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On Solvability of Some Quasilinear Parabolic Equations of Higher Order

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Abstract. We consider boundary value problems for quasilinear parabolic equations with a main quasilinear elliptic operator of order $2b \geq 2$ in Sobolev space $W_p^{2b,1}(Q_T)$.

Key Words and Phrases: quasilinear parabolic equation, interpolation method, Sobolev space.

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1. Introduction

Let $a \in R_+ \equiv \left\{ a \in R \middle| a \geq 0 \right\}$; $x = (x_1, \dots, x_n)$ be a point in the space $R^n, \Omega \subset R^n$ be a bounded domain with the boundary $\partial\Omega$ of class c^{2b}, $b \geq 1$; $Q_T = \Omega \times (0, T)$ be a cylinder of the given height $T > 0$; $Q_t = \Omega \times (a, a+t)$ be a cylindrical domain in the space R^{n+1}, $t \in R_+$; $\partial Q_t = \partial\Omega \times (a, a+t)$ be a lateral surface of the cylinder Q_t.

Throughout this paper, the functions are assumed to be real-valued. We will use the following function spaces [8, p.126], [9, p.118], [10] and [11]: the spaces of summable functions $L_p(Q_t)$, $p \geq 1$ with the norm

$$\|u\|_{p;Q_t} = \left(\int\limits_a^{a+t} \int\limits_\Omega |u(x,t)|^p \, dxdt \right)^{1/p};$$

the anisotropic Sobolev space $W_p^{2b,1}(Q_t)$ with the norm

$$\|u\|_{W_p^{2b,1}(Q_t)} = \|u\|_{p;Q_t} + \sum_{i=1}^n \left\| \frac{\partial^{2b} u}{\partial x_i^{2b}} \right\|_{p;Q_t} + \|u_t\|_{p;Q_t}.$$

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By $D^i u$ we denote a vector from partial derivatives $D^\alpha u$ of the function $u(x,t)$, $\alpha = (\alpha_1, \ldots, \alpha_n, 0)$, $|\alpha| = i$. This vector has n_i components, where n_i is the number of various multiindice $\alpha = (\alpha_1, \ldots, \alpha_n, 0)$, $|\alpha| = i$.

Each section of the paper has its own notations. Therefore, in different sections, C_1, C_2, C_3, etc. can denote different constants.

We deal with the generalized solutions $u \in W_p^{2b,1}(Q_T)$, $p > 1$ of the problem

$$\begin{cases} Lu + \frac{\partial u}{\partial t} = f\left(x, t, u, Du, \ldots, D^{2b-1}u\right), \ (x,t) \in Q_T, \\ B_i u\Big|_{\partial Q_T} = 0 \ \ (i = 0, 1, \ldots, b-1), \ (x,t) \in \partial Q_T, \\ u(x,0) = 0, \ \ x \in \Omega \end{cases} \tag{1}$$

provided that there exists a priori estimate $\|u\|_{W_2^{b,1}(Q_T)}$ in the space $W_2^{b,1}(Q_T)$ and $b > \frac{n+2}{2}$. Thus, the boundary value problem (1) is considered in the intersection of spaces $W_p^{2b,1}(Q_T) \cap W_2^{b,1}(Q_T)$. Note that [see. 17] for $[2b + (n+2)]\, p \geq 2(2b+n)$ we have the embedding $W_p^{2b,1}(Q_T) \subset W_2^{b,1}(Q_T)$.

Here $D^l u = \left\{ D_x^\gamma u \,\middle|\, \gamma \text{ is a multiindex with } |j| = l \right\}$, Lu is a quasilinear elliptic operator of order $2b \geq 2$ of the form

$$Lu = \sum_{|\alpha|=2b} a_\alpha \left(x, t, u, Du, \ldots, D^k u\right) D^\alpha u$$

with some k, $0 \leq k \leq 2b-1$, the coefficients $a_\alpha\left(x, t, u, \ldots, D^k u\right)$ $(|\alpha| = 2b)$ of the operator L are real and continuous functions on $\overline{Q_T} \times R \times R^n \times \cdots \times R^{n_k}$ and B_i $(i = 0, 1, \ldots, b-1)$ are linear boundary differential operators of orders $b_i \leq 2b-1$ with real coefficients.

In what follows, $\xi_l = \left\{ \xi_\gamma \,\middle|\, \gamma \text{ is a multiindex with } |\gamma| = l \right\}$ is such that $\xi_0 \in R$, $\xi_1 \in R^n, \ldots, \xi_l \in R^{n_l}$ with corresponding n_l and $|\xi_l| = \sum\limits_{|\gamma|=l} |\xi_\gamma|$.

The following conditions are to be fulfilled.

A.1) Let the function $f(x, t, \xi_0, \xi_1, \ldots, \xi_{2b-1})$ be defined on $\overline{Q_T} \times R \times R^n \times \cdots \times R^{n_{2b-1}}$ with the values in R and also be a Caratheodory function, i.e. measurable with respect to (x,t) for all $(\xi_0, \xi_1, \ldots, \xi_{2b-1}) \in R \times R^n \times \cdots \times R^{n_{2b-1}}$ and continuous with respect to $(\xi_0, \xi_1, \ldots, \xi_{2b-1})$ almost for all $(x,t) \in Q_T$.

A.2) Let $b > \frac{n+2}{2}$. By l_0 we denote the least positive integer larger or equal to $b - \frac{n+2}{2}$ (i.e. $l_0 \geq b - \frac{n+2}{n}$) and let $\xi_* = \left\{ \xi_\gamma \,\middle|\, |\gamma| < l_0 \right\}$.

Let

$$|f(x, t, \xi_0, \xi_1, \ldots, \xi_{2b-1})| \leq b(x, t, \xi_*) + \sum_{l=l_0}^{2b-1} b_l(x, t, \xi_*) \cdot |\xi_l|^{\mu_l}$$

almost for all $(x,t) \in Q_T$ and for all $\xi_0 \in R$, $\xi_1 \in R^n, \ldots, \xi_{2b-1} \in R^{n_{2b-1}}$ with non-negative Caratheodory function b, b_l, such that for any $r \geq 0$

$$\hat{b}_r(x,t) = \sup \left\{ b(x,t,\xi_*) \,\middle|\, |\xi_*| = \sum_{|\gamma| < b - \frac{n+2}{2}} |\xi_\gamma| \leq r \right\}$$

belongs to $L_p(Q_T)$ with $p > 1$ and $[2b + (n+2)]\, p > 2(n+2b)$, while the function

$$\hat{b}_{l,r}(x,t) = \sup \left\{ b_l(x,t,\xi_*) \,\middle|\, |\xi_*| \leq r \right\}$$

belongs to $L_{q_l}(Q_T)$ with $q_l > p$ for $l = l_0,\ l_0+1, \ldots, 2b-1$.

A.3) Let

$$\mu_l = \frac{2b+n+2}{n+2+2(l-b)} - \frac{2}{n+2+2(l-b)} \cdot \frac{n+2b}{q_l} \tag{2}$$

for $l = l_0,\ l_0+1, \ldots, 2b-1$ if $b > \frac{n+2}{2}$. Here μ_{l_0} is any positive number for $n+2+2(l_0-b) = 0$.

A.4) Let $b > \frac{n+2}{2}$ and the integer $k \geq 0$ be such that $b - k > \frac{n+2}{2}$.

Let the operator $u(x,t)$ linear with respect to $L_v u$, equal to

$$L_v u = \sum_{|\alpha|=2b} a_\alpha \left(x,t,v,\ldots,D^k v\right) D^\alpha u,$$

for any function $v \in C^{k,0}\left(\overline{Q_T}\right)$ be a linear elliptic operator, and $B_i\ (i = 0,1,\ldots,b-1)$ be linear boundary differential operators of orders $b_i \leq 2b-1$, respectively, with real coefficients such that the linear with respect to $u(x,t)$) boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} + L_v u(x,t) = g(x,t)\,, & (x,t) \in Q_T, \\ B_i u\Big|_{\partial Q_T} = 0 \ \ (i = 0,1,\ldots,b-1)\,, & (x,t) \in \partial\Omega \times (0,T)\,, \\ u(x,0) = 0, \ \ x \in \Omega \end{cases} \tag{3}$$

is coercive in the space $W_p^{2b,1}(Q_T)$, i.e. a priori estimate

$$\|u\|_{W_p^{2b,1}(Q_T)} \leq C \left(\|g\|_{p;Q_T} + \|u\|_{p;Q_T} \right) \tag{4}$$

with a positive constant C, independent of $g \in L_p(Q_T)$ and of the solution $u \in W_p^{2b,1}(Q_T)$ of linear problem (3) is true. This time, the constant C can depend on the modules of continuity of the coefficients a_α, ($|\alpha| = 2b$) and modules of continuity of the functions $v, Dv, \ldots, D^k v$. Note that sufficient conditions on the coefficients of the operator L and B_i $(i = 0, 1, \ldots, b-1)$, providing coerciveness of linear (with respect to $u(x,t)$) boundary value problem (3) were stated in the papers of S.Agmon, A. Duglis, L.Nierenberg [12], V.A.Solonnikov [9, p.112] and [10].

This paper organized as follows.

In Section 1, we give a theorem on a priori estimate $\|u\|_{W_p^{2b,1}(Q_T)}$, expressed by $\|u\|_{W_2^{b,1}(Q_T)}$. In Section 2, there is a counterexample showing the unimpovability of the growth of the function f with regard to the derivatives $D^\gamma u$ with $|\gamma| \leq 2b-1$ under the general conditions of the theorem from Section 1. A theorem on the solvability of boundary value problems for some quasilinear parabolic equations of arbitrary order is proved in Section 3, Section 4. The main attention is paid to the attainability of limits to growth for the nonlinear component of equations and to restrictions on its summability. As known, for second order equations of elliptic [1] and parabolic [4] types, the limit degrees are achieved. But as the examples constructed in [2] show, this is not true for higher order elliptic equations. In the parabolic case Wahl [5, 6] proved that subject to the coercive type conditions for higher order equations and systems, the maximum limit to growth is achievable.

The solvability of boundary value problems is proved on the basis of known theorems on a priori estimates using the Leray-Schauder method [7, p.235-236].

Quasilinear elliptic parabolic equations were considered by M.I Vishik [13], Yu. D. Dubinski [14], F.Browder [18], V. Wahl [15], S.I. Pohozhaev [1, 2], G.G. Laptev [3, 4] and other authors. In [13, 14, 15, 18], it was assumed that a nonlinear operator has a divergent structure. In [1, 2, 3, 4], and in the present paper this problem is studied without assumption of divergence of the structure of subordinate nonlinear operator. This enables us to consider more general nonlinear subordinated operators.

The paper is based on the following two moments: the use of an interpolation inequality for estimating $\left\|D^j u\right\|^{\mu_l}$, $|j| = l$, $2b > l \geq b - \frac{n+2}{2}$ through $\|u\|_{W_p^{2b,1}(Q_t)}$ and $\|u\|_{W_2^{b,1}(Q_t)}$, and application of the theorem on solvability of linear parabolic problems in the small cylinder Q_t [9, p.112-129]. By means of special constructions it is shown that the coefficients for $\|u\|_{W_p^{2b,1}(Q_t)}$ are independent of the cylinder height, when this height is low. This allows through the use of low-height cylinders to set the achievability of limiting degrees μ_l that are not achieved in the elliptic case.

Lemma 1. *Let* $u \in W_p^{2b,1}(Q_\tau)$, $\quad Q_\tau = \Omega \times (t_0, t_0 + \tau) \subset Q_T$, $u(x,t_0) = 0$ *and condition A.1) -A.3) be fulfilled. Then,*

$$\left\|D^j u\right\|_{S_l;Q_\tau} \leq C_1 \cdot \|u\|^{\theta_l}_{W_p^{2b,1}(Q_\tau)} \cdot \|u\|^{1-\theta_l}_{W_2^{b,1}(Q_\tau)} + C_2 \cdot \|u\|_{W_2^{b,1}(Q_\tau)} \tag{5}$$

with positive constants C_1 *and* C_2, *independent of the function* $u(x,t)$ *from* $W_p^{2b,1}(Q_\tau)$, t_0 *and* τ, *where*

$$\frac{1}{S_l}=\frac{2(l-b)+n+2}{2(n+2b)}+\theta_l\left(\frac{1}{p}-\frac{2b+n+2}{2(n+2b)}\right),$$

$$\frac{2(l-b)+n+2}{2b+n+2}\le\theta_l<1,\ |j|=l,\ 2b>l>k. \tag{6}$$

Proof. The function $u(x,t)$ is originally defined in the cylinder of low height τ. Let's continue it over the entire cylinder Q_T by extending it by zero on the segment $[0,t_0]$ and then redefining it elsewhere. In more detail,

$$\tilde{u}(x,t)=\begin{cases}0,\ \ 0\le t\le t_0,\\ u(x,t),\ \ t_0<t<t_0+\tau,\\ u(x,2(t_0+\tau)-t),\ \ t_0+\tau\le t<t_0+2\tau,\\ 0,\quad t\ge t_0+2\tau.\end{cases}$$

Note that according to the theorem on traces [9, p. 116], for each $t\in[t_0,t_0+\tau]$, the trace $u(x,t)$ is determined as a function continuous in $\overline{\Omega}$. In particular, the functions $u(x,t_0)$ and $u(x,t_0+\tau)$ are completely defined. Obviously, $\tilde{u}\in W_p^{2b,1}(Q_T)$, this time

$$\|\tilde{u}\|_{W_2^{b,1}(Q_T)}\le 2\cdot\|u\|_{W_2^{b,1}(Q_\tau)},$$

$$\|\tilde{u}\|_{W_p^{2b,1}(Q_T)}\le 2\|u\|_{W_p^{2b,1}(Q_\tau)},\quad \left\|D^j\tilde{u}\right\|_{S_l;Q_T}\ge\left\|D^ju\right\|_{S_l;Q_\tau}. \tag{7}$$

From the Galiiardo-Nirenberg interpolation inequality [16] for the derivative $\left\|D^j\tilde{u}\right\|$, $|j|=l,\ 2b>l>k$, we have

$$\left\|D^j\tilde{u}\right\|_{S_l;Q_T}\le C_1\cdot\|\tilde{u}\|^{\theta_l}_{W_p^{2b,1}(Q_T)}\cdot\|\tilde{u}\|^{1-\theta_l}_{W_2^{b,1}(Q_T)}+C_2\|\tilde{u}\|_{W_2^{b,1}(Q_T)}.$$

The direct calculation shows that $\theta_l=1/\mu_l$. Taking into account the inequalities , we obtain the statement of Lemma 1. ◄

Lemma 2. *Let the conditions A.1) - A.3) be fulfilled. Then the operator* $F(u)(x,t)\equiv f\left(x,t,u,Du,\dots,D^{2b-1}u\right)$ *acts completely continuosly from* $W_p^{2b,1}(Q_\tau)$ *to* $L_p(Q_\tau)$.

Proof. Estimate $\|F(u)\|_{p;Q_\tau}$ by means of conditions A.1)-A.3). It follows from condition A.2) that

$$\|F(u)\|_{p;Q_\tau}\le\left\|\hat{b_r}\right\|_{p;Q_\tau}+C_3\cdot\sum_{l=l_0}^{2b-1}\sum_{|j|=l}\left\|\hat{b_{l,r}}\right\|_{q_l;Q_\tau}\cdot\left\|D^ju\right\|^{\mu_l}_{S_l;Q_\tau}$$

with $r = \text{const}\,\|u\|_{W_2^{b,1}(Q_\tau)}$, $S_l = \frac{pq_l\cdot\mu_l}{q_l-p}$ and the positive constant C_3 independent of the function $u(x,t)$ from $W_p^{2b,1}(Q_\tau)$, t_0 and τ. Then, based on the equalities (2) and interpolation inequalities (5) with $|j| = l = l_0, l_0+1, \ldots, 2b-1$ and corresponding S_l and μ_l, defined by formula (6), we obtain

$$\|F(u)\|_{p;Q_\tau} \le \left\|\hat{b_r}\right\|_{p;Q_\tau} + C_3 \cdot \sum_{l=l_0}^{2b-1} \sum_{|j|=l} \left\|\hat{b_{l,r}}\right\|_{q_l;Q_\tau} \cdot$$

$$\cdot \left[C_1 \cdot \|u\|^{\theta_l}_{W_p^{2b,1}(Q_\tau)} \cdot \|u\|^{1-\theta_l}_{W_2^{b,1}(Q_\tau)} + C_2 \|u\|_{W_2^{b,1}(Q_\tau)}\right]^{\mu_l} \le$$

$$\le \left\|\hat{b_r}\right\|_{p;Q_\tau} + C_3 \cdot \sum_{l=l_0}^{2b-1} \left\|\hat{b_{l,r}}\right\|_{q_l;Q_\tau} \cdot 2^{\mu_l-1} \cdot C_1^{\mu_l} \cdot \|u\|_{W_p^{2b,1}(Q_\tau)} \cdot \|u\|^{\mu_l-1}_{W_2^{b,1}(Q_\tau)} +$$

$$+C_3 \cdot \sum_{l=l_0}^{2b-1} \left\|\hat{b_{l,r}}\right\|_{q_l;Q_\tau} \cdot 2^{\mu_l-1} \cdot C_2^{\mu_l} \cdot \|u\|^{\mu_l}_{W_2^{b,1}} \cdot =$$

$$= \left\|\hat{b_r}\right\|_{p;Q_\tau} + \Phi_1\left(\|u\|_{W_2^{b,1}(Q_\tau)}\right) \cdot \|u\|_{W_p^{2b,1}(Q_\tau)} + \Phi_2\left(\|u\|_{W_2^{b,1}(Q_\tau)}\right), \tag{8}$$

where

$$\Phi_1\left(\|u\|_{W_2^{b,1}(Q_\tau)}\right) = C_3 \cdot \sum_{l=l_0}^{2b-1} \left\|\hat{b_{l,r}}\right\|_{q_l;Q_\tau} \cdot 2^{\mu_l-1} \cdot C_1^{\mu_l} \cdot \|u\|^{\mu_l-1}_{W_2^{b,1}(Q_\tau)},$$

$$\Phi_2\left(\|u\|_{W_2^{b,1}(Q_\tau)}\right) = C_3 \cdot \sum_{l=l_0}^{2b-1} \left\|\hat{b_{l,r}}\right\|_{q_l;Q_\tau} \cdot 2^{\mu_l-1} \cdot C_2^{\mu_l} \cdot \|u\|^{\mu_l}_{W_2^{b,1}(Q_\tau)},$$

i.e. $\Phi_1, \Phi_2 : R_+ \to R_+$ are the increasing functions defined by the given data. This proves the boundedness of the operator $F(u)$.

Since $p[2b+(n+2)] > 2(n+2b)$, from Sobolev‘s embedding theorem [17, p.74-95] it follows that the embedding operator $F : W_p^{2b,1}(Q_\tau) \to L_{S_l}(Q_\tau)$ is completely continuous. By estimate (8), the operator $F : L_{s_l}(Q_\tau) \to L_p(Q_\tau)$ is bounded and, by the general properties of a superposition operator, is continuous. Then the operator $F : W_p^{2b,1}(Q_\tau) \to L_p(Q_\tau)$ is completely continuous as a composition of completely continuous and continuous operators. ◄

Theorem 1. *Let conditions A.1) - A.4) be fulfilled. Then there exists such an increasing function* $\Phi : R_+ \to R_+$ *that for any possible solution* $u \in W_p^{2b,1}(Q_T)$ *of problem* (1) *the following a priori estimate is true:*

$$\|u\|_{W_p^{2b,1}(Q_T)} \le \Phi\left(\|u\|_{W_2^{b,1}(Q_T)}\right). \tag{9}$$

The function Φ *depends only on unknown data included into the conditions of the theorem (including those on the quantities* $\left\|\hat{b}_r\right\|_{p,Q_T}$, $\left\|\hat{b}_{l_0,r}\right\|_{q_{l_0};Q_T}, \ldots, \left\|\hat{b}_{2b-1,r}\right\|_{q_{2b-1};Q_T}$ *with* $r = const\|u\|_{W_2^{b,1}(Q_T)}$.

Proof. In the cylinder $Q_\tau = \Omega \times (t_0, t_0 + \tau)$, we consider the following linear problem:

$$\begin{cases} L_v u + \frac{\partial u}{\partial t} = g(x,t), \ \ (x,t) \in Q_\tau, \\ B_i u\Big|_{\partial Q_\tau} = 0 \ \ (i = 0, 1, \ldots, b-1), \ (x,t) \in \partial Q_\tau, \\ u(x, t_0), \ \ x \in \Omega. \end{cases} \tag{10}$$

The linear boundary value problem (10) (with respect to $u(x,t)$) is coercive in the space $W_p^{2b,1}(Q_\tau)$, i.e. the following a priori estimate is true:

$$\|u\|_{W_p^{2b,1}(Q_\tau)} \le C_4\left(\|g\|_{p;Q_\tau} + \|u\|_{p;Q_\tau}\right) \tag{11}$$

with positive constant C_4, independent of $g \in L_p(Q_\tau)$, the solution $u \in W_p^{2b,1}(Q_\tau)$ and t_0, τ . On the other hand, from the Sobolev embedding theorem [17, p. 74] we have

$$\|u\|_{p;Q_\tau} \le C_5 \cdot \|u\|_{W_2^{b,1}(Q_\tau)} \tag{12}$$

for $2b > n+2$, with the constant $C_5 > 0$, independent of the function $u(x,t)$ and t_0, τ.

Now, using the coerciveness inequality (11) with

$$g(x,t) \equiv f\left(x, t, u, Du, \ldots, D^{2b-1}u\right) \equiv F(u)(x,t),$$

for the problem

$$\begin{cases} Lu + \frac{\partial u}{\partial t} = f\left(x, t, u, Du, \ldots, D^{2b-1}u\right), \ \ (x,t) \in Q_\tau, \\ B_i u\Big|_{\partial Q_\tau} = 0 \ \ (i = 0, 1, \ldots, b-1), \ (x,t) \in \partial Q_\tau, \\ u(x, t_0), \ \ x \in \Omega, \end{cases} \tag{13}$$

estimate (12) and inequality (8) we obtain

$$\|u\|_{W_p^{2b,1}(Q_\tau)} \le C_4 \cdot \left[\left\|\hat{b}_r\right\|_{p,Q_\tau} + \Phi_1\left(\|u\|_{W_2^{b,1}(Q_\tau)}\right)\cdot\right.$$

$$\cdot \|u\|_{W_p^{2b,1}(Q_\tau)} + \Phi_2\left(\|u\|_{W_2^{b,1}(Q_\tau)}\right) + C_5 \cdot \|u\|_{W_2^{b,1}(Q_\tau)}\Big]. \tag{14}$$

Based on the aforesaid, we choose τ so that the inequality $C_4 \cdot \Phi_1 \leq \frac{1}{2}$ is satisfied or

$$C_3 C_4 \cdot \sum_{l=l_0}^{2b-1} C_1^{\mu_l} \cdot 2^{\mu_l - 1} \cdot \|u\|_{W_2^{b,1}(Q_\tau)}^{\mu_l - 1} \cdot \left\|\hat{b_{l,r}}\right\|_{q_l;Q_\tau} \leq \frac{1}{2}. \tag{15}$$

Recall that the constants C_3 and C_4 are independent of the choice of t_0 and τ.

We decompose the domain Q_T into the cylinders $Q^0 = \Omega \times (0, \tau)$, $Q^1 = \Omega \times (\tau, 2\tau)$, $\cdot$, $Q^k = \Omega \times (k\tau, (k+1)\tau)$, $\ldots$, $Q^K = \Omega \times (K\tau, T)$ of height τ. Since τ is fixed, the number of these cylinders are finite.

Let $u_0 \in W_p^{2b,1}(Q_T)$ be the solution of (1). Let us consider problem (1) in the cylinder Q^0 as linear one with the right-hand side $f_0 = f\left(x, t, u_0, \ldots, D^{2b-1}u_0\right)$. According to the inequalities (11), (12) and (14)

$$\|u_0\|_{W_p^{2b,1}(Q^0)} \leq C_4\left(\|f_0\|_{p;Q^0} + \|u_0\|_{p;Q^0}\right) \leq$$

$$\leq C_4\left(\left\|\hat{b}_r\right\|_{p;Q^0} + \Phi_1\left(\|u_0\|_{W_2^{b,1}(Q^0)}\right) \cdot \|u_0\|_{W_p^{2b,1}(Q^0)} + \right.$$

$$\left. + \Phi_2\left(\|u_0\|_{W_2^{b,1}(Q^0)}\right) + C_5 \cdot \|u_0\|_{W_2^{b,1}(Q^0)}\right). \tag{16}$$

By virtue of (15), the coefficient for $\|u_0\|_{W_p^{2b,1}(Q^0)}$ does not exceed $\frac{1}{2}$, so the inequality (16) takes the form

$$\|u_0\|_{W_p^{2b,1}(Q^0)} \leq \frac{1}{2} \cdot \|u_0\|_{W_p^{2b,1}(Q^0)} + \frac{C^0}{2},$$

where the constant C^0 is defined by (16). Hence

$$\|u_0\|_{W_p^{2b,1}(Q^0)} \leq C^0.$$

Consider the cylinders Q^{k-1} and $Q^k, 1 \leq k \leq K$ and assume that the estimate

$$\|u_0\|_{W_p^{2b,1}\left(Q^{k-1}\right)} \leq C^{k-1}$$

is already obtained.

Let us make sure that it implies the estimate

$$\|u_0\|_{W_p^{2b,1}\left(Q^k\right)} \leq C^k.$$

Let us denote $V(x,t) = u_0(x, 2k\tau - t)$ and assume

$$\tilde{u}(x,t) = u_0(x,t) - V(x,t), \ \ (x,t) \in Q^k.$$

Obviously, $\tilde{u} \in W_p^{2b,1}(Q^k)$ and $\tilde{u}(x, k\tau) = 0$.

Consider in Q^k the following problem:

$$\begin{cases} L\tilde{u} + \frac{\partial \tilde{u}}{\partial t} = f\left(x, t, \tilde{u} + V, \ldots, D^{2b-1}(\tilde{u} + v)\right) - \left(LV + \frac{\partial V}{\partial t}\right), \ \ (x,t) \in Q^k, \\ \tilde{u}(x, k\tau) = 0, \ \ x \in \Omega, \\ B_i \tilde{u}\Big|_{\partial Q^k} = 0 \ \ (i = 0, 1, \ldots, b-1), \ (x,t) \in \partial Q^k, \\ \tilde{u}(x, k\tau) = 0, \ \ x \in \Omega. \end{cases} \tag{17}$$

The function $\tilde{u}(x,t)$ is the solution of the problem (17) since according to the assumption, the function $u_0(x,t)$ is the solution of problem (1).

Using the estimate

$$\|\tilde{u}\|_{W_2^{b,1}(Q^k)} \leq 2\|u_0\|_{W_2^{b,1}(Q_T)} \tag{18}$$

and the equality

$$\|V\|_{W_p^{2b,1}(Q^k)} = \|u_0\|_{W_p^{2b,1}(Q^{k-1})}, \tag{19}$$

similar to (16), we obtain

$$\|\tilde{u}\|_{W_p^{2b,1}(Q^k)} \leq$$

$$\leq C_4 \cdot \left(\left\| f\left(x, t, \tilde{u} + V, \ldots, D^{2b-1}(\tilde{u} + V)\right) - \left(\frac{\partial V}{\partial t} + LV\right) \right\|_{p;Q^k} + \|\tilde{u}\|_{p;Q^k} \right) \leq$$

$$\leq C_4 \left(\left\| \hat{b}_r \right\|_{p;Q^k} + C_3 \cdot \sum_{l=l_0}^{2b-1} \sum_{|j|=l} \left\| \hat{b}_{l;r} \right\|_{q_l;Q^k} \cdot \left\| D^j(\tilde{u} + V) \right\|_{S_l;Q^k}^{\mu_l} + \right.$$

$$\left. + C_6 \cdot \|V\|_{W_p^{2b,1}(Q^k)} + C_5 \cdot \|\tilde{u}\|_{W_2^{b,1}(Q^k)} \right) \leq$$

$$\leq C_4 \left(\left\| \hat{b}_r \right\|_{p;Q^k} + \Phi_1\left(\|\tilde{u} + V\|_{W_2^{b,1}(Q^k)} \right) \cdot \|\tilde{u} + V\|_{W_p^{2b,1}(Q^k)} + \right.$$

$$\left. + \Phi_2\left(\|\tilde{u} + V\|_{W_2^{b,1}(Q^k)} \right) + C_6 \cdot \|u_0\|_{W_p^{2b,1}(Q^{k-1})} + C_5 \cdot \|u_0 - V\|_{W_2^{b,1}(Q^k)} \right) \leq$$

$$\leq C_4 \left(\left\| \hat{b}_r \right\|_{p;Q^k} + \Phi_1\left(\|u_0\|_{W_2^{b,1}(Q^k)} \right) \cdot \|u_0\|_{W_p^{2b,1}(Q^k)} + \Phi_2\left(\|u_0\|_{W_2^{b,1}(Q^k)} \right) + \right.$$

$$\left. + C_6 \|u_0\|_{W_p^{2b,1}(Q^{k-1})} + 2C_5 \cdot \|u_0\|_{W_2^{b,1}(Q^k)} \right), \tag{20}$$

where $C_6 > 0$ is a constant independent of the function $u(x,t)$ and t_0, τ.

Since the coefficient $\|u_0\|_{W_p^{2b,1}(Q^k)}$ does not exceed $\frac{1}{2}$, the inequality (20) takes the form

$$\|\tilde{u}\|_{W_p^{2b,1}(Q^k)} \leq \frac{1}{2} \cdot \|u_0\|_{W_p^{2b,1}(Q^k)} + C_7 \cdot \|u_0\|_{W_p^{2b,1}(Q^{k-1})} + C_8, \tag{21}$$

where C_7 and C_8 are defined by (20).

Note that $u_0 = \tilde{u} + V$, so

$$\|u_0\|_{W_p^{2b,1}(Q^k)} \leq \|\tilde{u}\|_{W_p^{2b,1}(Q^k)} + \|V\|_{W_p^{2b,1}(Q^k)}.$$

Using (21) and (19), we obtain

$$\|u_0\|_{W_p^{2b,1}(Q^k)} \leq \frac{1}{2} \|u_0\|_{W_p^{2b,1}(Q^k)} + C_7 \cdot \|u_0\|_{W_p^{2b,1}(Q^{k-1})} +$$

$$+C_8 + \|V\|_{W_p^{2b,1}(Q^k)} \leq \frac{1}{2} \|u_0\|_{W_p^{2b,1}(Q^k)} +$$

$$+C_7 \cdot \|u_0\|_{W_p^{2b,1}(Q^{k-1})} + C_8 + \|u_0\|_{W_p^{2b,1}(Q^{k-1})}.$$

Hence

$$\|u_0\|_{W_p^{2b,1}(Q^k)} \leq 2(1 + C_7) \cdot \|u_0\|_{W_p^{2b,1}(Q^{k-1})} + 2C_8. \tag{22}$$

According to the assumption $\|u_0\|_{W_p^{2b,1}(Q^{k-1})} \leq C^{k-1}$, (22) implies

$$\|u_0\|_{W_p^{2b,1}(Q^k)} \leq C^k.$$

Application of induction completes the proof of Theorem 1. ◄

2. Unimprovability of growth indicators under the conditions of Theorem 1

In this section, we give an example of boundary value problem (1) for which all the conditions of Theorem 1, except condition A.3), i.e. equality (2), are fulfilled. An appropriate inequality for this counterexample is satisfied, and it is shown that the statement of Theorem 1 is not fulfilled.

Let us consider the function $u(x,t)$, dependent on the parameter ε, $0 < \varepsilon \leq 1$, defined by the relations

$$u(x,t) = \varepsilon^k v(y,\tau), \quad y = \frac{x}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon^{2b}}, \tag{23}$$

where $v \in C^{\infty}\left(R^{n+1}\right)$ and $k = b - \frac{n+2}{2}$. Assume that

$$\|u\|^2_{W_2^{b,1}(Q_T)} = \sum_{|\alpha|\le b} \int_{Q_T} |D^{\alpha} u\,(x,t)|^2\, dxdt + \int_{Q_T} \left|\frac{\partial u}{\partial t}\right|^2 dxdt \le C,\ \ \forall\, \varepsilon \in (0,1]\,, \tag{24}$$

where the constant C is independent of ε.

On the other hand, we have

$$\|u\|_{W_p^{2b,1}(Q_T)} = \left[\sum_{|\alpha|\le 2b} \int_{Q_T} |D^{\alpha} u|^p\, dxdt + \int_{Q_T} \left|\frac{\partial u}{\partial t}\right|^p dxdt\right]^{1/p} \ge$$

$$\ge \left[\sum_{|\alpha|= 2b} \int_{Q_T} |D^{\alpha} u|^p\, dxdt + \int_{Q_T} \left|\frac{\partial u}{\partial t}\right|^p dxdt\right]^{1/p} =$$

$$= \varepsilon^{k-2b+\frac{n+2b}{p}} \cdot \left[\sum_{|\alpha|= 2b} \int_{Q_T^{\varepsilon}} |D^{\alpha} v|^p\, dyd\tau + \int_{Q_T^{\varepsilon}} \left|\frac{\partial v}{\partial \tau}\right|^p dyd\tau\right]^{1/p},$$

where $Q_T^{\varepsilon} = \left\{(y,\tau)\,\middle|\, \varepsilon \cdot y \in \Omega,\ \tau \cdot \varepsilon^{2b} \in (0,T)\right\}$. Hence, for a domain Q_T such that

$$\Omega \subseteq \Omega_{\varepsilon} = \left\{y \in R^n \,\middle|\, \varepsilon y \in \Omega\right\} \ \text{ for } \varepsilon \in (0,1]\,, \tag{25}$$

we obtain

$$\|u\|_{W_p^{2b,1}(Q_T)} \le \varepsilon^{k-2b+\frac{n+2b}{p}} \cdot \left[\sum_{|\alpha|= 2b} \int_{Q_T} |D^{\alpha} v|^p\, dyd\tau + \int_{Q_T} \left|\frac{\partial v}{\partial \tau}\right|^p dyd\tau\right]^{1/p}.$$

Then for the function $v\,(y,\tau)$ with

$$\sum_{|\alpha|= 2b} \int_{Q_T} |D^{\alpha} v|^p\, dyd\tau + \int_{Q_T} \left|\frac{\partial v}{\partial \tau}\right|^p dyd\tau > 0 \tag{26}$$

and for $k - 2b + \frac{n+2b}{p} = b - \frac{n+2}{2} - 2b + \frac{n+2b}{p} = -b - \frac{n+2}{2} + \frac{n+2b}{p} > 0$, i.e. for

$$p > \frac{2\,(n+2b)}{2b+n+2}, \tag{27}$$

we obtain the relation

$$\|u\|_{W_p^{2b,1}(Q_T)} \to +\infty \ \text{ as } \varepsilon \to 0 \ (\varepsilon > 0)\,. \tag{28}$$

Note that the inequality (27) implies the embedding

$$W_p^{2b,1}(Q_T) \subset W_2^{b,1}(Q_T)\,. \tag{29}$$

Now for the function $u(x,t)$ we compose an equation of the form

$$\Delta^b u(x,t) - \frac{\partial u}{\partial t} = b_l(x,t) \cdot |L_l u(x,t)|^{\mu_l} \tag{30}$$

with a linear homogeneous differential operator L_l of order l with constant coefficients. Then,

$$b_l(x,t) = \frac{\Delta^b u(x,t)}{|L_l u(x,t)|^{\mu_l}} - \frac{\partial u(x,t)/\partial t}{|L_l u(x,t)|^{\mu_l}}$$

and

$$\|b_l\|_{q_l;Q_T} = \left(\int\limits_{Q_T} |b_l|^{q_l}\, dxdt \right)^{1/q_l} \le$$

$$\le \varepsilon^{k-2b-(k-l)\mu_l + \frac{n+2b}{q_l}} \cdot \left(\int\limits_0^T \int\limits_{\Omega_\varepsilon} \frac{\left|\Delta^b v\right|^{q_l} dyd\tau}{|L_l v|^{q_l \mu_l}} + \int\limits_0^T \int\limits_{\Omega_\varepsilon} \frac{|\partial v/\partial\tau|^{q_l} dyd\tau}{|L_l v|^{q_l\mu_l}} \right)^{1/q_l} .$$

Hence, for

$$\mu_l > \frac{2b+n+2}{n+2+2(l-b)} - \frac{2}{n+2+2(l-b)} \cdot \frac{n+2b}{q_l}, \quad l = l_0, l_0+1, \ldots, 2b-1 \tag{31}$$

we obtain

$$\|b_l\|_{q_l;Q_T} \le \left(\int\limits_{R^{n+1}} \frac{\left|\Delta^b v\right|^{q_l} dyd\tau}{|L_l v|^{q_l\mu_l}} + \int\limits_{R^{n+1}} \frac{|\partial v/\partial\tau|^{q_l} dyd\tau}{|L_l v|^{q_l\mu_l}} \right)^{1/q_l} . \tag{32}$$

Now, we assign to the equation (30) the initial-boundary conditions

$$\begin{cases} u\Big|_{\partial Q_T} = 0, \ \frac{\partial u}{\partial v}\Big|_{\partial Q_T} = 0, \ldots, \frac{\partial^{b-1} u}{\partial v^{b-1}}\Big|_{\partial Q_T} = 0, \ (x,t) \in \partial Q_T, \\ u(x,0) = 0, \ x \in \Omega, \end{cases} \tag{33}$$

where $\frac{\partial u}{\partial v}, \ldots, \frac{\partial^{b-1} u}{\partial v^{b-1}}$ are appropriate order derivatives of the function $u(x,t)$ on ∂Q_T in the direction of outer unit normal ν to ∂Q_T.

Thus, the construction of a counterexample in the domain Q_T, satisfying condition (25), for $k = b - \frac{n+2}{2}$ and $p > 1$, satisfying the inequality (27), is reduced to the construction of the function $v \in C^\infty\left(R^{n+1}\right)$ satisfying the inequality (25) and the inequality

$$\left(\int\limits_{R^{n+1}} \frac{\left|\Delta^b v\right|^{q_l} dy d\tau}{\left|L_l v\right|^{q_l \mu_l}} + \int\limits_{R^{n+1}} \frac{\left|\partial v / \partial \tau\right|^{q_l} dy d\tau}{\left|L_l v\right|^{q_l \mu_l}}\right)^{1/q_l} < \infty \tag{34}$$

and generating boundary conditions (33) for the function $u(x,t)$.

3. Conditional existence theorem

Based on Theorem 1 on a priori estimate, a general theorem on the solvability of boundary value problems for quasilinear parabolic equations subject to the existence of intermediate a priori estimate $\|u\|_{W_2^{b,1}(Q_T)}$ for solving a corresponding family of boundary value problems, was obtained.

Let us consider the boundary value problem (1) in the real Sobolev space $W_{2b,1}^p(Q_T)$, $p > 1$, provided that there exists a priori estimate for $\|u\|_{W_2^{b,1}(Q_T)}$ in the space $W_2^{b,1}(Q_T)$, and $b > \frac{n+2}{2}$.

Let us associate to the problem (1) the following parametric family of boundary value problems:

$$\begin{cases} \frac{\partial u}{\partial t} + \sum\limits_{|\alpha|=2b} \tilde{a}_\alpha\left(x,t,\lambda,u,\ldots,D^k u\right) D^\alpha u = \tilde{f}\left(x,t,\lambda,u,\ldots,D^{2b-1}u\right), \ (x,t) \in Q_T, \\ B_i u\Big|_{\partial Q_T} = 0, \ \ (i = 0,1,\ldots,b-1), \ \ x \in \partial\Omega, \ \ t \in (0,T) \\ u(x,0) = 0, \ \ x \in \Omega, \end{cases} \tag{35}$$

dependent on the parameter $\lambda \in [0,1]$.

We consider the boundary value problem (35) in the real Sobolev space $W_p^{2b,1}(Q_T)$ with $p > 1$ under the following conditions .

B.1) Let the function $\tilde{f}(x,t,\lambda,\xi_0,\ldots,\xi_{2b-1})$ be defined on $Q_T \times [0,1] \times R \times R^n \times \cdots \times R^{n_{2b-1}}$ with the values in R and also be a Caratheodory function, i.e. measurable with respect to (x,t) for all $(\lambda,\xi_0,\ldots,\xi_{2b-1}) \in [0,1] \times R \times R^n \times \cdots \times R^{n_{2b-1}}$ and continuous with respect to $(\lambda,\xi_0,\ldots,\xi_{2b-1})$ almost for all $(x,t) \in Q_T$.

B.2) Let $b > \frac{n+2}{2}$. By l_0 we denote the least positive integer either larger or equal to $b - \frac{n+2}{2}$ and let $\xi_* = \left\{\xi_\gamma \Big| \, |\gamma| < l_0\right\}$.

Let

$$\left|\tilde{f}(x,t,\lambda,\xi_0,\ldots,\xi_{2b-1})\right| \le b(x,t,\xi_*) + \sum_{l=l_0}^{2b-1} b_l(x,t,\xi_*) \cdot |\xi_l|^{\mu_l}$$

almost for all $(x,t) \in Q_T$ and for all $\lambda \in [0,1]$, $\xi_0 \in R$, $\xi_1 \in R^n, \ldots, \xi_{2b-1} \in R^{n_{2b-1}}$ with nonnegative Caratheodory functions b, b_l such that for any $r \geq 0$

$$\hat{b_r}(x,t) \equiv \sup \left\{ b(x,t,\xi_*) \,\Big|\, |\xi_*| = \sum_{|\xi_\gamma| < b - \frac{n+2}{2}} \leq r \right\}$$

belongs to $L_p(Q_T)$ with $p > 1$ and $[2b + (n+2)]\, p > 2(n+2b)$, while the functions

$$\hat{b}_{1,r}(x,t) \equiv \sup \left\{ b_l(x,t,\xi_*) \,\Big|\, |\xi_*| \leq r \right\}$$

belong to $L_{q_l}(Q_T)$, $\quad q_l > p$, $l = l_0, l_0+1, \ldots, 2b-1$.

B.3) Let

$$\mu_l = \frac{2b+n+2}{n+2+2(l-b)} - \frac{2}{n+2+2(l-b)} \cdot \frac{n+2b}{q_l}$$

for $l = l_0, l_0+1, \ldots, 2b-1$, of $b > \frac{n+2}{2}$

B.4) Let $b > \frac{n+2}{2}$ and the integer $k \geq 0$ be such that $b - k > \frac{n+2}{2}$. Let the functions $\tilde{a}_\alpha(x,t,\lambda,\xi_0,\ldots,\xi_k)$ $\;(|\alpha| = 2b)$ be real and continuous on $\overline{Q}_T \times [0,1] \times R \times \cdots \times R^{n_k}$. Let the operator

$$\tilde{L}_v u \equiv \sum_{|\alpha|=2b} a_\alpha \left(x,t,\lambda,v,\ldots,D^k v \right) D^\alpha u,$$

be linear with respect to $u(x,t)$ for any $\lambda \in [0,1]$ and $v \in C^{k,0}\left(\overline{Q}_T\right)$, be a linear parabolic operator, and B_i $(i = 0,1,\ldots,b-1)$ be linear boundary differential operators of orders $b_i \leq 2b-1$, respectively, with real coefficients such that for any $\lambda \in [0,1]$ and $v \in C^{k,0}\left(\overline{Q}_T\right)$ the linear (with respect to $u(x,t)$) boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} + \tilde{L}_v u = g(x,t)\,, \;\; (x,t) \in Q_T, \\ B_i u\Big|_{\partial Q_T} = 0, \;\; (i = 0,1,\ldots,b-1)\,, \;\; x \in \partial\Omega, \;\; t \in (0,T) \\ u(x,0) = 0, \;\; x \in \Omega, \end{cases} \tag{36}$$

is a parabolic coercive boundary value problem uniquely solvable in the space $W_p^{2b,1}(Q_T)$ for any function $g \in L_p(Q_T)$ with $p > 1$ so that the following estimate is true for the solutions $u(x,t)$ from $W_p^{2b,1}(Q_T)$ of problem (36):

$$\|u\|_{W_p^{2b,1}(Q_T)} \leq C_1 \cdot \|g\|_{L_p(Q_T)}$$

where the positive constant C_1 is independent of $\lambda \in [0,1]$.

Let all possible solutions $u(x,t)$ from the space $W_p^{2b,1}(Q_T)$ of boundary value problem (35) satisfy a priori estimate $\|u\|_{W_2^{b,1}(Q_T)} \leq M$ for all $\lambda \in [0,1]$, where the constant $M > 0$ is independent of $\lambda \in [0,1]$.

B.5) Let the boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} + \sum\limits_{|\alpha|=2b} \tilde{a}_\alpha \left(x,t,0,u,Du,\ldots,D^k u\right) D^\alpha u = \tilde{f}\left(x,t,0,u,Du,\ldots,D^{2b-1}u\right), \\ (x,t) \in Q_T, \\ B_i u\Big|_{\partial Q_T} = 0, \ \ (i = 0,1,\ldots,b-1), \ \ x \in \partial\Omega, \ \ t \in (0,T), \\ u(x,0) = 0, \ \ x \in \Omega, \end{cases}$$

have finitely many solutions in the space $W_p^{2b,1}(Q_T)$.

B.6) Let $\tilde{a}_\alpha(x,t,1,\xi_0,\xi_1,\ldots,\xi_k) = a_\alpha(x,t,\xi_0,\xi_1,\ldots,\xi_k)$ ($|\alpha| = 2b$) for all $(x,t) \in \overline{Q}_T$, $\xi_0 \in R, \xi_1 \in R^n,\ldots,\xi_k \in R^{n_k}$ and let $\tilde{f}(x,t,1,\xi_0,\xi_1,\ldots,\xi_{2b-1}) = f(x,t,\xi_0,\xi_1,\ldots,\xi_{2b-1})$ almost for all $(x,t) \in Q_T$ and for all $\xi_0 \in R, \xi_1 \in R^n,\ldots,$ $\xi_{2b-1} \in R^{n_{2b-1}}$.

Theorem 2. *Let conditions B.1) - B.6) be fulfilled and for the parametric family of problems* (35) *with* $\lambda \in [0,1]$ *there exist a priori estimate* $\|u\|_{W_2^{b,1}(Q_T)}$. *Then the boundary value problem (1) has a solution in the space* $W_p^{2b,1}(Q_T)$ *with* $p > 1$, *and* $[2b + (n+2)]\,p > 2(n+2b)$.

Proof. Let us consider a parametric family of problems (35) for $\lambda \in [0,1]$. For that family, all the conditions of Theorem 1 are fulfilled, by virtue of which there exists such a constant C_2, independent of λ, that for any possible solution $u(x,t) \in W_p^{2b,1}(Q_T)$ of problem (35) the following inequality is satisfied:

$$\|u\|_{W_p^{2b,1}(Q_T)} \leq C_2, \ \ \forall\,\lambda \in [0,1]. \tag{37}$$

It follows from condition B.4) that the linear boundary value problem (36) is uniquely solvable and for its solution $u(x,t)$ the relation $u = Ag$ is valid, where A is a linear continuous operator from $L_p(Q_T)$ to $W_p^{2b,1}(Q_T)$. Then the boundary value problem (35) is equivalent to the operator equation

$$V = P(V,\lambda), \tag{38}$$

considered in the Banach space

$$\overset{\circ}{W}{}_p^{2b,1}(Q_T;B_0) = \Big\{V \in W_p^{2b,1}(Q_T)\ |B_iV|_{\partial Q_T} = 0,\ (i = 0,1,\ldots,b-1)$$
$$x \in \partial\Omega,\ t \in (0,T);\ V(x,0) = V(x,T) = 0,\ x \in \Omega\Big\}.$$

Here $P(V,\lambda) = A\tilde{f}(V,\lambda)$ and $\tilde{f}(u,\lambda)(x,t) \equiv f\left(x,t,\lambda,u(x,t),\ldots,D^{2b,1}u(x,t)\right)$.

The operator $\tilde{f}(u,\lambda)$ is defined in the space $W_p^{2b,1}(Q_T)$ with the values in $L_p(Q_T)$ and is a completely continuous operator from $W_p^{2b,1}(Q_T)$ to $L_p(Q_T)$ [17]. Therefore, the operator P is a completely continuous operator acting in the space $\overset{\circ}{W}{}_p^{2b,1}(Q_T)$.

For possible solutions $V(x,t)$ from the space $\overset{\circ}{W}{}_p^{2b,1}(Q_T)$, by virtue of the estimate (37), the following inequality is satisfied:

$$\|V\|_{\overset{\circ}{W}{}_p^{2b,1}(Q_T;B_0)} \leq C_3, \ \ \forall\,\lambda \in [0,1].$$

where C_3 is a positive constant independent of neither V, nor λ. Then, by the Leray-Schauder principle [7], equation (38) for $\lambda = 1$ and, consequently, boundary value problem (1) have the solution $u_0(x,t) \in W_p^{2b,1}(Q_T)$.

Theorem 2 is proved. ◄

4. Solvability of some quasilinear parabolic problems

Let's represent the principal part of the quasilinear parabolic operator in the divergent form and consider the following boundary value problem:

$$\begin{cases} (-1)^b \sum\limits_{\|\alpha\|,\|\beta\|=b} D^\alpha\left(a_{\alpha\beta}\left(x,t,u,\ldots,D^k u\right)D^\beta u\right) + \frac{\partial u}{\partial t} = \\ = f\left(x,t,u,Du,\ldots,D^{2b-1}u\right), \ (x,t)\in Q_T, \\ u\Big|_{\partial Q_T} = \frac{\partial u}{\partial \nu}\Big|_{\partial Q_T} = \cdots = \frac{\partial^{b-1}u}{\partial \nu^{b-1}}\Big|_{\partial Q_T} = 0, \ \ (x,t)\in\partial Q_T, \\ u(x,0) = 0, \ \ x\in\Omega. \end{cases} \tag{39}$$

Here Ω is a bounded domain from R^n with the boundary $\partial\Omega$ of class C^{2b}, $Q_T = \Omega\times(0,T)$, $T>0$ and $\frac{\partial u}{\partial\nu},\ldots,\frac{\partial^{b-1}u}{\partial\nu^{b-1}}$ are the derivatives of corresponding order of the function $u(x,t)$ on the boundary ∂Q_T in the direction of the outer unit normal ν.

The boundary value problem (39) is considered in the real Sobolev space $W_p^{2b,1}(Q_T)$ with $p>1$ and $b>\frac{n+2}{2}$, and this time the integer $k\geq 0$ is such that $b-k>\frac{n+2}{2}$.

For the boundary value problem (39) we assume that the following conditions are fulllfilled.

C.1) Let the function $f(x,t,\xi_0,\xi_1,\ldots,\xi_{2b-1})$ be defined on $Q_T\times R\times R^n\times\cdots\times R^{n_{2b-1}}$ with the values in R and also be a Caratheodory function, i.e. measurable with respect to (x,t) for all $(\xi_0,\ldots,\xi_{2b-1})$ and continuous with respect to $(\xi_0,\ldots,\xi_{2b-1})$ almost for all $(x,t)\in Q_T$.

C.2) Let $b > \frac{n+2}{2}$. Denote by l_0 the least positive integer larger or equal to $b - \frac{n+2}{2}$ and let $\xi_* = \left\{ \xi_\gamma \middle| \, |\gamma| < l_0 \right\}$ and $|\xi_*| = \sum\limits_{|\gamma| < b - \frac{n+2}{2}} |\xi_\gamma|$.

Let

$$|f(x,t,\xi_0,\xi_1,\ldots,\xi_{2b-1})| \le b(x,t,\xi_*) + \sum_{l=l_0}^{2b-1} b_l(x,t,\xi_*) \cdot |\xi_l|^{\mu_l}$$

almost for all $(x,t) \in Q_T$ and for all, $\xi_0 \in R$, $\xi_1 \in R^n, \ldots, \xi_{2b-1} \in R^{n_{2b-1}}$ with nonnegative Caratheodory functions b, b_l such that for any $r \ge 0$

$$\hat{b_r}(x,t) = \sup \left\{ b(x,t,\xi_*) \middle| \, |\xi_*| \le r \right\}$$

belongs to $L_p(Q_T)$ with $p > 1$ and $[2b + (n+2)]\, p > 2(n+2b)$, while the function

$$\hat{b}_{1,r}(x,t) = \sup \left\{ b_l(x,t,\xi_*) \middle| \, |\xi_*| \le r \right\}$$

belongs to $L_{q_l}(Q_T)$, with $q_l > p$ for $l = l_0, l_0+1, \ldots, 2b-1$.

C.3) Let

$$\mu_l = \frac{2b+n+2}{2(l-b)+n+2} - \frac{2}{2(l-b)+n+2} \cdot \frac{n+2b}{q_l}$$

for $l = l_0, l_0+1, \ldots, 2b-1$, if $b > \frac{n+2}{2}$.

Here μ_{l_0} is any positive number for $2(l_0 - b) + n + 2 = 0$.

C.4) Let $b > \frac{n+2}{2}$, and the integer $k \ge 0$ be such that $b - k > \frac{n+2}{2}$.

Let $a_{\alpha\beta} = a_{\beta\alpha}$ $(|\alpha|, |\beta| = b)$ be real functions defined on $\overline{Q}_T \times R \times R^n \times \cdots \times R^{n_k}$, belonging to the class C^b.

Let the operator $L_v u$, linear with respect to $u(x,t)$, equal to

$$L_v u = (-1)^b \sum_{|\alpha|,\, |\beta| = b} a_{\alpha\beta}\left(x,t,v,Dv,\ldots,D^k v\right) D^{\alpha+\beta} u$$

for any function $v \in C^{k,0}\left(\overline{Q}_T\right)$, be such a linear elliptic operator that the linear (with respect to $u(x,t)$) boundary value problem

$$\begin{cases} L_v u + \frac{\partial u}{\partial t} = g(x,t)\,, \ (x,t) \in Q_T, \\ u\Big|_{\partial Q_T} = 0, \ \frac{\partial u}{\partial \nu}\Big|_{\partial Q_T} = 0, \ldots, \ \frac{\partial^{b-1} u}{\partial \nu^{b-1}}\Big|_{\partial Q_T} = 0, \ \ (x,t) \in \partial Q_T, \\ u(x,0) = 0, \ \ x \in \Omega, \end{cases}$$

for any function $v \in C^{k,0}\left(\overline{Q}_T\right)$ is coercive in the space $W_p^{2b,1}\left(Q_T\right)$ for any function $g \in L_p\left(Q_T\right)$ with $p > 1$ and $p\left(n+2+2b\right) > 2\left(n+2b\right)$, so that

$$\|u\|_{W_p^{2b,1}(Q_T)} \leq C \cdot \|g\|_{p;Q_T}.$$

Let there exist a constant $C_0 > 0$ such that

$$\sum_{|\alpha|,\,|\beta|=b} \int_{Q_T} a_{\alpha\beta}\left(x,t,v,Dv,\ldots,D^k v\right) D^\beta u \cdot D^\alpha u \, dx\, dt \geq$$

$$\geq C_0 \cdot \left(\sum_{|\alpha|=b} \int_{Q_T} |D^\alpha u|^2 \, dxdt + \int_{Q_T} \left|\frac{\partial u}{\partial t}\right|^2 dxdt \right), \ \forall\, u \in \overset{\circ}{W}{}_p^{2b,1}\left(Q_T; B_0\right) \quad (40)$$

for any function $v \in C^{k,0}\left(\overline{Q}_T\right)$.

Here $\overset{\circ}{W}{}_p^{2b,1}\left(Q_T\right)$ is a subspace of all functions from the real Sobolev space $W_p^{2b,1}\left(Q_T\right)$, satisfying homogeneous inital-boundary conditions from boundary value problem (39).

C.5) Let there exist $\xi \in R^n$ and the constants $C_1 > 0$ and $C_2 > 0$ such that

$$\lambda \int_{Q_T} u \cdot f\left(x,t,u,Du,\ldots,D^{2b-1}u\right) e^{(\xi,x)} dxdt \leq$$

$$\leq \sum_{|\alpha|,\,|\beta|=b} \sum_{\gamma\leq\alpha} \int_{Q_T} \frac{\alpha!}{(\alpha-\gamma)!\gamma!} \cdot \xi^\gamma \cdot a_{\alpha\beta}\left(x,t,u,\ldots,D^k u\right) D^\beta u \cdot D^{\alpha-\gamma} u \cdot e^{(\xi,x)} dxdt -$$

$$-C_1 \left[\sum_{|\alpha|=b} \int_{Q_T} |D^\alpha u|^2 \, dxdt + \int_{Q_T} \left|\frac{\partial u}{\partial t}\right|^2 dxdt \right] + C_2, \ \ \forall\, \lambda \in [0,1]\,, \forall\, u \in \overset{\circ}{W}{}_p^{2B,1}\left(Q_t\right).$$

Here $(\xi, x) = \xi_1 x_1 + \cdots + \xi_n x_n$; $\gamma \leq \alpha$ means the totality of all (integral non-negative) multiindices γ such that $\gamma_1 \leq \alpha_1, \gamma_2 \leq \alpha_2, \ldots, \gamma_n \leq \alpha_n$; $\alpha! = \alpha_1! \ldots \alpha_n!$; $\alpha - \gamma = (\alpha_1 - \gamma_1, \alpha_2 - \gamma_2, \ldots, \alpha_n - \gamma_n)$ and $\xi^\gamma = \xi_1^{\gamma_1} \ldots \gamma_n^{\gamma_n}$.

Theorem 3. *Let conditions C.1) - C.5) with* $p > 1$ *be fulfilled, and* $p\left(2b + (n+2)\right) > 2\left(n+2b\right)$. *Then there exists the solution* $u_0 \in W_p^{2b,1}\left(Q_T\right)$ *of the boundary value problem* (39).

Proof. Let us consider the parametric family of boundary value problems

$$(-1)^b \sum_{|\alpha|,\,|\beta|=b} D^\alpha \left(a_{\alpha\beta}\left(x,u,Du,\ldots,D^k u\right) D^\beta u \right) + \frac{\partial u}{\partial t} =$$

$$= \lambda \cdot f\left(x, t, u, Du, \ldots, D^{2b-1}u\right), \ (x,t) \in Q_T, \tag{41}$$

$$u\Big|_{\partial Q_T} = \frac{\partial u}{\partial \nu}\Big|_{\partial Q_T} = \cdots = \frac{\partial^{b-1} u}{\partial \nu^{b-1}}\Big|_{\partial Q_T} = 0, \ (x,t) \in \partial Q_T, \tag{42}$$

$$u(x,0) = 0, \ \ x \in \Omega, \tag{43}$$

for $\lambda \in [0,1]$ in the space $W_p^{2b,1}(Q_T)$.

We decompose the further proof into several steps.

Step I. A priori estimate for $\|u\|_{W_2^{b,1}(Q_T)}$.

We multiply equation (41) by the function $e^{(\xi,x)} \cdot u(x,t) \in \overset{\circ}{W}{}_{p^{2b,1}}(Q_T; B_0)$ and integrate over the domain Q_T allowing for boundary conditions (42) and (43). Then we obtain

$$\sum_{|\alpha|,\,|\beta|=b} \sum_{\gamma \le \alpha} \int_{Q_T} \frac{\alpha!}{(\alpha-\gamma)!\gamma!} \xi^\gamma \, a_{\alpha\beta}\left(x, t, u, \ldots, D^k u\right) D^\beta u \cdot D^{\alpha-\gamma} u \cdot e^{(\xi,x)} dxdt =$$

$$= \lambda \int_{Q_T} u\, f\left(x, t, u, \ldots, D^{2b-1}u\right) e^{(\xi,x)} dxdt.$$

Hence, by virtue of C.5), we have

$$\sum_{|\alpha|=b} \int_{Q_T} |D^\alpha u|^2 \, dxdt + \int_{Q_T} \left|\frac{\partial u}{\partial t}\right|^2 dxdt \le \frac{C_2}{C_1}, \tag{44}$$

for $\forall\, \lambda \in [0,1]$ and for any possible solution $u \in W_p^{2b,1}(Q_T)$ of boundary value problem (41)-(43).

Step II. A priori estimate for $\|u\|_{W_p^{2b,1}(Q_T)}$.

The belonging of functions $a_{\alpha\beta}(x, t, \xi_0, \xi_1, \ldots, \xi_k)$ $(|\alpha|, |\beta| = b)$ to the class $C^b\left(\overline{Q}_T \times R \times R^n \times \cdots \times R^{n_k}\right)$ makes possible to rewrite the equation (41) as follows:

$$(-1)^b \sum_{|\alpha|,\,|\beta|=b} a_{\alpha\beta}\left(x, t, u, \ldots, D^k u\right) D^{\alpha+\beta} u + \frac{\partial u}{\partial t} =$$

$$= \lambda f\left(x, t, u, \ldots, D^{2b-1}u\right) + \tilde{f}\left(x, t, u, \ldots, D^{2b-1}u\right). \tag{45}$$

Here the function $\tilde{f}$ is defined by the rule of differentiation of corresponding complex functions (composition of functions). This time the obtained function

$\tilde{f}(x,t,\xi_0,\xi_1,\ldots,\xi_{2b-1})$ is continuous on $\overline{Q}_T\times R\times R^n\times\cdots\times R^{n_{2b-1}}$ and satisfies the inequality

$$\left|\tilde{f}(x,t,\xi_0,\xi_1,\ldots,\xi_{2b-1})\right|\le\tilde{b}(x,t,\xi_*)+\sum_{l=l_0}^{2b-1}\tilde{b}_l(x,t,\xi_*)\cdot|\xi_l|^{\tilde{\mu}_l}$$

for all $(x,t)\in\overline{Q}_T$, $\xi_0\in R,\ldots,\xi_{2b-1}\in R^{n_{2b-1}}$ with continuous functions $\tilde{b}$ and $\tilde{b}_l$ $(l=l_0,\ l_0+1,\ldots,2b-1)$ and with $\xi_*=\left\{\xi_\gamma\Big|\,|\gamma|<b-\frac{n+2}{2}\right\}$. The growth indicators $\tilde{\mu}_l$ of this function $\tilde{f}$ relative to the derivatives D^ju of order $|j|=l$, $l=l_0,l_0+1,\ldots,2b-1$ satisfy the equality (2) with $\tilde{q}_l=\infty$, i.e.

$$\tilde{\mu}_l=\frac{2b+n+2}{2(l-b)+n+2}$$

for $l=l_0,l_0+1,\ldots,2b-1$, if $b>\frac{n+2}{2}$.

Thus, the right-hand side of the equation (45) is a linear function $\lambda\cdot f+\tilde{f}$ that satisfies conditions C.1), C.2), and C.3) (uniformly with respect to $\lambda\in[0,1]$).

From the condition C.4) it follows that the condition A.4) is fulfilled. This time, uniform parabolicity of the operator

$$(-1)^b\sum_{|\alpha|,\,|\beta|=b}a_{\alpha\beta}\left(x,t,v,Dv,\ldots,D^kv\right)D^{\alpha+\beta}u+\frac{\partial u}{\partial t}$$

in the domain $\overline{Q}_T$ for all $v\in C^{k,0}\left(\overline{Q}_T\right)$, follows from inequality (40).

Consequently, all the conditions of Theorem 1 with appropriate constants and functions independent of $\lambda\in[0,1]$ are fulfilled for boundary value problem (41)-(43). Then, using Theorem 1, taking into account estimate (44) and homogeneous initial-boundary conditions (42) and (43), we conclude that there exists such a constant $C_3>0$, independent of $\lambda\in[0,1]$, that

$$\|u\|_{W_p^{2b,1}(Q_T)}\le C_3 \tag{46}$$

for all possible solutions $u\in W_p^{2b,1}(Q_T)$ of boundary value problem (41)-(43) for any $\lambda\in[0,1]$.

Step III. Solvability of problem (39)**.**

Let's consider the boundary value problem

$$\begin{cases}(-1)^b\sum\limits_{|\alpha|,\beta=b}D^\alpha\left(a_{\alpha\beta}\left(x,t,v,\ldots,D^kv\right)D^\beta\tilde{u}\right)+\frac{\partial\tilde{u}}{\partial t}=\\=\lambda\cdot f\left(x,t,v,Dv,\ldots,D^{2b-1}v\right),\ (x,t)\in Q_T,\\ \tilde{u}\Big|_{\partial Q_T}=\frac{\partial\tilde{u}}{\partial\nu}\Big|_{\partial Q_T}=\cdots=\frac{\partial^{b-1}\tilde{u}}{\partial\nu^{b-1}}\Big|_{\partial Q_T}=0,\ \ (x,t)\in\partial Q_T,\\ \tilde{u}(x,0)=0,\ \ x\in\Omega,\end{cases} \tag{47}$$

linear with respect to $\tilde{u}(x,t)$ for any arbitrary fixed function $v(x,t)$ from $\overset{\circ}{W}{}_p^{2b,1}(Q_T)$ and for $\lambda \in [0,1]$. Then it follows from conditions C.1) - C.4) that this problem is uniquely solvable in $\overset{\circ}{W}{}_p^{2b,1}(Q_T; B_0)$ for any function $v(x,t)$ from $\overset{\circ}{W}{}_p^{2b,1}(Q_T)$, so that $\tilde{u} = P(v,\lambda)$, where P is an operator defined by the problem (47) and acting from $\overset{\circ}{W}{}_p^{2b,1}(Q_T) \times [0,1]$ to $\overset{\circ}{W}{}_p^{2b,1}(Q_T; B_0)$. The conditions of Theorem 3, by the virtue of Sobolev-Kondrashev [17] embedding theorem, imply the complete continuity of the operator

$$P : \overset{\circ}{W}{}_p^{2b,1}(Q_T) \times [0,1] \to \overset{\circ}{W}{}_p^{2b,1}(Q_T).$$

Then the boundary value problem (41)-(43) is equivalent to the operator equation

$$u = P(u,\lambda), \tag{48}$$

considered in the space $\overset{\circ}{W}{}_p^{2b,1}(Q_T)$. For possible solutions $u \in \overset{\circ}{W}{}_p^{2b,1}(Q_T)$ of this equation, a priori estimate (46) with the constant $C_3 > 0$, independent of $\lambda \in [0,1]$ is true.

Since the boundary value problem (41)-(43) for $\lambda = 0$ has a unique isolated solution, by the Leray Schauder [7] principle the equation (48) for $\lambda = 1$ and, consequently, the boundary value problem (39) has the solution $u_0 \in \overset{\circ}{W}{}_p^{2b,1}(Q_T)$, for which the following estimate is valid:

$$\|u_0\|_{W_p^{2b,1}(Q_T)} \le C_3.$$

Theorem 3 is proved. ◄

Let's present the corollary of Theorem 3. To this end, we consider the particular case of condition C.5), the condition of the existence of a priori estimate for $\|u\|_{W_2^{b,1}(Q_T)}$: we assume that the vector parameter $\xi \in R^n$ is equal to zero.

C.6). Let there exist constants $C_1 > 0$ and C_2 such that

$$\lambda \int_{Q_T} uf\left(x,t,u,Du,\ldots,D^{2b-1}u\right) dxdt \le$$

$$\le \sum_{|\alpha|,\,|\beta|=b} \int_{Q_T} a_{\alpha\beta}\left(x,t,u,Du,\ldots,D^k u\right) D^\beta u D^\alpha u dxdt -$$

$$-C_1 \left[\sum_{|\alpha|=b} \int_{Q_T} |D^\alpha u|^2 dxdt + \int_{Q_T} \left|\frac{\partial u}{\partial t}\right|^2 dxdt \right] + C_2,$$

$\forall\,\lambda \in [0,1]\,,\ \forall\, u \in \overset{\circ}{W}{}_p^{2b,1}\left(Q_T\right).$

For this condition to be fulfilled, it is sufficient that the condition C.4) holds, so that the following condition is fulfilled.

C. 7). Let the inequality (40) with the constant C_0 be satisfied, and let there exist positive constants $C_1 < C_0$ and C_2 such that

$$\int\limits_{Q_T} uf\left(x,t,u,Du,\ldots,D^{2b-1}u\right)dxdt \le$$

$$\le C_1\left[\sum_{|\alpha|=b}\int\limits_{Q_T}|D^\alpha u|^2\,dxdt + \int\limits_{Q_T}\left|\frac{\partial u}{\partial t}\right|^2 dxdt\right] + C_2,$$

$\forall\, u \in W_p^{2b,1}\left(Q_T;B_0\right).$

Corollary 1. *Let conditions C.1)-C.6) be fulfilled. Then there exists the solution* $u_0 \in W_p^{2b,1}\left(Q_T\right)$ *of boundary value problem* (39).

Corollary 2. *Let conditions C.1)-C.5) and C. 7) be fulfilled. Then there exists the solution* $u_0 \in W_p^{2b,1}\left(Q_T\right)$ *of the boundary value problem* (39).

As an application of Corollary 1, we give an example.

Let us consider in $Q_T \subset R^n$ the following boundary value problem:

$$\begin{cases} (-1)^b \sum\limits_{|\alpha|,\beta=b} D^\alpha\left(a_{\alpha\beta}\left(x,t,u,\ldots,D^k u\right)D^\beta u\right) + b\left(x,t\right)\cdot u\left(x,t\right)\times \\ \times \sum\limits_{|\gamma|=l}|D^\gamma u|^{\mu_l} - \frac{\partial u}{\partial t} = g\left(x,t\right),\ \left(x,t\right)\in Q_T, \\ u\Big|_{\partial Q_T} = \frac{\partial u}{\partial \nu}\Big|_{\partial Q_T} = \cdots = \frac{\partial^{b-1}u}{\partial \nu^{b-1}}\Big|_{\partial Q_T} = 0,\ \ \left(x,t\right)\in\partial Q_T, \\ u\left(x,0\right) = 0,\ \ x\in\Omega. \end{cases}$$

Let conditions C.4), $l = l_0,\ldots,2b-1$, $g\left(x,t\right)\in L_p\left(Q_T\right)$ with $p>1$, $b\left(x,t\right)\ge 0$ be fulfilled almost everywhere in Q_T, $b\left(x,t\right)\in L_q\left(Q_T\right)$ with $q>p$, and let in the case $l \ge b-\frac{n+2}{2}$, $q > \frac{n+2}{2b-l}$.

Then for

$$\mu_l = \frac{2b+n+2}{2\left(l-b\right)+n+2} - \frac{2}{2\left(l-b\right)+n+2}\cdot\frac{n+2b}{q},$$

if $l \ge b-\frac{n+2}{2}$, the boundary value problem (49) has a solution in the space $W_p^{2b,1}\left(Q_T\right)$.

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