 \int_{t_0}^{t_1} \sum_{k=1}^{n} A_k(t) \tilde{x}^{(k)}(t) h^{(k)}(t) dt + 2 \int_{t_0}^{t_1} A_0(t) h(t) \tilde{x}(t) dt.$$

Integrating by parts each term k times, we obtain

$$\int_{t_0}^{t_1} A_k(t)\tilde{x}^{(k)}(t)h^{(k)}(t)dt = \int_{t_0}^{t_1} A_k(t)h^{(k)}d\tilde{x}^{(k-1)} =$$

$$= A_k(t)h^{(k)}\tilde{x}^{(k-1)}\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} -\frac{d}{dt}(A_k(t)h^{(k)})\tilde{x}^{(k-1)}dt =$$

$$= A_k(t)h^{(k)}\tilde{x}^{(k-1)}\Big|_{t_0}^{t_1} - \frac{d}{dt}(A_k(t)h^{(k)})\tilde{x}^{(k-2)}\Big|_{t_0}^{t_1} +$$

$$+\int_{t_0}^{t_1} \frac{d^2}{dt^2}(A_k(t)h^{(k)})\tilde{x}^{(k-2)}dt = ... = \int_{t_0}^{t_1}(-1)^k\frac{d^k}{dt^k}(A_k(t)h^{(k)})\tilde{x}dt$$

$$+A_k(t)h^{(k)}\tilde{x}^{(k-1)}\Big|_{t_0}^{t_1} - \frac{d}{dt}(A_k(t)h^{(k)})\tilde{x}^{(k-2)}\Big|_{t_0}^{t_1} + ...+$$

$$+(-1)^{k+1}\frac{d^{k-1}}{dt^{k-1}}(A_k(t)h^{(k)})\tilde{x}\Big|_{t_0}^{t_1} =$$

$$= \int_{t_0}^{t_1}(-1)^k\frac{d^k}{dt^k}(A_k(t)h^{(k)})\tilde{x}dt+$$

$$+\sum_{j=1}^{k}(-1)^{j+1}\frac{d^{j-1}}{dt^{j-1}}(A_k(t)h^{(k)})\tilde{x}^{(k-j)}\Big|_{t_0}^{t_1}.$$

Thus,

$$J(\tilde{x}+h) - J(\tilde{x}) = 2\int_{t_0}^{t_1}\sum_{k=0}^{n}(-1)^k\frac{d^k}{dt^k}(A_k(t)h^{(k)})\tilde{x}dt+$$

$$+2\sum_{k=1}^{n}\sum_{j=1}^{k}(-1)^{j+1}\frac{d^{j-1}}{dt^{j-1}}(A_k(t)h^{(k)})\tilde{x}^{(k-j)}\Big|_{t_0}^{t_1}.$$

The expression under the integral sign is equal to 0. Thus,

$$J(\tilde{x}+h) - J(\tilde{x}) = 2\sum_{k=1}^{n}\sum_{j=1}^{k}(-1)^{j+1}\frac{d^{j-1}}{dt^{j-1}}(A_k(t)h^{(k)})\tilde{x}^{(k-j)}\Big|_{t_0}^{t_1} = 2G.$$

Then $J(\tilde{x}+\lambda h) = J(\tilde{x})+2\lambda G$. So, $J(\tilde{x}+\lambda h) \to -\infty$, as $\lambda \to +\infty$ or $\lambda \to -\infty$, i.e. $S_{\text{absmin}\, P} = -\infty$.

c) Suppose that the strong Jacobi condition is satisfied. Let's prove that feasible extrema exist and are unique.

Existence. We can find the extrema in the form of

$$\hat{x} = C_1 h_1(t) + ... + C_n h_n(t) + C_{n+1} h_{n+1}(t) ... + C_{2n} h_{2n}(t),$$

where $h_j(t)$ is the solution of the Jacobi equation. Let's use the following matrices:

$$H_0(t) = \begin{pmatrix} h_1(t) & \dots & h_n(t) \\ \dot{h}_1(t) & \dots & \dot{h}_n(t) \\ \dots & \dots & \dots \\ h_1^{(n-1)}(t) & \dots & h_n^{(n-1)}(t) \end{pmatrix},$$

$$H_1(t) = \begin{pmatrix} h_{n+1}(t) & \dots & h_{2n}(t) \\ \dot{h}_{n+1}(t) & \dots & \dot{h}_{2n}(t) \\ \dots & \dots & \dots \\ h_{n+1}^{(n-1)}(t) & \dots & h_{2n}^{(n-1)}(t) \end{pmatrix},$$

$H_0(t_0) = 0$, $H_0^{(n)}(t_0) = I$, $H_1(t_1) = 0$, $H_1^{(n)}(t_1) = I$. Such solutions of the Jacobi equation exist by Existence and Uniqueness Theorem (EUT) for the differential equations of order 2n. As the strong Jacobi condition is satisfied, we have $\det H_0(t_1) \neq 0$, $\det H_1(t_0) \neq 0$. Consider the system of algebraic linear equations:

$$\begin{cases} \hat{x}(t_0) = C_{n+1} h_{n+1}(t_0) + \ldots + C_{2n} h_{2n}(t_0) = x_{00} \\ \ldots \\ \hat{x}^{(n-1)}(t_0) = C_{n+1} h_{n+1}^{(n-1)}(t_0) + \ldots + C_{2n} h_{2n}^{(n-1)}(t_0) = \\ = x_{n-10}. \end{cases}$$

This system has the following matrix:

$$\begin{pmatrix} h_{n+1}(t_0) & \dots & h_{2n}(t_0) \\ \dot{h}_{n+1}(t_0) & \dots & \dot{h}_{2n}(t_0) \\ \dots & \dots & \dots \\ h_{n+1}^{(n-1)}(t_0) & \dots & h_{2n}^{(n-1)}(t_0) \end{pmatrix}.$$

The determinant of this matrix is $\det H_1(t_0) \neq 0$. Then we can find $C_{n+1}, \ldots, C_{2n}$

applying Cramer's method:

$$C_{n+1} = \frac{\det \begin{pmatrix} x_{00} & \dots & h_{2n}(t_0) \\ x_{10} & \dots & \dot{h}_{2n}(t_0) \\ \dots & \dots & \dots \\ x_{n-10} & \dots & h_{2n}^{(n-1)}(t_0) \end{pmatrix}}{\det H_1(t_0)},$$

$$\dots$$

$$C_{2n} = \frac{\det \begin{pmatrix} h_{n+1}(t_0) & \dots & x_{00} \\ \dot{h}_{n+1}(t_0) & \dots & x_{10} \\ \dots & \dots & \dots \\ h_{n+1}^{(n-1)}(t_0) & \dots & x_{n-10} \end{pmatrix}}{\det H_1(t_0)}.$$

Now let's consider the following system of algebraic linear equations:

$$\begin{cases} \hat{x}(t_1) = C_1 h_1(t_1) + \ldots + C_n h_n(t_1) = x_{01} \\ \dots \\ \hat{x}^{(n-1)}(t_1) = C_1 h_1^{(n-1)}(t_1) + \ldots + C_n h_n^{n-1}(t_1) = x_{n-11}. \end{cases}$$

It has the following matrix:

$$\begin{pmatrix} h_1(t_1) & \dots & h_n(t_1) \\ \dot{h}_1(t_1) & \dots & \dot{h}_n(t_1) \\ \dots & \dots & \dots \\ h_1^{(n-1)}(t_1) & \dots & h_n^{(n-1)}(t_1) \end{pmatrix}.$$

The determinant of this matrix is $\det H_0(t_1) \neq 0$. Then we can find $C_1, \ldots, C_n$ by Cramer's method:

$$C_1 = \frac{\det \begin{pmatrix} x_{01} & \dots & h_n(t_1) \\ x_{11} & \dots & \dot{h}_n(t_1) \\ \dots & \dots & \dots \\ x_{n-11} & \dots & h_n^{(n-1)}(t_1) \end{pmatrix}}{\det H_0(t_1)},$$

$$\dots$$

$$C_n = \frac{\det \begin{pmatrix} h_1(t_1) & \dots & x_{01} \\ \dot{h}_1(t_1) & \dots & x_{11} \\ \dots & \dots & \dots \\ h_1^{(n-1)}(t_1) & \dots & x_{n-11} \end{pmatrix}}{\det H_0(t_1)}.$$

Uniqueness. Let $\overline{x}$ be different feasible extrema. Then $h = \widehat{x} - \overline{x}$ is a nontrivial solution of the Jacobi equation with zero boundary conditions. This fact contradicts the strong Jacobi condition.

By strong Legendre and Jacobi conditions feasible extrema can be embedded in a central field of extremals. Let $x \in C^n([t_0;\, t_1])$ be an arbitrary feasible function. Then, by the formula for the Weierstrass E-Function in quadratic case,

$$J(x) - J(\widehat{x}) =$$

$$= \int_{t_0}^{t_1} E(t, x, ..., x^{(n-1)}, u, x^{(n)})dt = \int_{t_0}^{t_1} A_n(x^{(n)} - u)^2 dt \geq 0.$$

Thus, $\hat{x} \in \operatorname{absmin} P$.

d) Assume that the Jacobi condition isn't satisfied. Then, according to the necessary conditions for weak local minimum function, $\overline{h} \equiv 0 \notin \operatorname{absmin} P''$ in the following problem:

$$\frac{1}{2}J''(h) = J(h(\cdot)) = \int_{t_0}^{t_1} \left(\sum_{k=0}^{n} A_k(t)(h^{(k)}(t))^2 dt \right) \to \min$$

$$x^{(k)}(t_0) = 0,\ x^{(k)}(t_1) = 0,\ k = 0, \ldots, n-1. \quad (P'')$$

Hence, $S_{\operatorname{absmin} P''} < 0$. Consequently, there exists a function $h \in C_0^n([t_0; t_1])$ such that $J(h) < 0$ and we obtain

$$J(x + \lambda h) = J(x) + \lambda J'(x)[h] + \frac{1}{2}\lambda^2 J''(h)$$

as $\lambda \to +\infty$, i.e. $S_{\operatorname{absmin} P} = -\infty$.

Theorem 1 is proved. ◄

4. Conclusion

In this paper, we have comprehensively analyzed the problem of the calculus of variations involving higher-order derivatives and a quadratic functional. By formulating and proving a key theorem, we addressed the conditions under which a feasible extremum provides an absolute minimum. Our work extends previous research by Alekseev, Galeev, and Tikhomirov, particularly by focusing on the scenario where the Jacobi condition is satisfied, but the strong Jacobi condition is not. This nuanced investigation is crucial because quadratic functionals often appear in practical optimization problems, especially in fields like physics and engineering, where they describe system energies and their minimization points to stable states. Our findings not only contribute to the theoretical framework, but also have significant implications for practical applications, such as space trajectory calculations. By modifying and applying methods previously used by Galeev, we have provided a more complete understanding of the above conditions, thereby advancing both the theory and its practical applications in the calculus of variations.

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