Azerbaijan Journal of Mathematics
V. 15, No 2, 2025, July
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2025.2.150

Commutators of Anisotropic Maximal Operators with BMO Functions on Anisotropic Total Morrey Spaces

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Abstract. In this paper we consider the anisotropic maximal commutator M_b^d and the commutator of the anisotropic maximal operator $[b, M^d]$ on the anisotropic total Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$. We obtain necessary and sufficient conditions for the boundedness of the operators M_b^d and $[b, M^d]$ on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ when b belongs to the bounded mean oscillation space $BMO(\mathbb{R}^n)$. We also obtain new characterizations for some subclasses of $BMO(\mathbb{R}^n)$.

Key Words and Phrases: anisotropic total Morrey spaces, anisotropic maximal function, commutators, BMO.

2010 Mathematics Subject Classifications: 42B20, 42B25, 42B35

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1. Introduction

The aim of this paper is to study anisotropic maximal commutators M_b^d and commutators of the anisotropic maximal operator $[b, M^d]$ in anisotropic total Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ when b belongs to anisotropic BMO spaces $BMO(\mathbb{R}^n)$.

Let $\mathbb{R}^n$ be an n-dimension Euclidean space with the norm $|x|$ for each $x \in \mathbb{R}^n$, and S^{n-1} denote the unit sphere on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $r > 0$, let $\mathcal{E}(x, r)$ denote the open ball centered at x of radius r and ${}^{\complement}\mathcal{E}(x, r)$ denote the set $\mathbb{R}^n \backslash \mathcal{E}(x, r)$. Let $d = (d_1, \dots, d_n)$, $d_i \geq 1$, $i = 1, \dots, n$, $|d| = \sum_{i=1}^n d_i$ and $t^d x \equiv \left(t^{d_1} x_1, \dots, t^{d_n} x_n\right)$. By [5, 8], the function $F(x, \rho) = \sum_{i=1}^n x_i^2 \rho^{-2d_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x, \rho) = 1$ is uniquely solvable. This unique solution will be denoted by $\rho(x)$. It is easy to verify that $\rho(x - y)$ defines a distance between any two points x, $y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space ([5, 7, 8]). The balls with respect to ρ, centered at x of radius r, are just the ellipsoids

$$\mathcal{E}_d(x,r) = \left\{ y \in \mathbb{R}^n : \frac{(y_1 - x_1)^2}{r^{2d_1}} + \cdots + \frac{(y_n - x_n)^2}{r^{2d_n}} < 1 \right\},$$

with the Lebesgue measure $|\mathcal{E}_d(x,r)| = v_n r^{|d|}$, where v_n is the volume of the unit ball in $\mathbb{R}^n$. Let also $\Pi_d(x,r) = \{y \in \mathbb{R}^n : \max_{1 \le i \le n} |x_i - y_i|^{1/d_i} < r\}$ denote the parallelepiped, and ${}^{\complement}\mathcal{E}_d(x,r) = \mathbb{R}^n \setminus \mathcal{E}_d(x,r)$ be the complement of $\mathcal{E}_d(0,r)$. If $d = \mathbf{1} \equiv (1,\dots,1)$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_{\mathbf{1}}(x,r) = \mathcal{E}(x,r)$. Note that in the standard parabolic case $d = (1,\dots,1,2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The anisotropic maximal operator M^d is given by

$$M^d f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |f(y)| dy$$

and the sharp anisotropic maximal operator $M^{\sharp,d} f$ is defined by

$$M^{\sharp,d} f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |f(y) - f_{\mathcal{E}(x,t)}| dy,$$

where $|\mathcal{E}(x,t)|$ is the Lebesgue measure of the ellipsoid $\mathcal{E}(x,t)$. If $d = \mathbf{1}$, then $M \equiv M^d$ is the classical Hardy-Littlewood maximal operator and $M^\sharp \equiv M^{\sharp,d}$ is the sharp maximal operator. For a fixed $q \in (0,1)$, any suitable function h and $x \in \mathbb{R}^n$, let $M_q^{\sharp,d} h(x) = \big(M^{\sharp,d}\big(|h|^q\big)(x)\big)^{1/q}$ and $M_q^d h(x) = \big(M^d\big(|h|^q\big)(x)\big)^{1/q}$.

The anisotropic maximal commutator generated by the operator M^d and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$M_b^d f(x) = \sup_{r>0} |\mathcal{E}(x,r)|^{-1} \int_{\mathcal{E}(x,r)} |b(x) - b(y)|\, |f(y)|\, dy.$$

The commutator generated by the operator M^d and a suitable function b is defined by

$$[b, M^d] f(x) = b(x) M^d f(x) - M^d(bf)(x).$$

Obviously, the operators M_b^d and $[b, M^d]$ essentially differ from each other since M_b^d is positive and sublinear and $[b, M^d]$ is neither positive nor sublinear [1, 2, 3].

Morrey spaces, introduced by C. B. Morrey [23], play important roles in the regularity theory of PDE, including heat equations and Navier-Stokes equations, see also [20]. In [15], Guliyev introduced a variant of Morrey spaces called total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. In [1], the anisotropic total Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$, have been considered, their basic properties and some embeddings into the Morrey space $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ have been studied. In [16], necessary and sufficient conditions were found for the boundedness of the fractional maximal operator M_α in the total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$. Necessary and sufficient conditions for the boundedness of the anisotropic maximal operator M^d on anisotropic total Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ were obtained in [17] .

The operators M^d, M^d_b and $[b, M^d]$ play an important role in real and harmonic analysis and applications (see, for instance, [1, 2, 3, 9, 21, 22, 25, 26, 27]). The nonlinear commutator of Hardy-Littlewood maximal function $[b, M]$ can be used in studying the product of a function in H^1 and a function in BMO [6]. The boundedness of the anisotropic maximal operator M^d on $L^p(\mathbb{R}^n)$ is one of the most fundamental results in harmonic analysis. It has been extended to a range of other function spaces, and to many variations of the standard maximal operator. The commutator estimates play an important role in studying the regularity of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and their boundedness can be used to characterize some function spaces (see, for instance, [4, 10, 11, 18, 19, 24]).

This paper is organized as follows. In Section 2 we give some definitions and auxiliary results. In Section 3 we obtain necessary and sufficient conditions for the boundedness of the anisotropic maximal commutator M^d_b on the anisotropic total Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$. In Section 4 we find necessary and sufficient conditions for the boundedness of the commutator of anisotropic maximal operator $[b, M^d]$ on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of corresponding quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Fractional maximal operator in total anisotropic Morrey spaces

In this section we find necessary and sufficient conditions for the boundedness of the anisotropic fractional maximal operator M^d_α in the anisotropic total Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.

Definition 1. *Let* $d = (d_1, \ldots, d_n)$, $d_i \ge 1$, $i = 1, \ldots, n$. *Let also* $0 < p < \infty$,

$\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. We denote by $L^d_{p,\lambda}(\mathbb{R}^n)$ the anisotropic Morrey space, by $\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n)$ the modified anisotropic Morrey space [12, 14], and by $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ the total anisotropic Morrey space [1, 15], the set of all classes of locally integrable functions f, with the finite norms

$$\|f\|_{L^d_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L_p(\mathcal{E}(x,t))},$$

$$\|f\|_{\widetilde{L}^d_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(\mathcal{E}(x,t))},$$

$$\|f\|_{L^d_{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L_p(\mathcal{E}(x,t))},$$

respectively.

Definition 2. *Let $d = (d_1, \ldots, d_n)$, $d_i \geq 1$, $i = 1, \ldots, n$. Let also $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak anisotropic Morrey space $WL^d_{p,\lambda}(\mathbb{R}^n)$, the weak modified anisotropic Morrey space $W\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n)$ [12, 14] and the weak anisotropic total Morrey space $WL^d_{p,\lambda,\mu}(\mathbb{R}^n)$ [1, 15] as the set of all locally integrable functions f with the finite norms*

$$\|f\|_{WL^d_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{WL_p(\mathcal{E}(x,t))},$$

$$\|f\|_{W\widetilde{L}^d_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{WL_p(\mathcal{E}(x,t))},$$

$$\|f\|_{WL^d_{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{WL_p(\mathcal{E}(x,t))},$$

respectively.

Lemma 1. *[17, Lemma 2.1] If $0 < p < \infty$, $0 \leq \mu \leq \lambda \leq |d|$, then*

$$L^d_{p,\lambda,\mu}(\mathbb{R}^n) = L^d_{p,\lambda}(\mathbb{R}^n) \cap L^d_{p,\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{L^d_{p,\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{L^d_{p,\lambda}}, \|f\|_{L^d_{p,\mu}}\right\}.$$

Lemma 2. *[17, Lemma 2.2] If $0 < p < \infty$, $0 \le \mu \le \lambda \le |d|$, then*

$$WL^d_{p,\lambda,\mu}(\mathbb{R}^n) = WL^d_{p,\lambda}(\mathbb{R}^n) \cap WL^d_{p,\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{WL^d_{p,\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{WL^d_{p,\lambda}}, \|f\|_{WL^d_{p,\mu}}\right\}.$$

Remark 1. *Let $0 < p < \infty$. If $\mu < 0$ or $\lambda > |d|$, then*

$$L^d_{p,\lambda,\mu}(\mathbb{R}^n) = WL^d_{p,\lambda,\mu}(\mathbb{R}^n) = \Theta(\mathbb{R}^n),$$

where $\Theta \equiv \Theta(\mathbb{R}^n)$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

The following Guliyev local estimate is valid (see also [13]).

Lemma 3. *[13, Lemma 4.1] Let $1 \le p < \infty$. Then, for $p > 1$ the inequality*

$$\|M^d f\|_{L_p(\mathcal{E}(x,r))} \lesssim r^{\frac{|d|}{p}} \sup_{t>2r} t^{-\frac{|d|}{p}} \|f\|_{L_p(\mathcal{E}(x,t))} \tag{1}$$

holds for all $\mathcal{E}(x,r)$ and for all $f \in L^p_{\rm loc}(\mathbb{R}^n)$.

Moreover, if $p = 1$, then the inequality

$$\|M^d f\|_{WL_1(\mathcal{E}(x,r))} \lesssim r^{|d|} \sup_{t>2r} t^{-|d|} \|f\|_{L_1(\mathcal{E}(x,t))} \tag{2}$$

holds for all $\mathcal{E}(x,r)$ and for all $f \in L^{\rm loc}_1(\mathbb{R}^n)$.

Theorem 1. *[17, Corollary 2.1] Let $1 \le p < \infty$, $0 \le \lambda < |d|$ and $0 \le \mu < |d|$.*

1. If $p > 1$, $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$, then $M^d f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ and

$$\|M^d f\|_{L^d_{p,\lambda,\mu}} \le C_{p,\lambda,\mu} \|f\|_{L^d_{p,\lambda,\mu}},$$

where $C_{p,\lambda,\mu}$ depends only on p, λ, μ and $|d|$.

2. If $f \in L^d_{1,\lambda,\mu}(\mathbb{R}^n)$, then $M^d f \in WL^d_{1,\lambda,\mu}(\mathbb{R}^n)$ and

$$\|M^d f\|_{WL_{1,\lambda,\mu}} \le C_{1,\lambda,\mu} \|f\|_{L^d_{1,\lambda,\mu}},$$

where $C_{1,\lambda,\mu}$ depends only on p, λ, μ and $|d|$.

3. Anisotropic maximal commutator M_b^d on spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$

In this section we find necessary and sufficient conditions for the boundedness of the anisotropic maximal commutator M_b^d on the anisotropic total Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ when b belongs to BMO spaces $BMO(\mathbb{R}^n)$.

Definition 3. *We define the bounded mean oscillation space $BMO(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite norm*

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |f(y) - f_{\mathcal{E}(x,t)}| dy < \infty,$$

where $f_{\mathcal{E}(x,t)} = |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} f(y)dy$.

Theorem 2. *[21, Lemma 1] If $b \in BMO(\mathbb{R}^n)$, then for any $q \in (0,1)$, there exists a positive constant C such that*

$$M_q^{\sharp,d}\big(M_b^d f\big)(x) \le C\|b\|_* M^d\big(M^d f\big)(x) \tag{3}$$

for every $x \in \mathbb{R}^n$ and for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

The following theorem is the first of our main results.

Theorem 3. *Let $1 < p < \infty$, $0 \le \lambda < |d|$ and $0 \le \mu < |d|$. The following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$.

(ii) The operator M_b^d is bounded on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.

(iii) There exists a constant $C > 0$ such that

$$\sup_{\mathcal{E}} \frac{\big\|\big(b(\cdot) - b_{\mathcal{E}}\big)\chi_{\mathcal{E}}\big\|_{L^d_{p,\lambda,\mu}(\mathbb{R}^n)}}{\|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}(\mathbb{R}^n)}} \le C. \tag{4}$$

(iv) There exists a constant $C > 0$ such that

$$\sup_{\mathcal{E}} \frac{\big\|\big(b(\cdot) - b_{\mathcal{E}}\big)\chi_{\mathcal{E}}\big\|_{L^1(\mathbb{R}^n)}}{|\mathcal{E}|} \le C. \tag{5}$$

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$. Combining Theorems 1 and 2, we get

$$\begin{aligned}\|M_b^d f\|_{L^d_{p,\lambda,\mu}} &\lesssim \|b\|_* \, \|M_q^{\sharp,d}\big(M_b^d f\big)\|_{L^d_{p,\lambda,\mu}} \lesssim \|b\|_* \, \|M^d\big(M^d f\big)\|_{L^d_{p,\lambda,\mu}} \\ &\lesssim \|b\|_* \|M^d f\|_{L^d_{p,\lambda,\mu}} \lesssim \|b\|_* \|f\|_{L^d_{p,\lambda,\mu}}.\end{aligned}$$

$(ii) \Rightarrow (i)$. Assume that M_b^d is bounded on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$. Let $\mathcal{E} = \mathcal{E}(x,r)$ be a fixed ellipsoid. We consider $f = \chi_{\mathcal{E}}$. It is easy to compute that

$$\|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}} \approx r^{\frac{|d|}{p}} \, [r]_1^{\frac{-\lambda}{p}} \, [1/r]_1^{\frac{\mu}{p}}. \tag{6}$$

On the other hand, for all $x \in B$ we have

$$\begin{aligned} \left|b(x) - b_{\mathcal{E}}\right| &\le \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |b(x) - b(y)| dy \\ &= \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |b(x) - b(y)| \, \chi_{\mathcal{E}}(y) dy \le M_b^d(\chi_{\mathcal{E}})(x). \end{aligned}$$

Since M_b^d is bounded on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$, by (6) we obtain

$$\frac{\|\big(b - b_{\mathcal{E}}\big)\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}}}{\|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}}} \le \frac{\|M_b^d(\chi_{\mathcal{E}})\|_{L^d_{p,\lambda,\mu}}}{\|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}}} \lesssim \frac{\|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}}}{\|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}}} = 1, \tag{7}$$

which implies that (4) holds because the ball $\mathcal{E} \subset \mathbb{R}^n$ is arbitrary.

$(iii) \Rightarrow (iv)$. Assume that (4) holds. Let's prove (5). For any fixed ellipsoid $\mathcal{E}$, by (4), (6), it is easy to see that

$$\begin{aligned} \frac{1}{|\mathcal{E}|} \left\|\big(b(\cdot) - b_{\mathcal{E}}\big)\chi_{\mathcal{E}}\right\|_{L_1(\mathbb{R}^n)} &= \frac{1}{|\mathcal{E}|} \left\|\big(b(\cdot) - b_{\mathcal{E}}\big)\chi_{\mathcal{E}}\right\|_{L^1(\mathcal{E})} \\ &\le \frac{1}{|\mathcal{E}|} \left\|\big(b(\cdot) - b_{\mathcal{E}}\big)\chi_{\mathcal{E}}\right\|_{L_p(\mathcal{E})} \|\chi_{\mathcal{E}}\|_{L_{p'}(\mathcal{E})} \\ &\lesssim r^{-|d|} \, [r]_1^{\frac{\lambda}{p}} \, [1/r]_1^{-\frac{\mu}{p}} \, \|\big(b - b_{\mathcal{E}}\big)\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}} \, r^{\frac{|d|}{p'}} \\ &\approx \frac{\|\big(b - b_{\mathcal{E}}\big)\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}}}{\|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}}} \lesssim 1. \end{aligned}$$

$(iv) \Rightarrow (i)$. For any fixed ellipsoid $\mathcal{E}$, we have

$$\begin{aligned} \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |b(x) - b_{\mathcal{E}}| dy &= \frac{\|\big(b - b_{\mathcal{E}}\big)\chi_{\mathcal{E}}\|_{L^1}}{|\mathcal{E}|} \\ &\le \sup_{\mathcal{E}} \frac{\|\big(b - b_{\mathcal{E}}\big)\chi_{\mathcal{E}}\|_{L^1}}{|\mathcal{E}|} \lesssim 1, \end{aligned}$$

which implies that $b \in BMO(\mathbb{R}^n)$. Thus the proof of Theorem 3 is completed. ◀

4. Commutator of anisotropic maximal operator $[b, M^d]$ on spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$

In this section we find necessary and sufficient conditions for the boundedness of the commutator of the anisotropic maximal operator $[b, M^d]$ on the anisotropic total Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ when b belongs to BMO spaces $BMO(\mathbb{R}^n)$.

Let b be a function defined on $\mathbb{R}^n$ and denote

$$b^-(x) := \begin{cases} 0\ , & \text{if } b(x) \geq 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The relations below between $[b, M^d]$ and M^d_b are valid.

Let b be any non-negative locally integrable function. Then for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ the following inequality is valid:

$$\begin{aligned} \big|[b, M^d]f(x)\big| &= \big|b(x)M^d f(x) - M^d(bf)(x)\big| \\ &= \big|M^d(b(x)f)(x) - M^d(bf)(x)\big| \\ &\leq M^d(|b(x) - b|f)(x) \\ &= M^d_b f(x). \end{aligned}$$

If b is any locally integrable function on $\mathbb{R}^n$, then

$$|[b, M^d]f(x)| \leq M^d_b f(x) + 2b^-(x)\, M^d f(x), \qquad x \in \mathbb{R}^n, \tag{8}$$

holds for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ (see, for example, [15, 21]).

Obviously, the operators M^d_b and $[b, M^d]$ are essentially different from each other because M^d_b is positive and sublinear and $[b, M^d]$ is neither positive nor sublinear.

Let $\mathcal{E} = \mathcal{E}(x, r)$ be a fixed ellipsoid. Denote by $M^d_{\mathcal{E}} f$ the local maximal function of f:

$$M^d_{\mathcal{E}} f(x) := \sup_{\mathcal{E}' \ni x:\, \mathcal{E}' \subset \mathcal{E}} \frac{1}{|\mathcal{E}'|} \int_{\mathcal{E}'} |f(y)|\, dy,\ \ x \in \mathbb{R}^n.$$

Applying Theorem 3, we obtain the following result which is the second of our main results.

Theorem 4. *Let* $1 < p < \infty$, $0 \leq \lambda < |d|$ *and* $0 \leq \mu < |d|$. *The following assertions are equivalent:*

(*i*) $b \in BMO(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.

(ii) The operator $[b, M^d]$ *is bounded on* $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.

(iii) There exists a constant $C > 0$ *such that*

$$\sup_{\mathcal{E}} \frac{\big\| \big(b(\cdot) - M^d_{\mathcal{E}}(b)(\cdot)\big)\chi_{\mathcal{E}} \big\|_{L^d_{p,\lambda,\mu}(\mathbb{R}^n)}}{\|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}(\mathbb{R}^n)}} \le C. \tag{9}$$

(iv) There exists a constant $C > 0$ *such that*

$$\sup_{\mathcal{E}} \frac{\big\| \big(b(\cdot) - M^d_{\mathcal{E}}(b)(\cdot)\big)\chi_{\mathcal{E}} \big\|_{L^1(\mathbb{R}^n)}}{|\mathcal{E}|} \le C. \tag{10}$$

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$. Combining Theorems 1, 3, and inequality (8), we get

$$\begin{aligned} \|[b, M^d]f\|_{L^d_{p,\lambda,\mu}} &\le \|M^d_b f + 2b^- \, M^d f\|_{L^d_{p,\lambda,\mu}} \\ &\le \|M^d_b f\|_{L^d_{p,\lambda,\mu}} + \|b^-\|_{L^\infty} \, \|M^d f\|_{L^d_{p,\lambda,\mu}} \\ &\lesssim \big(\|b\|_* + \|b^-\|_{L^\infty}\big) \, \|f\|_{L^d_{p,\lambda,\mu}}. \end{aligned}$$

Hence it follows that $[b, M^d]$ is bounded on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.

$(ii) \Rightarrow (iii)$. Assume that $[b, M^d]$ is bounded on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$. Since

$$M^d(b\chi_{\mathcal{E}})\chi_{\mathcal{E}} = M^d_{\mathcal{E}}(b) \quad \text{and} \quad M^d(\chi_{\mathcal{E}})\chi_{\mathcal{E}} = \chi_{\mathcal{E}},$$

we have

$$\begin{aligned} |M^d_{\mathcal{E}}(b) - b\chi_{\mathcal{E}}| &= |M^d(b\chi_{\mathcal{E}})\chi_{\mathcal{E}} - bM^d(\chi_{\mathcal{E}})\chi_{\mathcal{E}}| \\ &\le |M^d(b\chi_{\mathcal{E}}) - bM^d(\chi_{\mathcal{E}})| = |[b, M^d]\chi_{\mathcal{E}}|. \end{aligned}$$

Hence

$$\|M^d_{\mathcal{E}}(b) - b\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}(\mathbb{R}^n)} \le \|[b, M^d]\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}(\mathbb{R}^n)}.$$

Thus we get

$$\frac{\|\big(b - M^d_{\mathcal{E}}(b)\big)\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}}}{\|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}}} \le \frac{\|[b, M^d](\chi_{\mathcal{E}})\|_{L^d_{p,\lambda,\mu}}}{\|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}}} \lesssim \frac{\|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}}}{\|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}}} = 1,$$

which implies (iii).

$(iii) \Rightarrow (iv)$. Assume that (9) holds. Then for any fixed ellipsoid $\mathcal{E}$, by (6), we conclude that

$$\frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |b(x) - M^d_{\mathcal{E}}(b)(x)| dx \lesssim \frac{1}{|\mathcal{E}|} \, \|\big(b - M^d_{\mathcal{E}}(b)\big)\chi_{\mathcal{E}}\|_{L_p(\mathcal{E})} \, \|\chi_{\mathcal{E}}\|_{L_{p'}(\mathcal{E})}$$

$$\lesssim r^{-|d|}\,[r]_1^{\frac{\lambda}{p}}\,[1/r]_1^{-\frac{\mu}{p}}\,\|\big(b-M_{\mathcal{E}}^d(b)\big)\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}}\,r^{\frac{|d|}{p'}}$$
$$\lesssim \frac{\|\big(b-M_{\mathcal{E}}^d(b)\big)\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}}}{\|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}}} \lesssim 1.$$

$(iv) \Rightarrow (i)$. Assume that (10) holds. Let's prove that $b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.

Denote

$$E := \{y \in \mathcal{E} : b(y) \le b_{\mathcal{E}}\}, \quad F := \{y \in \mathcal{E} : b(y) > b_{\mathcal{E}}\}.$$

Since

$$\int_E |b(y) - b_{\mathcal{E}}|\,dy = \int_F |b(y) - b_{\mathcal{E}}|\,dy,$$

in view of the inequality $b(x) \le b_{\mathcal{E}} \le M_{\mathcal{E}}^d(b)$, $x \in E$, we get

$$\begin{aligned}\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} |b(y) - b_{\mathcal{E}}|\,dy &= \frac{2}{|\mathcal{E}|}\int_E |b(y) - b_{\mathcal{E}}|\,dy \\ &\le \frac{2}{|\mathcal{E}|}\int_E |b(y) - M_{\mathcal{E}}^d(b)(y)\,dy| \\ &\le \frac{2}{|\mathcal{E}|}\int_{\mathcal{E}} |b(y) - M_{\mathcal{E}}^d(b)(y)\,dy| \lesssim 1.\end{aligned}$$

Consequently, $b \in BMO(\mathbb{R}^n)$.

In order to show that $b^- \in L^\infty(\mathbb{R}^n)$, note that $M_{\mathcal{E}}^d(b) \ge |b|$. Hence

$$0 \le b^- = |b| - b^+ \le M_{\mathcal{E}}^d(b) - b^+ + b^- = M_{\mathcal{E}}^d(b) - b.$$

Thus

$$(b^-)_{\mathcal{E}} \le c,$$

and by the Lebesgue differentiation theorem we get

$$0 \le b^-(x) = \lim_{|\mathcal{E}|\to 0} \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} b^-(y)dy \le c \quad \text{for a.e. } x \in \mathbb{R}^n.$$

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Acknowledgements

The author thanks the referee(s) for carefully reading the paper and useful comments.

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Received 05 January 2025
Accepted 12 March 2025