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On the Dispersion of Axisymmetric Waves Propagating in a Two-layer Hollow Cylinder Containing an Inviscid Liquid (Lucite + water case)

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Abstract. This work deals with the investigation of axisymmetric longitudinal waves propagating in a two-layer hollow cylinder with inviscid fluid. The corresponding problem is formulated in the framework of the piecewise homogeneous body model, using the exact equations and relations of elastodynamics to describe the motion of the cylinder and the linearized Euler equations to describe the flow of the fluid. By solving the corresponding eigenvalue problem, analytical expressions for the unknown functions are obtained and with these expressions and the corresponding boundary, contact and compatibility conditions, the dispersion equation is obtained. The roots of this equation are found numerically and thus the dispersion curves for the zeroth, first and second mod are constructed. Concrete numerical results are obtained for the case where the material of the inner layer is Lucite, but the material of the outer layer is assumed to be a hypothetical material whose Poisson's ratio and density (Young's modulus) are the same as those of the material of the inner layer, but the Young's modulus (density) of the outer layer is different from that of the inner layer. Water is chosen as the fluid in the cylinder.

Key Words and Phrases: inviscid fluid, two-layer hollow cylinder, Lucite, axisymmetric longitudinal waves, wave dispersion, pipe-in-pipe system.

2010 Mathematics Subject Classifications: 74B05, 76D05

1. Introduction

One of the liquid pipe systems widely used in many areas of modern industry and mechanical engineering is the two-layer hollow cylinder (also known as pipe-in-pipe systems). The successful application of pipe-in-pipe systems can be guaranteed by the results of the relevant theoretical studies as listed and reviewed in [1]. We remember that in a pipe-in-pipe system, the inner layer is in contact

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with a liquid, but the outer layer serves to insulate the inner layer and to optimize and control the rigidity of this system, which plays an important role in real working processes.

The question arises as to how the ratio of the mechanical and acoustoelastic properties of the outer and inner layers in pipe-in-pipe systems transporting a fluid affects the dispersion of waves propagating in these systems. Note that the results of the studies related to this question enable the successful use of non-destructive ultrasonic testing methods for defects in such structural elements. In addition, the results of these studies allow the correct selection of the materials of the layers from various points of view with regard to the dynamics of these systems. The study conducted in this paper relates to such a question, namely the investigation of the dispersion of longitudinal axisymmetric waves propagating in a pipe-in-pipe system modeled as a two-layer hollow cylinder containing a barotropic compressible, non-viscous fluid. In this study, the motion of the two-layer hollow cylinder is described within the piecewise homogeneous body model using the exact equations of elastodynamics, but the flow of the fluid is described using the linearized Euler equations for compressible, inviscid fluids.

Now we give a brief overview of the related investigations, starting with the work by Lamb [2], in which the dispersion of axisymmetric waves propagating in a thin cylindrical shell containing a non-viscous compressible fluid is investigated. The paper [3] made a new contribution to the study in [2], which consisted in the use of refined first-order shell theory to describe the motion of the shell in the aforementioned hydro-elastic system. At the same time, the paper [3] presents and discusses specific numerical results on the influence of shear deformation and rotational inertia on the corresponding dispersion curves.

An overview of these studies, which were carried out until the last decade of the last century, was given in the papers [4] and [5]. In [4], the dispersion of the axisymmetric longitudinal wave propagating in a hollow cylinder containing an incompressible, non-viscous fluid was studied theoretically by using the exact equations and relations of elastodynamics. We also mention the more recent investigations in the papers [6, 7, 8, 9, 10, 11] and many others in which related studies were carried out for the cases in which the materials of the cylinder and fluid were complex.

We also focus on recent researches carried out in the papers [12, 13, 14, 15, 16] which directly relate to the study of the dispersion of longitudinal axisymmetric waves propagating in the hollow cylinder + fluid system when the cylinder has inhomogeneous initial stresses. In these studies, however, a single-layer hollow cylinder containing a liquid was considered. Therefore, the investigation carried out in the present work, can also be considered as a starting point for the further development of the investigations carried out in the works [12, 13, 14, 15, 16].

2. Formulation of the problem

We introduce to the consideration a two-layer hollow cylinder containing an inviscid barotropic compressible fluid and associate the cylindrical coordinate system $Or\theta z$ with the central axis of the cylinder. Assume that the Oz axis is aligned along this central axis and the inner and outer layers of the cylinder occupy the regions $R < r < R + h_i$ and $R + h_i < r < R + h_i + h_o$, respectively, under $0 \leq \theta \leq 2\pi$, $-\infty < z < +\infty$, where h_i and h_o are the thicknesses of the inner and outer layers. In addition, we assume that the fluid occupies the interior of the inner cylinder, i.e. the region $0 < r < R$ under $0 \leq \theta \leq 2\pi$, $-\infty < z < +\infty$. The materials of the inner and outer layers of the cylinder are assumed to be linearly elastic and isotropic. To study the dispersion of the axisymmetric waves propagating in this hydro-elastic system, we employ the exact equations and relations of elastodynamics to describe the motion of the cylinder and the linearized Euler equations to describe the flow of the fluid. The motion of the cylinder will be considered within the model of a piecewise homogeneous body. Thus, we write the field equations describing the motion of the cylinder.

The equations of motion:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho \frac{\partial^2 u_r}{\partial t^2}, \frac{\partial \sigma_{zr}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{zr} = \rho \frac{\partial^2 u_z}{\partial t^2}. \quad (1)$$

The elasticity relations:

$$\sigma_{rr} = \lambda e + 2\mu\varepsilon_{rr}, \sigma_{\theta\theta} = \lambda e + 2\mu\varepsilon_{\theta\theta}, \sigma_{zz} = \lambda e + 2\mu\varepsilon_{zz},$$

$$\sigma_{rz} = 2\mu\varepsilon_{rz}, e = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}. \quad (2)$$

The strain-displacement relations:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \varepsilon_{\theta\theta} = \frac{u_r}{r}, \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_z}{\partial r}\right). \quad (3)$$

In (1)–(3) conventional notation is used and below the symbols in these equations will be supplied by the upper or lower index i and o, when they refer to the inner and outer layers of the cylinder, respectively.

According to the monograph [17], we use the following linearized field equations (or linearized Euler equations) for barotropic, compressible, inviscid fluids to describe the fluid flow.

The linearized continuity equation:

$$\frac{\partial p'}{\partial t} + \rho_0 \left(\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{\partial V_z}{\partial z}\right) = 0. \quad (4)$$

The linearized equations of the fluid flow:

$$\frac{\partial V_r}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial r}, \frac{\partial V_z}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z}. \tag{5}$$

The state equation:

$$p' = a_0^2 \rho', a_0^2 = \left(\frac{\partial p'}{\partial \rho'}\right)_0, \tag{6}$$

where a_0 is the sound speed in the fluid.

We add to the foregoing equations the following boundary, contact and compatibility conditions.

The boundary condition on the external surface of the outer layer:

$$\sigma_{rr}^{(o)}\Big|_{r=R+h_i+h_o} = 0, \sigma_{rz}^{(o)}\Big|_{r=R+h_i+h_o} = 0. \tag{7}$$

The contact conditions between the inner and outer layers of the cylinder:

$$\sigma_{rr}^{(o)}\Big|_{r=R+h_i} = \sigma_{rr}^{(i)}\Big|_{r=R+h_i}, \sigma_{rz}^{(o)}\Big|_{r=R+h_i} = \sigma_{rz}^{(i)}\Big|_{r=R+h_i},$$
$$u_r^{(o)}\Big|_{r=R+h_i} = u_r^{(i)}\Big|_{r=R+h_i}, u_z^{(o)}\Big|_{r=R+h_i} = u_z^{(i)}\Big|_{r=R+h_i}. \tag{8}$$

The compatibility conditions on the interface surface between the fluid and inner layer of the cylinder, i.e. internal layer of the cylinder:

$$\sigma_{rr}^{(i)}\Big|_{r=R} = -p', \sigma_{rz}^{(i)}\Big|_{r=R} = 0, \frac{\partial u_r^{(i)}}{\partial t}\Bigg|_{r=R} = V_r|_{r=R}. \tag{9}$$

We also write the condition for the boundedness of the quantities relating to the fluid in the central axis of the cylinder:

$$\left\{|p'|, |\rho'|, |V_r|, |V_z|\right\}\Big|_{r=0} < \infty. \tag{10}$$

This completes the mathematical formulation of the problem under consideration.

3. Method of solution

For the solution to the field equations related to the motion of the two-layer hollow cylinder, we use the classical Lame decomposition [18], which can be represented as follows for the axisymmetric case:

$$u_r = \frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Psi}{\partial r \partial z}, u_z = \frac{\partial \Phi}{\partial z} - \frac{\partial^2 \Psi}{\partial r^2} - \frac{\partial \Psi}{r \partial r}, \tag{11}$$

with the potentials $\varPhi$ and $\varPsi$ satisfying the following equations:

$$\frac{\partial^2 \varPhi}{\partial r^2} + \frac{\partial \varPhi}{r\partial r} + \frac{\partial^2 \varPhi}{\partial z^2} = \frac{1}{(c_1)^2}\frac{\partial^2 \varPhi}{\partial t^2}, \frac{\partial^2 \varPsi}{\partial r^2} + \frac{\partial \varPsi}{r\partial r} + \frac{\partial^2 \varPsi}{\partial z^2} = \frac{1}{(c_2)^2}\frac{\partial^2 \varPsi}{\partial t^2}, \tag{12}$$

where $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_2 = \sqrt{\mu/\rho}$ are the speeds of the dilatation and distortion wave propagation velocities, respectively, of the cylinder's material.

Thus, assuming that the functions $\varPhi$,u_r,u_θ, σ_{rr}, $\sigma_{\theta\theta}$ and σ_{zz} (ψ, u_z and σ_{rz}) have the multiplying in the form $\sin(kz - \omega t)$(the form$\cos(kz - \omega t)$) and denoting the amplitudes of these quantities by the same symbols, we obtain the following equations for the amplitudes of $\varPhi$ and $\varPsi$ from equations in (12):

$$\frac{d^2 \varPhi}{d(r_2)^2} + \frac{1}{r_2}\frac{d\varPhi}{dr_2} + \varPhi = 0, \frac{d^2 \varPsi}{d(r_1)^2} + \frac{1}{r_1}\frac{d\varPsi}{dr_1} + \varPsi = 0, \tag{13}$$

where

$$r_2 = kr\sqrt{\frac{c^2}{(c_1)^2} - 1}, r_1 = kr\sqrt{\frac{c^2}{(c_2)^2} - 1}. \tag{14}$$

The solution to the equations (13) are determined as follows.

For the inner layer of the cylinder:

$$\varPhi^{(i)} = A_1^{(i)} E_0(r_2^{(i)}) + A_2^{(i)} F_0(r_2^{(i)}), \Psi^{(i)} = B_1^{(i)} E_0(r_1^{(i)}) + B_2^{(i)} F_0(r_1^{(i)}). \tag{15}$$

For the outer layer of the cylinder:

$$\varPhi^{(o)} = A_1^{(o)} E_0(r_2^{(o)}) + A_2^{(o)} F_0(r_2^{(o)}), \Psi^{(o)} = B_1^{(o)} E_0(r_1^{(o)}) + B_2^{(o)} F_0(r_1^{(o)}). \tag{16}$$

where

$$E_0(r_m^{(q)}) = \begin{cases} J_0(r_m^{(q)}) \text{ if } (r_m^{(q)})^2/r^2 > 0 \\ I_0(r_m^{(q)}) \text{ if } (r_m^{(q)})^2/r^2 < 0 \end{cases}, F_0(r_m^{(q)}) = \begin{cases} Y_0(r_m^{(q)}) \text{ if } (r_m^{(q)})^2/r^2 > 0 \\ K_0(r_m^{(q)}) \text{ if } (r_m^{(q)})^2/r^2 < 0 \end{cases}. \tag{17}$$

In (17) $m = 1, 2$; $(q) = (i)$ and (o). Moreover, in (17) $J_0(x)$ and $I_0(x)$ are the Bessel and modified Bessel functions of the first kind and of the zeroth order. However, $Y_0(x)$ and $K_0(x)$ are also Bessel and modified Bessel functions of the second kind and of the zeroth order.

Substituting the expressions in (17) into (16) and using the relations (11), (3) and (2), we obtain the expressions for the amplitudes of the displacements and stresses existing within each layer, through which the contact, boundary and compatibility conditions are expressed satisfied between the layers and between the inner layer and fluid, respectively. These expressions are:

$$u_r^{(q)}(r) = A_1^{(q)} \frac{dr_2^{(q)}}{dr} \frac{dE_0(r_2^{(q)})}{dr_2^{(q)}} + A_2^{(q)} \frac{dr_2^{(q)}}{dr} \frac{dF_0(r_2^{(q)})}{dr_2^{(q)}} +$$

$$+B_1^{(q)}\frac{dr_1^{(q)}}{dr}\frac{dE_0(r_1^{(q)})}{dr_1^{(q)}}+B_2^{(q)}\frac{dr_1^{(q)}}{dr}\frac{dF_0(r_1^{(q)})}{dr_1^{(q)}},$$

$$u_z^{(q)}(r)=A_1^{(q)}E_0(r_2^{(q)})+A_2^{(q)}F_0(r_2^{(q)})-$$

$$-B_1^{(q)}\left[\frac{1}{r}\frac{dr_1^{(q)}}{dr}\frac{dE_0(r_1^{(q)})}{d(r_1^{(q)})}+\left(\frac{dr_1^{(q)}}{dr}\right)^2\frac{d^2E_0(r_1^{(q)})}{d(r_1^{(q)})^2}\right]-$$

$$B_2^{(q)}\left[\frac{1}{r}\frac{dr_1^{(q)}}{dr}\frac{dF_0(r_1^{(q)})}{d(r_1^{(q)})}+\left(\frac{dr_1^{(q)}}{dr}\right)^2\frac{d^2F_0(r_1^{(q)})}{d(r_1^{(q)})^2}\right],$$

$$\frac{\sigma_{rr}^{(q)}(r)}{\mu^{(q)}}=A_1^{(q)}\left\{\left(\frac{dr_2^{(q)}}{dr}\right)^2 2\left(1+\frac{\lambda^{(q)}}{2\mu^{(q)}}\right)\frac{d^2E_0(r_2^{(q)})}{d(r_2^{(q)})^2}+\right.$$

$$\left.+\frac{\lambda^{(q)}}{\mu^{(q)}}\frac{1}{r_2^{(q)}}\left(\frac{dr_2^{(q)}}{dr}\right)^2 r_2^{(q)}\frac{dE_0(r_2^{(q)})}{d(r_2^{(q)})}+\frac{\lambda^{(q)}}{\mu^{(q)}}E_0(r_2^{(q)})\right\}+$$

$$A_2^{(q)}\left\{\left(\frac{dr_2^{(q)}}{dr}\right)^2 2\left(1+\frac{\lambda^{(q)}}{2\mu^{(q)}}\right)\frac{d^2F_0(r_2^{(q)})}{d(r_2^{(q)})^2}+\right.$$

$$\left.\frac{\lambda^{(q)}}{\mu^{(q)}}\frac{1}{r_2^{(q)}}\left(\frac{dr_2^{(q)}}{dr}\right)^2 r_2^{(q)}\frac{dF_0(r_2^{(q)})}{d(r_2^{(q)})}+\frac{\lambda^{(q)}}{\mu^{(q)}}F_0(r_2^{(q)})\right\}+$$

$$B_1^{(q)}\left\{\left(\frac{dr_1^{(q)}}{dr}\right)^2 2\left(1+\frac{\lambda^{(q)}}{2\mu^{(q)}}\right)\frac{d^2E_0(r_1^{(q)})}{d(r_1^{(q)})^2}+\frac{\lambda^{(q)}}{\mu^{(q)}}\left[\left(\frac{dr_1^{(q)}}{dr}\right)^2\frac{dE_0(r_1^{(q)})}{d(r_1^{(q)})}+\right.\right.$$

$$\left.\left.+\frac{1}{r_1^{(q)}}\left(\frac{dr_1^{(q)}}{dr}\right)^2\frac{dE_0(r_1^{(q)})}{d(r_1^{(q)})}+\left(\frac{dr_1^{(q)}}{dr}\right)^2\frac{d^2E_0(r_1^{(q)})}{d(r_1^{(q)})^2}\right]\right\}+$$

$$B_2^{(q)}\left\{\left(\frac{dr_1^{(q)}}{dr}\right)^2 2\left(1+\frac{\lambda^{(q)}}{2\mu^{(q)}}\right)\frac{d^2F_0(r_1^{(q)})}{d(r_1^{(q)})^2}+\frac{\lambda^{(q)}}{\mu^{(q)}}\left[\left(\frac{dr_1^{(q)}}{dr}\right)^2\frac{dF_0(r_1^{(q)})}{d(r_1^{(q)})}+\right.\right.$$

$$\left.\left.+\frac{1}{r_1^{(q)}}\left(\frac{dr_1^{(q)}}{dr}\right)^2\frac{dF_0(r_1^{(q)})}{d(r_1^{(q)})}+\left(\frac{dr_1^{(q)}}{dr}\right)^2\frac{d^2F_0(r_1^{(q)})}{d(r_1^{(q)})^2}\right]\right\},$$

$$\frac{\sigma_{rz}^{(q)}(r)}{\mu^{(q)}}=A_1^{(q)}2\frac{dr_2^{(q)}}{dr}\frac{dE_0(r_2^{(q)})}{d(r_2^n)}+A_2^{(q)}2\frac{dr_2^{(q)}}{dr}\frac{dF_0(r_2^{(q)})}{d(r_2^n)}$$

$$B_1^{(q)}\left[\left(\frac{dr_1^{(q)}}{dr}\right)^3\frac{d^3E_0(r_1^{(q)})}{d(r_1^{(q)})^3}-\frac{1}{r_1^{(q)}}\left(\frac{dr_1^{(q)}}{dr}\right)^3\frac{d^2E_0(r_1^{(q)})}{d(r_1^{(q)})^2}+\frac{1}{r_1^{(q)}}\frac{dr_1^{(q)}}{dr}\frac{dE_0(r_1^{(q)})}{d(r_1^{(q)})}\right]+$$

$$B_2^{(q)}\left[\left(\frac{dr_1^{(q)}}{dr}\right)^3\frac{d^3F_0(r_1^{(q)})}{d(r_1^{(q)})^3}-\frac{1}{r_1^{(q)}}\left(\frac{dr_1^{(q)}}{dr}\right)^3\frac{d^2F_0(r_1^{(q)})}{d(r_1^{(q)})^2}+\frac{1}{r_1^{(q)}}\frac{dr_1^{(q)}}{dr}\frac{dF_0(r_1^{(q)})}{d(r_1^{(q)})}\right].\tag{18}$$

This completes the consideration of the solution procedure of the field equations related to the cylinder's motion.

According to [17], for the solution of the field equations (4)–(6) related to the fluid flow we use the following representations:

$$\rho' = -a_0^{-2}\rho_0\frac{\partial}{\partial t}\Phi_f,\, p' = -\rho_0\frac{\partial}{\partial t}\Phi_f,\, V_r = \frac{\partial}{\partial r}\Phi_f,\, V_z = \frac{\partial}{\partial z}\Phi_f,\tag{19}$$

where the function Φ_f satisfies the equation

$$\left[\Delta - \frac{1}{a_0^2}\frac{\partial^2}{\partial t^2}\right]\Phi_f = 0,\, \Delta = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.\tag{20}$$

Assuming that the functions V_z, p' and ρ' (the functions Φ_f and V_r) have the multiplying $\sin(kz-\omega t)$ (the multiplying $\cos(kz-\omega t)$), and denoting the amplitudes of the considered quantities with the same symbols, we obtain the following equation for the amplitude:

$$\left(\frac{d^2}{dr_3^2}+\frac{1}{r_3}\frac{d}{dr_3}+1\right)\Phi_f(r) = 0,\tag{21}$$

where

$$r_3 = kr\sqrt{\left(\frac{c}{a_0}\right)^2-1}.\tag{22}$$

Taking the condition (10) into consideration, we find the solution to equation (22) as follows:

$$\Phi_f(r) = \begin{cases} FJ_0(r_3)\ if\ r_3^2 > 0, \\ FI_0(r_3)\ if\ r_3^2 < 0, \end{cases}\tag{23}$$

where $J_0(r_3)$ ($I_0(r_3)$) is the first kind of Bessel (modified Bessel) function of the zeroth order and F is an unknown constant.

Thus, substituting the function (23) (after multiplying that with $\cos(kz - \omega t)$) into the expressions in (19), the following expressions are obtained for the quantities related to the fluid:

$$p' = \rho_0 \left(V_z^0 k + \omega\right) \sin(kz - \omega t) \begin{cases} F J_0(r_3) \; if \; r_3^2 > 0 \\ F I_0(r_3) \; if \; r_3^2 < 0 \end{cases},$$

$$\rho' = a_0^{-2} \rho_0 \left(V_z^0 k + \omega\right) \sin(kz - \omega t) \begin{cases} F J_0(r_3) \; if \; r_3^2 > 0 \\ F I_0(r_3) \; if \; r_3^2 < 0 \end{cases},$$

$$V_r = k \frac{dr_3}{dr} \cos(kz - \omega t) \begin{cases} -F J_1(r_3) \; if \; r_3^2 > 0 \\ F I_1(r_3) \; if \; r_3^2 < 0 \end{cases},$$

$$V_z = -k \sin(kz - \omega t) \begin{cases} F J_0(r_3) \; if \; r_3^2 > 0 \\ F I_0(r_3) \; if \; r_3^2 < 0 \end{cases}. \quad (24)$$

Thus, this completes the determination of the analytical expressions for all the quantities related to the hollow cylinder and fluid.

Finally, we consider obtaining the corresponding dispersion equation and for this purpose, using the foregoing expressions obtained for the sought values which contain 9 number of unknown constants $A_1^{(i)}$, $A_2^{(i)}$, $B_1^{(i)}$, $B_2^{(i)}$, $A_1^{(o)}$, $A_2^{(o)}$, $B_1^{(o)}$, $B_2^{(o)}$ and F, we satisfy the boundary (7), contact (8) and compatibility conditions (9). In this way, we obtain the system of linear homogeneous equations with respect to these unknowns and according to the usual procedure, equating to zero the determinant of the coefficient matrix of these equations' system, we obtain the dispersion equation which formally can be presented as follows:

$$\det(a_{nm}(c/c_2^{(i)}, kR, \rho_i/\rho_0, \rho_o/\rho_i, E_o/E_i, h_i/R, h_o/R, a_0/c_2^{(i)})) = 0, n; m = 1, 2, ..., 9. \quad (25)$$

Note that the explicit expressions of the components a_{nm} of the coefficient matrix (a_{nm}) can be easily determined from the expressions in (18) and (24) and therefore these expressions are not given here. Moreover, note that this dispersion equation is solved numerically by employing the well-known "bi-section" method.

4. Numerical results and discussions

First of all, we note that the main aim of the present numerical study is to determine how the ratio of the mechanical and acoustoelastic properties of the outer and inner layers in a two-layer hollow cylinder conveying a fluid affects the velocity of the waves (or the dispersion of the waves) propagating in these systems. In other words, we want to determine how the ratios E_o/E_i (where $E_i\,(E_o)$ is the modulus of elasticity of the material of the inner (outer) layer) and ρ_o/ρ_i (where ρ_i (ρ_o) is the density of the material of the inner (outer) layer) affect the dispersion curves of the axisymmetric longitudinal waves propagating in this two-layer hollow cylinder containing a compressible inviscid fluid.

To determine these influences unambiguously, we choose the material of the inner layer as a real material, but we choose the material of the outer layer of the cylinder as a hypothetical material whose density (the modulus of elasticity) is different from the density (the modulus of elasticity) of the material of the inner layer and assume that the Poisson's ratio of the layers' materials are the same. Consequently, in this framework, we will consider the influence of the ratio E_o/E_i(under $\rho_o/\rho_i = 1$) and the ratio ρ_o/ρ_i (under $E_o/E_i = 1$) on the dispersion curves related to the zeroth, first and second modes.

First, we test the PC programs and the calculation algorithm within which the corresponding numerical results are obtained. For this purpose, as in [4], we consider the case where the material of the inner layer is steel with the Lame constants $\lambda = 1.075 \times 10^{11} Pa$ and $\mu = 0.77 \times 10^{11} Pa$, and with the material density $\rho = 7910 kg/m^3$, and the fluid is water with sound speed $a_0 = 1495 m/\sec$ and the density $\rho_0 = 1000 kg/m^3$. Assume that $h_i/R = h_o/R = 0.1$, for which, as in [4], $h/R = 0.2$ where $h = h_i + h_o$. Thus, consider the dispersion diagrams shown in Fig. 1, i.e., the graph of the dependence between $\omega h/c_2^i$ and kh obtained for the zeroth and first modes by using the present computational algorithm and PC programs. Note that the same plots for the case where $E_o/E_i = 1$ and $\rho_o/\rho_i = 1$ are also obtained in the paper [4], which fully coincide with the corresponding dispersion diagrams in Fig. 1 drawn with dashed lines. Consequently, this agreement proves the reliability of the calculation algorithm and the PC programs used in obtaining the numerical results presented in this paper. Moreover, the results presented in Fig. 1 show that an increase in the values of the ratio E_o/E_i (under $\rho_o/\rho_i = 1$) (Fig. 1a) and a decrease in the ratio ρ_o/ρ_i (under$E_o/E_i = 1$) (Fig. 1b) lead to an increase in the wave propagation velocity in the considered hydro-elastic system.

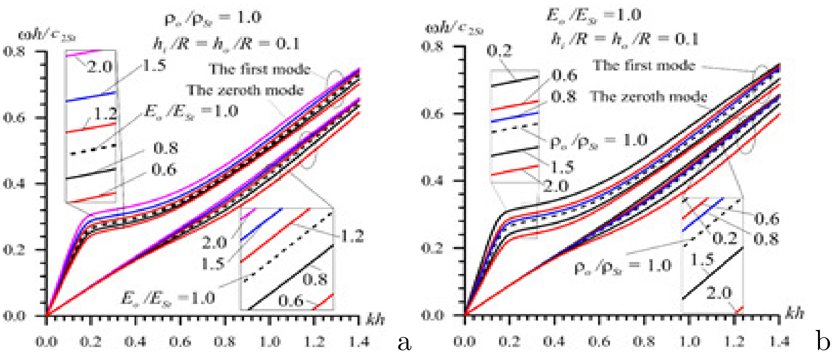

Fig. 1

Now we analyze the dispersion curves obtained for the case where the material of the inner layer of the cylinder is Lucite with the mechanical constants $\mu = 1.86 \times 10^9 Pa$, $\nu = 0.3976$ (Poisson ratio), density $\rho = 1160 kg/m^3$ and shear wave propagation velocity $c_2 = 1265\, m/s$. We choose water as the liquid with the speed of sound and density given above.

Thus, consider the dispersion curves for the zeroth mode (also known as the quasi-Scholte mode) which are shown in Fig. 2 and are obtained for different values of the ratio E_o/E_iunder $\rho_o/\rho_i = 1$ (Fig. 2a) and for different values of the ratio ρ_o/ρ_i under $E_o/E_i = 1$ (Fig. 2 b). We point out once again that such a selection of the values of the problem parameters makes it possible to clearly determine the character of the influence of the change in the modulus of elasticity and the density of the material of the outer (or covering) layer on the wave propagation speed of the quasi-Scholte waves in the two-layer hollow cylinder containing a fluid.

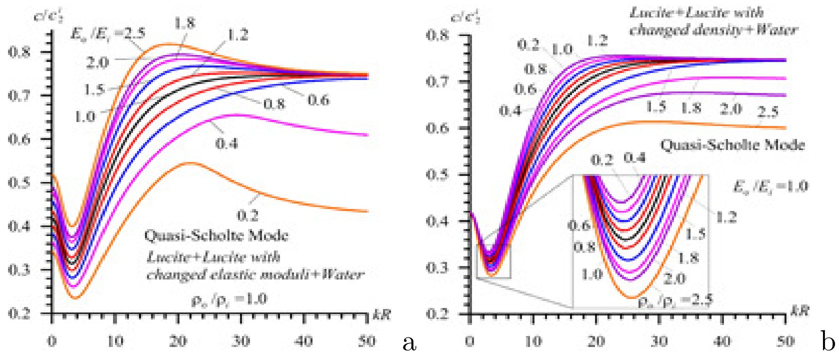

Fig. 2

Observation of the graphs in Fig. 2 shows that, as in the case where the material of the inner layer of the cylinder is steel, an increase in the values of the modulus of elasticity of the outer layer and a decrease in the density of the material of the outer layer lead to an increase in the speed of propagation of the quasi-Scholte waves in the system under consideration.

It follows from Fig. 2a that the propagation speed on the dispersion curves of the quasi-Scholte waves has limit values under lower wave numbers and these values increase monotonically with increasing elastic modulus of the outer layer material. However, Fig. 2b shows that changing the density of the outer layer material has no influence on these limit values.

The analysis of the graphs in Fig. 2 shows that the influence of the material properties of the outer layer material on the limiting values of the wave propagation velocity of the quasi-Scholte waves in the two-layer hollow cylinder containing the fluid under higher wave numbers has a complicated character. This complication can be explained by the following provisions.

According to the physical and mechanical knowledge of wave propagation, we can write the following relation for the limiting values of velocities under the cases of higher wave numbers for the waves with respect to the zeroth mode in the considered system:

$$c = \min(c_R^0, c_{St}, c_{Sch}) \quad \text{as} \quad kR \to \infty. \tag{26}$$

where c_R^o is the propagation speed of the Rayleigh waves in the material of the outer layer, c_{St} is the speed of the Stoneley waves near the interface between the inner and outer layers of the cylinder, and finally c_{Sch} is the speed of the Scholte waves between the fluid and the material of the inner layer.

From Fig. 2, it can be seen that in the cases of $E_o/E_i \geq 0.6$ under $\rho_o/\rho_i = 1.0$ and in the cases of $\rho_o/\rho_i \leq 1.5$ under $E_o/E_i = 1.0$, the above limiting values of c/c_2^i for the waves in the zeroth mode are the velocity of the corresponding Scholte waves, i.e. the c_{Sch}/c_2^i. However, in the cases $E_o/E_i \leq 0.4$ under $\rho_o/\rho_i = 1.0$ and in the cases $\rho_o/\rho_i \geq 1.8$ under $E_o/E_i = 1.0$, the limiting values of c/c_2^i are the Rayleigh velocity for the outer layer material, i.e. the c_R^o/c_2^i or the velocity of the Stonoley wave, i.e. the c_{St}/c_2^i (if this wave exists). Consequently, these statements can be used to explain the character of the dispersion curves obtained in the zeroth mode.

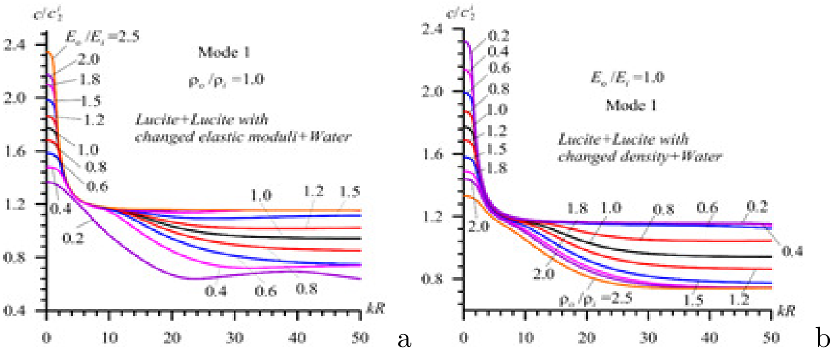

Fig. 3

Consider the results in Fig. 3, which show the dispersion curves with respect to the first mode obtained for different values of the ratio E_o/E_i in the case $\rho_o/\rho_i = 1.0$ (Fig. 3a) and for different ρ_o/ρ_i in the case $E_o/E_i = 1.0$ (Fig. 3a). These results show that, as in the zeroth mode, an increase in the values of the ratio E_o/E_i and a decrease in the values of the ratio ρ_o/ρ_i lead to an increase in the propagation speed of the waves in the first mode. These results also show that, in contrast to the zeroth mode, in the first mode, not only an increase in the values of the ratio E_o/E_i, but also a decrease in the values of the ratio ρ_o/ρ_i leads to an increase in the limit values of the wave propagation speed when approaching lower wave numbers. Moreover, the results in Fig. 3 show that there is such an interval for the change of the dimensionless wave number kR in which the influence of the ρ_o/ρ_i and E_o/E_i ratios becomes insignificant. For the case considered here, this interval is approximately $5 \leq kR \leq 15$.

We also note the unusual nature of the limits of wave propagation speed under higher wave numbers. This unusualness can be explained, as in the zeroth mode, by the relation (26), according to which in the case $E_o/E_i \geq 0.8$ under $\rho_o/\rho_i = 1.0$ and in the cases $\rho_o/\rho_i \leq 1.5$ under $E_o/E_i = 1.0$ the limit value of c/c_2^i is c_R^o/c_2^i, but in the cases $E_o/E_i \leq 0.6$ under $\rho_o/\rho_i = 1.0$ and in the cases $\rho_o/\rho_i \geq 1.8$ under $E_o/E_i = 1.0$ the limit value of c/c_2^i is $\min(c_{Sch}/c_2^i, c_{St}/c_2^i)$.

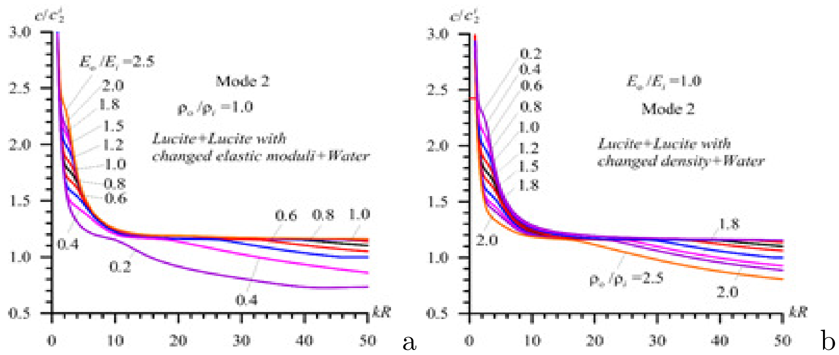

Fig. 4

The above formulated character of the influence of the ratios E_o/E_i and ρ_o/ρ_i is also observed for the dispersion curves of the second mode, which are shown in Fig. 4. The dispersion curves obtained for the second mode for the different values of E_o/E_i under $\rho_o/\rho_i = 1.0$ (for the different values of ρ_o/ρ_i under $E_o/E_i = 1.0$) are shown in Fig. 4a (in Fig. 4b). It can be seen from Fig. 4 that the influence of the change in the ratios E_o/E_i and ρ_o/ρ_i on the cut-off values of the frequencies of the waves in the second mode is negligible. Moreover, it can be seen from Fig. 4 that, as in the first mode, there is such an interval for the change of the dimensionless wave number kR in which the influence of the ratios E_o/E_i and ρ_o/ρ_i on the wave propagation speed in the second mode is insignificant.

5. Conclusions

The present paper deals with the study the influence of the mechanical and acoustoelastic properties of the outer layer of the two-layer hollow cylinder, which contains a compressible, inviscid fluid on the dispersion of the longitudinal axisymmetric waves propagating in this cylinder. The motion of the cylinder is described within the piecewise homogeneous body model by using the exact equations, and relations of elastodynamics, but the flow of the fluid is described by the linearized Euler equations. The solution of the corresponding field equations is found analytically and the dispersion equation is established using the boundary, contact and compatibility conditions. This equation is solved numerically, which produces the dispersion curves. In order to clearly estimate the influence of the mechanical properties of the outer layer material on the dispersion curves, this material is selected as a hypothetical material, the Poisson's ratio and density (modulus of elasticity) of which are equal to the Poisson's ratio and density (modulus of elasticity) of the inner layer material, but the modulus of elasticity (density) is different from that of the inner layer material. However, the inner layer material is selected as a real material, which is Lucite in the present study. The water is selected as the liquid in the cylinder.

The dispersion curves of the zeroth, first and second modes are presented and discussed. Many concrete conclusions were drawn, which are listed in the text of the paper. The most important of them is the increase in the wave propagation speed of axisymmetric waves with an increase in the elastic modulus and with a decrease in the density of the material of the outer layer. Other conclusions were also drawn about the influence of the ratios of the elastic moduli and densities of the materials of the cylinder layers on the character of the dispersion curves of the zero, first and second modes.

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