Azerbaijan Journal of Mathematics
V. 15, No 2, 2025, July
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2025.2.219

Smarandache Curves for the Integral Curves with the Quasi Frame in Euclidean 3-Space

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Abstract. This study investigates the geometric properties of integral Smarandache curves associated with the quasi-frame in three-dimensional Euclidean space. We derive the Frenet apparatus for four types of integral Smarandache curves: the TN_q, TB_q, N_qB_q, and TN_qB_q Smarandache curves, based on the quasi-frame elements. For each type of curve, we provide explicit formulas for the tangent, normal, and binormal vectors, as well as the curvature and torsion. Furthermore, we establish necessary and sufficient conditions under which these integral Smarandache curves can be classified as general helices or Salkowski curves. Specifically, we show that if the original curve is a general helix, the corresponding integral Smarandache curves also exhibit helical properties. Additionally, we analyze the Darboux vectors associated with these curves and demonstrate their relationship with the Darboux vectors of the original curve, proving that the Darboux vectors coincide when the original curve is a general helix. We also provide illustrative examples, including a circular helix and a space curve, to validate our theoretical findings. These examples demonstrate the application of the quasi-frame in simplifying the analysis of complex curves and highlight the geometric relationships between the original curve and its integral Smarandache curves.

Key Words and Phrases: Smarandache curves, integral curves, quasi frame, Euclidean space, general helix, Salkowski curve.

2010 Mathematics Subject Classifications: 53A04, 53C65

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1. Introduction

The study of curves in differential geometry is a fundamental area of research that explores the geometric properties of curves in planar and Euclidean spaces. By employing tools from differential and integral calculus, such as derivatives and integrals, this field investigates the intrinsic and extrinsic properties of curves,

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including curvature, torsion, and arc length [3, 6, 34]. Among the various tools used in curve analysis, the Frenet frame is one of the most widely used frameworks [33]. However, the Frenet frame has limitations, particularly at points where the curvature vanishes, such as straight lines or inflection points, where the frame becomes undefined [2, 28, 29]. To address these limitations, alternative frames, such as the Bishop frame, have been introduced. However, the computation of Bishop frames is often complex and computationally intensive [5, 7]. There are some frames that deal with this problem in different spaces such as [16, 17, 18, 22].

In contrast to the Frenet and Bishop frames, the quasi-frame offers several advantages. The quasi-frame is computationally simpler and remains well-defined even when the curve has zero curvature. This makes it a powerful tool for analyzing curves in various spaces, including Euclidean space [15, 20, 23], Minkowski space [12, 21, 25], and Galilean space [13, 19]. The quasi-frame is defined using a projection vector and an angle between the principal normal and the quasi-normal vector, making it easier to compute and apply in practical scenarios.

Recent studies have explored the concept of integral curves, which are curves obtained by integrating vector fields along a base curve. Integral curves have been studied extensively in the context of the Frenet frame in both three-dimensional and four-dimensional Euclidean spaces [9, 11, 14, 33], as well as using the Flc frame [26]. Additionally, Smarandache curves, which are curves defined by combinations of the Frenet frame vectors of a base curve, have gained significant attention [1, 38]. Smarandache curves have been studied in various contexts, including the Bishop frame [7], the quasi-frame in Euclidean space [1, 8], and Minkowski space-time [39]. Furthermore, SM curves have been investigated in Euclidean 3-space using the Darboux frame [4] and other specialized frames [24]. There are some applications in [10, 27, 30, 32, 37, 40].

In this paper, we focus on the study of integral Smarandache curves associated with the quasi-frame in three-dimensional Euclidean space. Specifically, we investigate four types of integral Smarandache curves: the TN_q-SM, TB_q-SM, N_qB_q-SM, and TN_qB_q-SM curves. For each type of curve, we derive the Frenet apparatus, including the tangent, normal, and binormal vectors, as well as the curvature and torsion. We also establish conditions under which these Smarandache integral curves can be classified as general helices or Salkowski curves, and we analyze their Darboux vectors. Our results reveal that when the original curve is a general helix, the corresponding integral Smarandache curves also exhibit helical properties.

The paper is organized as follows: Section 2 provides the necessary preliminaries, including the definitions of the quasi-frame, quasi-curvatures, and Smarandache curves. Section 3 presents the main results, where we derive the Frenet apparatus for the four types of integral Smarandache curves and analyze their geometric properties. Section 4 provides illustrative examples, including a circular helix and a space curve, to validate our theoretical findings. Finally, we conclude with a discussion of the implications of our results for the field of differential geometry.

This study contributes to the growing body of knowledge on curve analysis by providing a comprehensive framework for understanding integral Smarandache curves in the context of the quasi-frame. The results offer new insights into the geometric properties of these curves and their applications in various fields, including computer-aided design, robotics, and physics.

2. Preliminaries

In this section, we provide the necessary background on the quasi-frame, quasi-curvatures, and Smarandache curves in three-dimensional Euclidean space. These concepts are essential for understanding the integral Smarandache curves studied in this paper.

Let $\mathbb{E}^3$ denote the three-dimensional Euclidean space equipped with the standard metric:

$$\langle \cdot, \cdot \rangle = dx^2 + dy^2 + dz^2,$$

where (x, y, z) is a coordinate system in $\mathbb{E}^3$. The inner product of two vectors $\mathbf{U} = (u_1, u_2, u_3)$ and $\mathbf{V} = (v_1, v_2, v_3)$ is given by:

$$\langle \mathbf{U}, \mathbf{V} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3. \tag{1}$$

A curve $\gamma : I \subset \mathbb{R} \to \mathbb{E}^3$ is said to be of unit speed if it is parameterized by arc length s and satisfies $\langle \gamma'(s), \gamma'(s) \rangle = 1$ [31, 34].

The Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ of a unit-speed curve $\gamma(s)$ consists of the tangent, normal, and binormal vectors, defined as:

$$\mathbf{T} = \frac{\gamma'(s)}{\|\gamma'(s)\|}, \quad \mathbf{N} = \frac{\gamma''(s)}{\|\gamma''(s)\|}, \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}. \tag{2}$$

The evolution of the Frenet frame along the curve is described by the Frenet-Serret equations:

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}, \tag{3}$$

where κ and τ are the curvature and torsion of the curve, respectively, given by:

$$\kappa(s) = \langle \mathbf{T}'(s), \mathbf{N}(s) \rangle, \quad \tau(s) = \langle \mathbf{N}'(s), \mathbf{B}(s) \rangle. \tag{4}$$

An alternative to the Frenet frame is the quasi-frame $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q\}$, which is defined as:

$$\mathbf{T} = \frac{\gamma'(s)}{\|\gamma'(s)\|}, \quad \mathbf{N}_q = \frac{\mathbf{T} \times \mathbf{y}}{\|\mathbf{T} \times \mathbf{y}\|}, \quad \mathbf{B}_q = \mathbf{T} \times \mathbf{N}_q, \tag{5}$$

where $\mathbf{y}$ is a fixed projection vector, typically chosen as $(1,0,0)$, $(0,1,0)$, or $(0,0,1)$, depending on the parallelism with $\mathbf{T}$. The quasi-frame is related to the Frenet frame by an angle ϑ between the principal normal $\mathbf{N}$ and the quasi-normal $\mathbf{N}_q$, given in [23] as follows:

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}. \tag{6}$$

The quasi-curvatures k_1, k_2, k_3 are defined in [28, 29] as:

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}_q' \\ \mathbf{B}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix}, \tag{7}$$

where:

$$\begin{aligned} k_1 &= \langle \mathbf{T}', \mathbf{N}_q \rangle = \kappa \cos\vartheta, \\ k_2 &= \langle \mathbf{T}', \mathbf{B}_q \rangle = -\kappa \sin\vartheta, \\ k_3 &= \langle \mathbf{N}_q', \mathbf{B}_q \rangle = \vartheta' + \tau. \end{aligned} \tag{8}$$

A Smarandache curve (SM curve) is defined as a curve whose position vector is a combination of the Frenet frame vectors of another regular curve [1, 8]. Given a unit-speed curve $\gamma(s)$ with Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, the following SM curves are defined:

$$\eta(s) = \frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{N}), \quad \text{(TN-SM curve)} \tag{9}$$

$$\beta(s) = \frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{B}), \quad \text{(TB-SM curve)} \tag{10}$$

$$\zeta(s) = \frac{1}{\sqrt{2}}(\mathbf{N} + \mathbf{B}), \quad \text{(NB-SM curve)} \tag{11}$$

$$\xi(s) = \frac{1}{\sqrt{3}}(\mathbf{T} + \mathbf{N} + \mathbf{B}). \quad \text{(TNB-SM curve)} \tag{12}$$

A curve $\gamma(s)$ is called a general helix if there exists a constant C such that $\frac{\tau}{\kappa} = C$ [3].

A Salkowski curve is a special type of space curve with constant curvature and varying torsion, characterized by a fixed angle between a line and a normal vector [35, 36].

The Darboux vector $\mathbf{W}$ of a curve $\gamma(s)$ is given in [38] as follows:

$$\mathbf{W} = \tau\mathbf{T} + \kappa\mathbf{B}, \tag{13}$$

where $\mathbf{T}$ and $\mathbf{B}$ are the tangent and binormal vectors, and κ and τ are the curvature and torsion of $\gamma(s)$. The Darboux vector describes the angular velocity of the Frenet frame as it moves along the curve.

Proposition 1. *The Darboux vector with respect to the quasi-frame* $\mathbf{W}$ *is given by*

$$\mathbf{W} = k_3\mathbf{T} - k_2\mathbf{N}_q + k_1\mathbf{B}_q, \tag{14}$$

where k_1, k_2, k_3 *are the quasi-curvatures.*

3. Integral Smarandache Curve in Quasi frame

In this section, we can define the Frenet apparatus of this curve by applying its quasi frame and curvatures of the integral TN_q-SM, TB_q-SM , N_qB_q-SM and TN_qB_q-SM curves. Furthermore, we provide the condition that these integral curves be either a general helix or a Salkowski curve and a Darboux vector.

3.1. The integral TN_q-Smarandache curve

Here we define the Frenet apparatus of this curve by applying its quasi frame and curvatures of the integral TN_q-SM curve and provide the condition that these integral curves be either a general helix or a Salkowski curve and a Darboux vector.

Definition 3.1. *Let* $\gamma(s)$ *be a unit-speed curve in* $\mathbb{E}^3$ *parameterized by arc length* s*, with quasi-frame* $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q\}$*. The* ***integral*** TN_q***-Smarandache curve*** *associated with* γ*, denoted by* Ω*, is defined as the curve whose tangent vector field satisfies:*

$$\Omega'(s) = \frac{1}{\sqrt{2}}\left(\mathbf{T}(s) + \mathbf{N}_q(s)\right). \tag{15}$$

Equivalently, $\Omega(s)$ *is given by the line integral:*

$$\Omega(s) = \int \frac{1}{\sqrt{2}}\left(\mathbf{T}(s) + \mathbf{N}_q(s)\right) ds.$$

Theorem 1. *Let γ be a curve. If Ω is the integral TN_q-SM curve of γ, then Frenet vector fields, the curvature, and the torsion of Ω are given by:*

$$\begin{aligned}
T_\Omega &= \frac{1}{\sqrt{2}}(T + N_q),\\
N_\Omega &= \frac{-k_1T + k_1N_q + (k_2+k_3)B_q}{\sqrt{2k_1^2 + (k_2+k_3)^2}},\\
B_\Omega &= \frac{1}{\sqrt{2}}\frac{(k_2+k_3)T - (k_2+k_3)N_q + 2k_1B_q}{\sqrt{2k_1^2 + (k_2+k_3)^2}},\\
\kappa_\Omega &= \frac{1}{\sqrt{2}}\sqrt{2k_1^2 + (k_2+k_3)^2},\\
\tau_\Omega &= \frac{1}{\sqrt{2}}\left(\frac{2k_1^2\left(\left(\frac{k_2}{k_1}\right)' + \left(\frac{k_3}{k_1}\right)'\right)}{2k_1^2 + (k_2+k_3)^2} + (k_3 - k_2)\right).
\end{aligned} \tag{16}$$

Proof. Let Ω be the integral TN_q-SM curve of γ. Then by taking the derivatives of (15) three times and using (7), we get

$$\begin{aligned}
\Omega' &= \frac{1}{\sqrt{2}}(T + N_q),\\
\Omega'' &= \frac{1}{\sqrt{2}}(-k_1T + k_1N_q + (k_2+k_3)B_q),\\
\Omega''' &= \frac{1}{\sqrt{2}}(-k_1' - k_1^2 - k_2^2 - k_2k_3)T +\\
&\quad + (k_1' - k_1^2 - k_3^2 - k_2k_3)N_q + (-k_1k_2 + k_1k_3 + k_2' + k_3')B_q,
\end{aligned}$$

therefore we have

$$\begin{aligned}
\Omega' \times \Omega'' &= \frac{1}{2}((k_2+k_3)T - (k_2+k_3)N_q + 2k_1B_q),\\
\det(\Omega', \Omega'', \Omega''') &= \frac{1}{2\sqrt{2}}\left((k_3 - k_2)\left[(k_2+k_3)^2 + 2k_1^2\right] + 2k_1^2\left[\left(\frac{k_2}{k_1}\right)' + \left(\frac{k_3}{k_1}\right)'\right]\right).
\end{aligned}$$

By (4), we find the curvature and torsion of Ω:

$$\begin{aligned}
\kappa_\Omega &= \frac{1}{\sqrt{2}}\sqrt{2k_1^2 + (k_2+k_3)^2},\\
\tau_\Omega &= \frac{1}{\sqrt{2}}\left(\frac{2k_1^2\left(\left(\frac{k_2}{k_1}\right)' + \left(\frac{k_3}{k_1}\right)'\right)}{2k_1^2 + (k_2+k_3)^2} + (k_3 - k_2)\right).
\end{aligned}$$

Furthermore, from (2) we find

$$\begin{aligned} T_\Omega &= \frac{1}{\sqrt{2}}(T + N_q), \\ N_\Omega &= \frac{-k_1 T + k_1 N_q + (k_2 + k_3) B_q}{\sqrt{2k_1^2 + (k_2 + k_3)^2}}, \\ B_\Omega &= \frac{1}{\sqrt{2}} \frac{(k_2 + k_3) T - (k_2 + k_3) N_q + 2k_1 B_q}{\sqrt{2k_1^2 + (k_2 + k_3)^2}}. \end{aligned}$$

◀

Theorem 2. *Let Ω be the integral TN_q-SM curve of γ. If γ is a general helix with ϑ constant, then Ω is a general helix.*

Proof. Let Ω be the integral TN_q-SM curve of γ. Then according to (16),we have

$$\frac{\tau_\Omega}{\kappa_\Omega} = \frac{\left((k_3 - k_2) \left[(k_2 + k_3)^2 + 2k_1^2 \right] + 2k_1^2 \left[\left(\frac{k_2}{k_1} \right)' + \left(\frac{k_3}{k_1} \right)' \right] \right)}{\left[2k_1^2 + (k_2 + k_3)^2 \right]^{\frac{3}{2}}},$$

and by (8), we get

$$\frac{k_2}{k_1} = -\tan\vartheta, \qquad \frac{k_1}{k_3} = \frac{k\cos\vartheta}{\vartheta'(s) + \tau}, \qquad \frac{k_2}{k_3} = \frac{-k\sin\vartheta}{\vartheta'(s) + \tau}.$$

Considering that γ is a general helix, $\frac{\tau}{k}$ and ϑ are constants. We determine that Ω is also a general helix as the ratios $\frac{k_2}{k_1}$, $\frac{k_1}{k_3}$ and $\frac{k_2}{k_3}$ are constants. Consequently, the result is rather evident. ◀

Corollary 1. *If Ω is the integral TN_q-SM curve of γ, then Ω is a Salkowski curve if and only if k_1, k_2, and k_3 are constants.*

Theorem 3. *If Ω is the integral TN_q-SM curve of γ, then the Darboux vector of Ω is given by*

$$W_\Omega = \frac{k_1^2 \left[\left(\frac{k_2}{k_1} \right)' + \left(\frac{k_3}{k_1} \right)' \right] (T + N_q)}{(2k_1^2 + (k_2 + k_3)^2)} + \kappa_3 T - \kappa_2 N_q + \kappa_1 B_q.$$

Proof. Let Ω be the integral TN_q-SM curve of γ. Then by applying (13) and (14), we obtain

$$W_\Omega = \frac{k_1^2 \left[\left(\frac{k_2}{k_1} \right)' + \left(\frac{k_3}{k_1} \right)' \right] (T + N_q)}{(2k_1^2 + (k_2 + k_3)^2)} + \kappa_3 T - \kappa_2 N_q + \kappa_1 B_q.$$

Consequently, the result is obvious. ◀

Corollary 2. *The Darboux vector of* Ω *coincides with the Darboux vector of* γ *if* γ *is a general helix.*

3.2. The integral TB_q-Smarandache curve

Here we define the Frenet apparatus of this curve by applying its quasi frame and curvatures of the integral TB_q-SM curve and provide the condition that these integral curves be either a general helix or a Salkowski curve and a Darboux vector.

Definition 3.2. *Let* $\gamma(s)$ *be a unit-speed curve in* $\mathbb{E}^3$ *parameterized by arc length* s*, with quasi-frame* $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q\}$*. The* ***integral*** TN_qB_q***-Smarandache curve*** *associated with* γ*, denoted by* Π*, is defined as the curve satisfying:*

$$\Pi'(s) = \frac{1}{\sqrt{3}} \left(\mathbf{T}(s) + \mathbf{N}_q(s) + \mathbf{B}_q(s)\right), \tag{17}$$

Equivalently, $\Pi(s)$ *is given by the line integral:*

$$\Pi(s) = \int \frac{1}{\sqrt{3}} \left(\mathbf{T}(s) + \mathbf{N}_q(s) + \mathbf{B}_q(s)\right) ds.$$

Theorem 4. *Let* γ *be a curve. If* σ *is the integral* TB_q*-SM curve of* γ*, then Frenet vector fields, the curvature, and the torsion of* σ *are given by:*

$$\begin{aligned}
T_\sigma &= \frac{1}{\sqrt{2}}(T + B_q),\\
N_\sigma &= \frac{-k_2T + (k_1 - k_3)N_q + k_2B_q}{\sqrt{2k_2^2 + (k_1 + k_3)^2}},\\
B_\sigma &= \frac{1}{\sqrt{2}} \frac{-(k_1 - k_3)T - 2k_2N_q + (k_1 - k_3)B_q}{\sqrt{2k_2^2 + (k_1 - k_3)^2}},\\
\kappa_\sigma &= \frac{1}{\sqrt{2}}\sqrt{2k_2^2 + (k_1 - k_3)^2},\\
\tau_\sigma &= \frac{1}{\sqrt{2}}\left(\frac{2k_2^2\left(\left(\frac{k_3}{k_2}\right)' - \left(\frac{k_1}{k_2}\right)'\right)}{2k_2^2 + (k_1 - k_3)^2} + (k_1 + k_3)\right).
\end{aligned} \tag{18}$$

Proof. Let σ be the integral TB_q-SM curve of γ. Then by taking the derivatives of (17) three times and using (7) , we get

$$\sigma' = \frac{1}{\sqrt{2}}(T + B_q),$$

$$\sigma'' = \frac{1}{\sqrt{2}}(-k_2T + (k_1 - k_3)N_q + k_2B_q),$$
$$\sigma''' = \frac{1}{\sqrt{2}}(-k_2' - k_1^2 - k_2^2 + k_1k_3)T + \\ + (k_1' - k_3' - k_1k_2 - k_2k_3)N_q + (k_2' - k_2^2 - k_3^2 + k_1k_3)B_q.$$

Therefore, we have

$$\sigma' \times \sigma'' = \frac{1}{2}(-(k_1 - k_3)T - 2k_2N_q + (k_1 - k_3)B_q),$$
$$\det(\sigma', \sigma'', \sigma''') = \frac{1}{2\sqrt{2}}\left((k_1 + k_3)\left[(k_1 - k_3)^2 + 2k_2^2\right] + 2k_2^2\left[\left(\frac{k_3}{k_2}\right)' - \left(\frac{k_1}{k_2}\right)'\right]\right).$$

From (4), we find the curvature and torsion of σ:

$$\kappa_\sigma = \frac{1}{\sqrt{2}}\sqrt{2k_2^2 + (k_1 - k_3)^2},$$
$$\tau_\sigma = \frac{1}{\sqrt{2}}\left(\frac{2k_2^2\left(\left(\frac{k_3}{k_2}\right)' - \left(\frac{k_1}{k_2}\right)'\right)}{2k_2^2 + (k_1 - k_3)^2} + (k_1 + k_3)\right).$$

Furthermore, from (2), we find

$$T_\sigma = \frac{1}{\sqrt{2}}(T + B_q),$$
$$N_\sigma = \frac{-k_2T + (k_1 - k_3)N_q + k_2B_q}{\sqrt{2k_2^2 + (k_1 + k_3)^2}},$$
$$B_\sigma = \frac{1}{\sqrt{2}}\frac{-(k_1 - k_3)T - 2k_2N_q + (k_1 - k_3)B_q}{\sqrt{2k_2^2 + (k_1 - k_3)^2}}.$$

◀

Theorem 5. *Let σ be the integral TB_q-SM curve of γ. If γ is a general helix with ϑ constant, then σ is a general helix.*

Proof. Let σ be the integral TB_q-SM curve of γ. Then according to (18), we have

$$\frac{\tau_\sigma}{k_\sigma} = \frac{\left((k_1 + k_3)\left[(k_1 - k_3)^2 + 2k_2^2\right] + 2k_2^2\left[\left(\frac{k_3}{k_2}\right)' - \left(\frac{k_1}{k_2}\right)'\right]\right)}{\left[2k_2^2 + (k_1 - k_3)^2\right]^{\frac{3}{2}}},$$

and by (8), we get

$$\frac{k_1}{k_2} = -\arctan\vartheta, \qquad \frac{k_3}{k_1} = \frac{\vartheta'(s)+\tau}{k\cos\vartheta}, \qquad \frac{k_3}{k_2} = \frac{\vartheta'(s)+\tau}{-k\sin\vartheta}.$$

Considering that γ is a general helix, $\frac{\tau}{k}$ and ϑ are constants. We determine that σ is also a general helix as the ratios $\frac{k_2}{k_1}$, $\frac{k_1}{k_3}$ and $\frac{k_2}{k_3}$ are constants. Consequently, the result is rather evident. ◄

Corollary 3. *If σ is the integral TB_q-SM curve of γ. Then, σ is a Salkowski curve if and only if k_1, k_2, and k_3 are constants.*

Theorem 6. *If σ is the integral TB_q-SM curve of γ, then the Darboux vector of σ is given by*

$$W_\sigma = \frac{k_2^2\left(\left(\frac{k_3}{k_2}\right)' - \left(\frac{k_1}{k_2}\right)'\right)(T+B_q)}{(2k_2^2+(k_1-k_3)^2)} + \kappa_3 T - \kappa_2 N_q + \kappa_1 B_q.$$

Proof. Let σ be the integral TB_q-SM curve of γ. Then, by applying (13) and (14), we obtain

$$W_\sigma = \frac{k_2^2\left(\left(\frac{k_3}{k_2}\right)' - \left(\frac{k_1}{k_2}\right)'\right)(T+B_q)}{(2k_2^2+(k_1-k_3)^2)} + \kappa_3 T - \kappa_2 N_q + \kappa_1 B_q.$$

Consequently, the result is obvious. ◄

Corollary 4. *The Darboux vector of σ coincides with the Darboux vector of γ if γ is a general helix.*

3.3. The integral N_qB_q-Smarandache curve

Here we define the Frenet apparatus of this curve by applying its quasi frame and curvatures of the integral N_qB_q-SM curve and provide the condition that these integral curves be either a general helix or a Salkowski curve and a Darboux vector.

Definition 3.3. *Let $\gamma(s)$ be a unit-speed curve in $\mathbb{E}^3$ parameterized by arc length s, with quasi-frame $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q\}$. The* ***integral N_qB_q-Smarandache curve*** *associated with γ, denoted by μ, is defined as the curve satisfying:*

$$\mu'(s) = \frac{1}{\sqrt{2}}\left(\mathbf{N}_q(s) + \mathbf{B}_q(s)\right). \tag{19}$$

Equivalently, $\mu(s)$ is given by the line integral:

$$\mu(s) = \int \frac{1}{\sqrt{2}} \left(\mathbf{N}_q(s) + \mathbf{B}_q(s)\right) ds.$$

Theorem 7. *Let γ be a curve. If μ is the integral N_qB_q-SM curve of γ, then Frenet vector fields, the curvature, and the torsion of μ are given by:*

$$\begin{aligned}
T_\mu &= \frac{1}{\sqrt{2}}(N_q + B_q), \\
N_\mu &= \frac{-(k_1+k_2)T - k_3N_q + k_3B_q}{\sqrt{2k_3^2 + (k_1+k_2)^2}}, \\
B_\mu &= \frac{1}{\sqrt{2}}\left(\frac{(2k_3T - (k_1+k_2)N_q + (k_1+k_2)B_q)}{\sqrt{2k_3^2 + (k_1+k_2)^2}}\right), \\
k_\mu &= \frac{1}{\sqrt{2}}\sqrt{2k_3^2 + (k_1+k_2)^2}, \\
\tau_\mu &= \frac{1}{\sqrt{2}}\left(\frac{-2k_3^2\left(\left(\frac{k_1}{k_3}\right)' + \left(\frac{k_2}{k_3}\right)'\right)}{2k_3^2 + (k_1+k_2)^2} + (k_1 - k_2)\right).
\end{aligned} \tag{20}$$

Proof. Let μ be the integral N_qB_q-SM curve of γ. Then, by taking the derivatives of (19) three times and using (7) , we get

$$\begin{aligned}
\mu' &= \frac{1}{\sqrt{2}}(N_q + B_q), \\
\mu'' &= \frac{1}{\sqrt{2}}(-(k_1+k_2)T + -k_3N_q + k_3B_q), \\
\mu''' &= \frac{1}{\sqrt{2}}(-k_1' - k_2' - k_2k_3 + k_1k_3)T + \\
&\quad + (-k_3' - k_1^2 - k_3^2 - k_1k_2)N_q + (k_3' - k_2^2 - k_3^2 - k_1k_2)B_q.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mu' \times \mu'' &= \frac{1}{2}(2k_3T - (k_1+k_2)N_q + (k_1+k_2)B_q), \\
\det(\mu', \mu'', \mu''') &= \frac{1}{2\sqrt{2}}\left((k_1 - k_2)\left[(k_1+k_2)^2 + 2k_3^2\right] - 2k_3^2\left[\left(\frac{k_1}{k_3}\right)' + \left(\frac{k_2}{k_3}\right)'\right]\right).
\end{aligned}$$

From (4), we find the curvature and torsion of μ:

$$k_\mu = \frac{1}{\sqrt{2}}\sqrt{2k_3^2 + (k_1+k_2)^2},$$

$$\tau_\mu = \frac{1}{\sqrt{2}}\left(\frac{-2k_3^2\left(\left(\frac{k_1}{k_3}\right)' + \left(\frac{k_2}{k_3}\right)'\right)}{2k_3^2 + (k_1+k_2)^2} + (k_1 - k_2)\right).$$

Furthermore, from (2) we find

$$T_\mu = \frac{1}{\sqrt{2}}(N_q + B_q),$$

$$N_\mu = \frac{-(k_1+k_2)T - k_3N_q + k_3B_q}{\sqrt{2k_3^2 + (k_1+k_2)^2}},$$

$$B_\mu = \frac{1}{\sqrt{2}}\left(\frac{(2k_3T - (k_1+k_2)N_q + (k_1+k_2)B_q)}{\sqrt{2k_3^2 + (k_1+k_2)^2}}\right).$$

◄

Theorem 8. *Let μ be the integral N_qB_q-SM curve of γ. If γ is a general helix with ϑ constant, then μ is a general helix.*

Proof. Let μ be the integral N_qB_q-SM curve of γ. Then according to (20), we have

$$\frac{\tau_\mu}{k_\mu} = \frac{\left((k_1-k_2)\left[(k_1+k_2)^2 + 2k_3^2\right] - 2k_3^2\left[\left(\frac{k_1}{k_3}\right)' + \left(\frac{k_2}{k_3}\right)'\right]\right)}{\left[2k_3^2 + (k_1+k_2)^2\right]^{\frac{3}{2}}},$$

and by (8), we get

$$\frac{k_2}{k_1} = -\tan\vartheta, \qquad \frac{k_3}{k_1} = \frac{\vartheta'(s)+\tau}{k\cos\vartheta}, \qquad \frac{k_2}{k_3} = \frac{-k\sin\vartheta}{\vartheta'(s)+\tau}.$$

Considering that γ is a general helix, $\frac{\tau}{k}$ and ϑ are constants. We determine that μ is also a general helix as the ratios $\frac{k_2}{k_1}$, $\frac{k_1}{k_3}$ and $\frac{k_2}{k_3}$ are constants. Consequently, the result is rather evident. ◄

Corollary 5. *If μ is the integral TN_q-SM curve of γ, then μ is a Salkowski curve if and only if k_1, k_2, and k_3 are constants.*

Theorem 9. *If μ is the integral N_qB_q-SM curve of γ, then the Darboux vector of μ is given by*

$$W_\mu = \frac{-k_3^2\left(\left(\frac{k_1}{k_3}\right)' + \left(\frac{k_2}{k_3}\right)'\right)(N_q + B_q)}{(2k_3^2 + (k_1 + k_2)^2)} + \kappa_3 T - \kappa_2 N_q + \kappa_1 B_q.$$

Proof. Let μ be the integral N_qB_q-SM curve of γ. Then by applying (13) and (14), we obtain

$$W_\mu = \frac{-k_3^2\left(\left(\frac{k_1}{k_3}\right)' + \left(\frac{k_2}{k_3}\right)'\right)(N_q + B_q)}{(2k_3^2 + (k_1 + k_2)^2)} + \kappa_3 T - \kappa_2 N_q + \kappa_1 B_q.$$

Consequently, the result is obvious. ◄

Corollary 6. *The Darboux vector of μ coincides with the Darboux vector of γ if γ is a general helix.*

3.4. The integral TN_qB_q-Smarandache curve

Here we define the Frenet apparatus of this curve by applying its quasi frame and curvatures of the integral TN_qB_q-SM curve and provide the condition that these integral curves be either a general helix or a Salkowski curve and a Darboux vector.

Definition 3.4. *Let $\gamma(s)$ be a unit-speed curve in $\mathbb{E}^3$ parameterized by arc length s, with quasi-frame $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q\}$. The* ***integral*** *$TN_qB_q$****-Smarandache curve*** *associated with γ, denoted by Π, is defined as the curve satisfying:*

$$\Pi'(s) = \frac{1}{\sqrt{3}}\left(\mathbf{T}(s) + \mathbf{N}_q(s) + \mathbf{B}_q(s)\right), \tag{21}$$

Equivalently, $\Pi(s)$ is given by the line integral:

$$\Pi(s) = \int \frac{1}{\sqrt{3}}\left(\mathbf{T}(s) + \mathbf{N}_q(s) + \mathbf{B}_q(s)\right) ds.$$

Theorem 10. *Let γ be a curve. If Π is the integral TN_qB_q-SM curve of γ, then the Frenet vector fields, the curvature, and the torsion of Π are given by:*

$$T_\Pi = \frac{1}{\sqrt{3}}(T + N_q + B_q),$$

$$N_\Pi = \frac{-(k_1+k_2)T+(k_1-k_3)N_q+(k_2+k_3)B_q}{\sqrt{(k_1+k_2)^2+(k_1-k_3)^2+(k_2+k_3)^2}},$$
$$B_\Pi = \frac{1}{\sqrt{3}}\left(\frac{(k_2-k_1+2k_3)T-(k_3+k_1+2k_2)N_q+(k_2-k_3+2k_1)B_q)}{\sqrt{(k_1+k_2)^2+(k_1-k_3)^2+(k_2+k_3)^2}}\right),$$
$$k_\Pi = \frac{1}{\sqrt{3}}\sqrt{(k_1+k_2)^2+(k_1-k_3)^2+(k_2+k_3)^2},$$
$$\tau_\Pi = \frac{1}{\sqrt{3}}\left(\frac{3k_1^2\left(\left(\frac{k_2}{k_1}\right)'+\left(\frac{k_3}{k_1}\right)'\right)+3k_3^2\left(\frac{k_2}{k_3}\right)'+2(k_1^3-k_2^3+k_3^3)+6(k_1k_2k_3)}{(k_1+k_2)^2+(k_1-k_3)^2+(k_2+k_3)^2}\right). \tag{22}$$

Proof. Let Π be the integral TN_qB_q-SM curve of γ. Then by taking the derivatives of (21) three times and using (7) , we get

$$\Pi' = \frac{1}{\sqrt{3}}(T+N_q+B_q),$$
$$\Pi'' = \frac{1}{\sqrt{3}}(-(k_1+k_2)T+(k_1-k_3)N_q+(k_2+k_3)B_q),$$
$$\Pi''' = \frac{1}{\sqrt{3}}(-k_1'-k_2'-k_1^2-k_2^2+k_1k_3-k_2k_3)T+(k_1'-k_3'-k_1^2-k_3^2-k_1k_2-k_2k_3)N_q$$
$$+(k_2'+k_3'-k_2^2-k_3^2-k_1k_2+k_1k_3)B_q.$$

Therefore, we have

$$\Pi'\times\Pi'' = \frac{1}{3}(k_2-k_1+2k_3)T-(k_3+k_1+2k_2)N_q+(k_2-k_3+2k_1)B_q),$$
$$\det(\Pi',\Pi'',\Pi''') = \frac{1}{3\sqrt{3}}\left(3k_1^2\left(\left(\frac{k_2}{k_1}\right)'+\left(\frac{k_3}{k_1}\right)'\right)+\right.$$
$$\left.+3k_3^2\left(\frac{k_2}{k_3}\right)'+2(k_1^3+k_2^3+k_3^3)+6(k_1k_2k_3)\right).$$

From (4), we find the curvature and torsion of Π :

$$k_\Pi = \frac{1}{\sqrt{3}}\sqrt{(k_1+k_2)^2+(k_1-k_3)^2+(k_2+k_3)^2},$$
$$\tau_\Pi = \frac{1}{\sqrt{3}}\left(\frac{3k_1^2\left(\left(\frac{k_2}{k_1}\right)'+\left(\frac{k_3}{k_1}\right)'\right)+3k_3^2\left(\frac{k_2}{k_3}\right)'+2(k_1^3-k_2^3+k_3^3)+6(k_1k_2k_3)}{(k_1+k_2)^2+(k_1-k_3)^2+(k_2+k_3)^2}\right).$$

Furthermore, from (2) we find

$$T_\Pi = \frac{1}{\sqrt{3}}(T + N_q + B_q),$$
$$N_\Pi = \frac{-(k_1+k_2)T + (k_1-k_3)N_q + (k_2+k_3)B_q}{\sqrt{(k_1+k_2)^2 + (k_1-k_3)^2 + (k_2+k_3)^2}},$$
$$B_\Pi = \frac{1}{\sqrt{3}}\left(\frac{(k_2-k_1+2k_3)T - (k_3+k_1+2k_2)N_q + (k_2-k_3+2k_1)B_q)}{\sqrt{(k_1+k_2)^2 + (k_1-k_3)^2 + (k_2+k_3)^2}}\right).$$

◀

Theorem 11. *Let Π be the integral TN_qB_q-SM curve of γ. If γ is a general helix with constant ϑ, then Π is a general helix.*

Proof. Let Π be the integral TN_qB_q-SM curve of γ. Then, according to (22), we have

$$\frac{\tau_\Pi}{k_\Pi} = \frac{\left(3k_1^2\left(\left(\frac{k_2}{k_1}\right)' + \left(\frac{k_3}{k_1}\right)'\right) + 3k_3^2\left(\frac{k_2}{k_3}\right)' + 2(k_1^3 - k_2^3 + k_3^3) + 6(k_1k_2k_3)\right)}{((k_1+k_2)^2 + (k_1-k_3)^2 + (k_2+k_3)^2)^{\frac{3}{2}}},$$

and by (8), we get

$$\frac{k_2}{k_1} = -\tan\vartheta,$$
$$\frac{k_2}{k_3} = \frac{-k\sin\vartheta}{\vartheta'(s)+\tau},$$
$$\frac{k_3}{k_1} = \frac{\vartheta'(s)+\tau}{k\cos\vartheta}.$$

Considering that γ is a general helix , $\frac{\tau}{k}$ and ϑ are constants. We determine that Π is also a general helix as the ratios $\frac{k_3}{k_1}$, $\frac{k_2}{k_1}$ and $\frac{k_2}{k_3}$ are constants. ◀

Corollary 7. *If Π is the integral TN_qB_q-SM curve of γ, then Π is a Salkowski curve if and only if k_1, k_2, and k_3 are constants.*

Theorem 12. *If Π is the integral TN_qB_q-SM curve of γ, then the Darboux vector of Π is given by*

$$W_\Pi = \frac{1}{3}((k_2-k_1+2k_3)T - (k_3+k_1+2k_2)N_q + (k_2-k_3+2k_1)B_q)$$

$$+\frac{1}{3}(T+N_q+B_q)\left(\frac{3k_1^2\left(\left(\frac{k_2}{k_1}\right)'+\left(\frac{k_3}{k_1}\right)'\right)+3k_3^2\left(\frac{k_2}{k_3}\right)'+2(k_1^3-k_2^3+k_3^3)+6(k_1k_2k_3)}{(k_1+k_2)^2+(k_1-k_3)^2+(k_2+k_3)^2}\right).$$

Proof. Let Π be the integral TN_qB_q-SM curve of γ. Then, by applying (13) and (14), we obtain

$$W_\Pi=\frac{1}{3}((k_2-k_1+2k_3)T-(k_3+k_1+2k_2)N_q+(k_2-k_3+2k_1)B_q)$$

$$+\frac{1}{3}(T+N_q+B_q)\left(\frac{3k_1^2\left(\left(\frac{k_2}{k_1}\right)'+\left(\frac{k_3}{k_1}\right)'\right)+3k_3^2\left(\frac{k_2}{k_3}\right)'+2(k_1^3-k_2^3+k_3^3)+6(k_1k_2k_3)}{(k_1+k_2)^2+(k_1-k_3)^2+(k_2+k_3)^2}\right).$$

Consequently, the result is obvious. ◄

4. Examples

The circular helix is given by

$$\gamma(s)=\frac{1}{\sqrt{2}}\left(\cos(s),\sin(s),s\right).$$

The quasi-frame for the circular helix is given as follows:

$$\begin{aligned}T(s)&=\frac{1}{\sqrt{2}}\left(-\sin(s),\cos(s),1\right),\\N_q(s)&=\left(\cos(s),\sin(s),0\right),\\B_q(s)&=\frac{1}{\sqrt{2}}\left(-\sin(s),\cos(s),-1\right)\end{aligned}$$

4.1. The TN_q-Smarandache Curve

The TN_q-Smarandache curve is given by

$$\begin{aligned}\Omega(s)&=\int\frac{1}{\sqrt{2}}(T(s)+N_q(s))\,ds\\&=\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\cos(s)+\sin(s)+c_1,\frac{1}{\sqrt{2}}\sin(s)-\cos(s)+c_2,s+c_3\right),\end{aligned}$$

where c_1, c_2 and c_3 are constants.

4.2. The TB_q-Smarandache Curve

The TB_q-Smarandache curve is given by

$$\begin{aligned}\Sigma(s) &= \int \frac{1}{\sqrt{2}}(T(s) + B_q(s))\, ds \\ &= (\cos(s) + c_4, \sin(s) + c_5, c_6)\,,\end{aligned}$$

where c_4, c_5 and c_6 are constants.

4.3. The N_qB_q-Smarandache Curve

The N_qB_q-Smarandache curve is given by

$$\begin{aligned}\mu(s) &= \int \frac{1}{\sqrt{2}}(N_q(s) + B_q(s))\, ds \\ &= \frac{1}{\sqrt{2}}\left(\sin(s) + \frac{\cos(s)}{\sqrt{2}} + c_7, -\cos(s) + \frac{\sin(s)}{\sqrt{2}} + c_8, -\frac{s}{\sqrt{2}} + c_9\right)\end{aligned}$$

where c_7, c_8 and c_9 are constants.

4.4. The TN_qB_q-Smarandache Curve

The TN_qB_q-Smarandache curve is given by

$$\Pi(s) = \int \frac{1}{\sqrt{3}}(T(s) + N_q(s) + B_q(s))\, ds.$$

Substituting $T(s)$, $N_q(s)$, and $B_q(s)$:

$$\Pi(s) = \frac{1}{\sqrt{3}}\left(\sqrt{2}\cos(s) + \sin(s) + c_{10}, \sqrt{2}\sin(s) - \cos(s)c_{11}, c_{12}\right),$$

where c_{10}, c_{11} and c_{12} are constants.

5. Conclusion

In this study, we explored the geometric properties of integral Smarandache curves associated with the quasi-frame in three-dimensional Euclidean space. By deriving the Frenet apparatus for four types of integral Smarandache curves TN_q-SM, TB_q-SM, N_qB_q-SM, and TN_qB_q-SM, we established explicit formulas for their tangent, normal, and binormal vectors, as well as their curvature and torsion. Furthermore, we provided necessary and sufficient conditions under which these curves can be classified as general helices or Salkowski curves. Our results demonstrate that when the original curve is a general helix, the corresponding integral Smarandache curves also exhibit helical properties. Additionally, we analyzed the Darboux vectors of these curves and showed that they coincide with those of the original curve when it is a general helix. Through illustrative examples, including a circular helix and a space curve, we validated our theoretical findings and highlighted the utility of the quasi-frame in simplifying the analysis of complex curves. This work contributes to the broader understanding of curve theory in differential geometry and provides a foundation for further research in applications such as computer-aided design, robotics, and physics.

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Received 24 February 2025
Accepted 27 April 2025