Azerbaijan Journal of Mathematics
V. 15, No 2, 2025, July
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2025.2.264

Inversion Formula for Solving Parametric Problems with Irregular Boundary Conditions for Variable Coefficients Equations

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Abstract. In this paper, a parametric problem is solved in some part of the complex plane. This problem involves a parabolic equation in the sense of Petrovsky. The irregular boundary conditions include integro-differential derivatives of higher order. An analytical expression for the solution of the considered problem is obtained, along with some transformation formulas for this solution.

Key Words and Phrases: parametric problem, irregular boundary conditions, analytical expression for the solution, transformation formulas.

2010 Mathematics Subject Classifications: 35L35, 44Axx

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1. Introduction

At the beginning of the 19th century, Fourier and later Cauchy [2], in their attempts to integrate some linear homogeneous partial differential equations with given homogeneous boundary conditions and nonhomogeneous initial conditions (mixed problem 1), arrived—through mathematically unjustified methods—at a corresponding parametric (often so-called "spectral") problem, which depends on some parameter λ, which is any number from the complex plane (parametric problem 2).

Fourier, Cauchy [2], Tamarkin [12], Naimark [10], Rasulov [6], Shkalikov [14] and many other mathematicians studied "parametric problem 2" with the purpose of solving "mixed problem 1". Their solution method was based on expansion formulas

$$f(x) = \sum c_k \varphi_k(x) \tag{1}$$

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for an arbitrary function $f(x)$ from some class of functions (here, $\varphi_k(x)$ are eigenfunctions and adjoint functions of "parametric problem 2").

To derive this formula (1), it is necessary to have the appropriate asymptote of the fundamental solution system of the parametric equation throughout the entire plane (for example, see Birkhoff [1], Vahabov [7]). For the validity of the expansion formula (1), the conditions of "regularity and weak regularity of boundary conditions" are sufficient (see, e.g., [6, 10, 12, 14], etc.). There is a sufficiently broad class of problems (with irregular boundary conditions) from both scientific and practical perspectives, for which the expansion formula (1) is not valid, and therefore, these problems cannot be solved using the methods of Fourier, Cauchy [2], Tamarkin [12], Naimark [10], Rasulov [6].

In the second half of the 20th century, a new method-the method of finite integral transformation—to solve "mixed problem 1" (in homogeneous and non-homogeneous cases; regular and wide ranges of irregular mixed problems) was proposed in [8]. As a result of applying this method [8] to the solution of "mixed problem 1", a corresponding "parametric problem 2" (for all complex parameters λ) was obtained in a strictly mathematically justified manner.

The above method showed that to solve "mixed problem 1", it is not necessary to study "parametric problem 2" on the whole complex plane (i.e., it is not necessary to rely on the expansion formula (1)), and it is sufficient to solve it within the irregular boundary conditions in some part of the complex plane, obtain the analytical expression for the solution, and derive the corresponding transformation formula for it. Using this method in [8, 3, 4, 5], helped to obtain the analytical expression for the solution of "mixed problem 1". It is worth mentioning that, in practice, one may encounter mixed problems of type 1 where the boundary conditions involve integro-differential terms (e.g., Ionkin [9]) and derivatives of the sought function of an order exceeding the order of the equation (e.g., Tikhonov [13]). Note that investigation of similar problems was carried out in [15] -[22] and other studies.

The main goal in solving the parametric problem (and obtaining some results for it) in this paper is to make it possible to solve the corresponding mixed problem in a short, clear, and precise manner by using the obtained results.

1.1. Main Results

1. The parametric equation corresponding to the parabolic equation in the sense of Petrovsky [11]

$$\frac{\partial u}{\partial t} = (a(x) + ib(x))\,\frac{\partial^2 u}{\partial x^2},$$

where $a(x) > 0$, has the following form:

$$(a(x) + ib(x))\, y'' - \lambda^2 y = 0. \tag{2}$$

In previous works, the asymptotics for the fundamental solution system of this equation (2) have been obtained (see, e.g., Rasulov [6]) for the case where

$$\frac{b(x)}{a(x)} = \text{const}. \tag{3}$$

However, for the general case

$$a(x) > 0, \quad \frac{b(x)}{a(x)} \neq \text{const}., \tag{4}$$

the asymptote for the fundamental solution system of equation (2) has not been studied. In this work, the equation (5), under the conditions (4), encompasses equation (2).

2. We study the parametric problem not on the entire complex plane, but only on a suitable part of it, and we obtain results that allow us to derive the analytical expression for the solution to the corresponding mixed problem

3. The boundary conditions (6) include local and global integro-differential expressions of the sought function, as well as higher-order derivatives resulting from the formulation of equation. The concept of "correctness" of boundary conditions is introduced, and this concept is more general than the notions of "regularity" and "weak regularity"; that is, if the conditions (6) are "regular" or "weakly regular", then they are "correct" in the sense of our definition, but the converse is not true.

4. In classical works, the formula for the solution of the parametric problem (5)–(6) (via the Green's function) exists if and only if the order of the derivative included in the boundary conditions is less than the order of the differential equation under consideration. In this work, the order of the derivative in the boundary conditions may be higher than that of the differential equation, and in this case, a new formula is provided—a generalization of the classical one (see (12)).

5. In general, the expansion formula (1) is not valid for the solution of the parametric problem (5)–(6) under consideration. Instead, the validity of the transformation formula (23) is proven for the solution.

6. The results obtained in this work for the parametric problem (5)–(6) are sufficient to solve the corresponding mixed problem.

In this work, the following parametric problem is solved in some part of the λ-complex plane.

1.2. Problem Statement

Find the solution of the equation

$$\sum_{j=0}^{2} a_j(x)\frac{d^j y}{dx^j} - \lambda^2 y = \psi(x), \quad x \in (0,1) \tag{5}$$

satisfying the conditions

$$U_i(y) = \sum_{k=0}^{n} \sum_{\mu=-m_i}^{m_i} \lambda^{\mu} \left[\sum_{s=0}^{q} \gamma_{ks}^{i\mu} \left(\frac{d^k y}{dx^k} \right)_{x=v_s} + \int_0^1 \gamma_k^{i\mu}(x) \left(\frac{d^k y}{dx^k} \right) dx \right] = \gamma_i \quad i = 1, 2, \tag{6}$$

where $a_j(x), \psi(x), \gamma_k^{i\mu}$ are the known functions, $\gamma_{ks}^{i\mu}, \gamma_i, v_s\,(0 = v_0 < v_1 < \cdots < v_q = 1)$ are the known numbers; n, m_i, q are non-negative integers, and λ is a complex parameter.

2. Solution of Parametric Problem

Let us accept the following constraints 1° and 2° to solve the parametric problem (5)–(6) in some part of the λ complex plane.

1°. Let $a_j(x) \in C^{m+N+j}([0,1])$ for $j = 0, 1, 2,$ where $a_2(x) > 0$ for all $x \in [0,1]$, and $N \geq 0$ is an integer.

2°. Let $a_2(x) = |a_2(x)| e^{\sqrt{-1}\arg a_2(x)}$, $|\arg a_2(x)| \leq \frac{\pi}{2} - 4\varphi$ for $x \in [0,1]$, where $\varphi \left(0 < \varphi \leq \frac{\pi}{8}\right)$ is a constant.

Assume

$$\theta(x) = \frac{1}{\sqrt{a_2(x)}},$$

$$R_{\varphi} = \left\{ \lambda : |\lambda| \geq R, |\arg \lambda| \leq \frac{\pi}{4} + \varphi \right\}.$$

Although it is not specifically stated in this article, it is understood that the number λ belongs to the domain R_{φ} of the complex plane, i.e. throughout this article $\lambda \in R_{\varphi}$.

Theorem 1. *Under the constraint 1°, there exist the functions*

$$y_j^0(x,\lambda) = \exp(-\lambda\theta_j(x))\left[g_j^{0,0}(x) + \frac{1}{\lambda}g_j^{0,1}(x) + \cdots + \frac{1}{\lambda^m}g_j^{0,m}(x)\right], \quad (j=1,2)$$

for which we have

$$\left(\sum_{i=0}^{2} a_i(x)\frac{d^i}{dx^i} - \lambda^2\right) y_j^0(x,\lambda) = \frac{1}{\lambda^m} q_{jm}(x)\exp(-\lambda\theta_j(x)),$$

where

$$\theta_1(x) = \int_0^x \theta(\xi)d\xi, \quad \theta_2(x) = \int_x^1 \theta(\xi)d\xi,$$

$$g_0^{j,0}(x) = \frac{1}{\sqrt{\theta(x)}}\exp\left(-\frac{1}{2}\int_0^x \frac{a_1(\xi)}{a_2(\xi)}d\xi\right);$$

$g_0^{j,\nu}(x)$ $(\nu = 1,\dots,m)$, $q_{jm}(x)$ *are some functions from* $C^n([0,1])$.

Proof. The proof of the existence and the method of finding the $g_0^{j,\nu}(x)$ functions that satisfy the conditions of Theorem 1 are the same as in [1, 3, 6, 10, 12]. ◄

Assume

$$W_0(\xi,\lambda) = y_1^0(\xi,\lambda)\frac{d}{d\xi}y_2^0(\xi,\lambda) - y_2^0(\xi,\lambda)\frac{d}{d\xi}y_1^0(\xi,\lambda),$$

$$P_0(x,\xi,\lambda) = -\frac{1}{a_2(\xi)W_0(\xi,\lambda)}\begin{cases} y_1^0(x,\lambda)y_2^0(\xi,\lambda), & \text{if } 0 \le \xi \le x \le 1, \\ y_2^0(x,\lambda)y_1^0(\xi,\lambda), & 0 \le x \le \xi \le 1. \end{cases}$$

Theorem 2. *Under the constraints 1° and 2° and for* $\lambda \in R_\varphi$ *(R is a rather large positive number), the equation (5) has a fundamental solution represented by the formula*

$$P(x,\xi,\lambda) = P_0(x,\xi,\lambda) + P_1(x,\xi,\lambda), \tag{7}$$

where $P_1(x,\xi,\lambda)$ *is some function satisfying the inequality*

$$\left|\frac{\partial^k}{\partial x^k}P_1(x,\xi,\lambda)\right| \le \frac{C}{|\lambda|^{m-k}}e^{-\varepsilon|\lambda||x-\xi|}, x,\xi \in [0,1], \lambda \in R_\varphi, k = 0,1,2,$$

where C *and* ε *are some positive constants.*

Proof. The method of finding the fundamental solution $P(x,\xi,\lambda)$ given in this theorem is the same as in [3]:

$$P(x,\xi,\lambda) = P_0(x,\xi,\lambda) + \int_0^1 P_0(x,\eta,\lambda)h(\eta,\xi,\lambda)d\eta.$$

The method of finding the leading part of the fundamental solution $P_0(x,\xi,\lambda)$ in this work differs from the known methods in other papers (for example, see Rasulov [6]). Choosing $P_0(x,\xi,\lambda)$ by this method allows determining the asymptote of the fundamental solution, i.e., $P_0(x,\xi,\lambda)$, with an accuracy up to $\frac{1}{\lambda^m}$. The theorem is proved by a method analogous to the ones in [3, 5]. ◄

Theorem 3. *Under the constraints 1° and 2° and for $\lambda \in R_\varphi$, the homogeneous equation corresponding to (5) has a system of fundamental particular solutions $y_j(x,\lambda)$, represented by the formula*

$$y_j(x,\lambda) = y_j^0(x,\lambda) + q_j(x,\lambda) \tag{8}$$

where $q_j(x,\lambda)$ are some functions satisfying the inequalities

$$\left|\frac{d^k}{dx^k}q_1(x,\lambda)\right| \le \frac{C}{|\lambda|^{m+1-k}}\exp(-\varepsilon|\lambda||x|),$$

$$\left|\frac{d^k}{dx^k}q_2(x,\lambda)\right| \le \frac{C}{|\lambda|^{m+1-k}}\exp(-\varepsilon|\lambda|(1-x)),$$

$$x \in [0,1], \lambda \in R_\varphi, k = 0,1,2,$$

Proof. The method of finding the fundamental solution system of the equation (5) in this work is similar to that in [3, 5], unlike the methods in [1, 6, 10, 12], and is as follows:

$$y_j(x,\lambda) = y_j^0(x,\lambda) + \int_0^1 P_0(x,\xi,\lambda)y_j^0(\xi,\lambda)d\xi.$$

Using this formula, Theorems 1, 2, and 3 can be easily proven. ◄

Assume

$$\Delta(\lambda) = \begin{vmatrix} U_1(y_1) & U_1(y_2) \\ U_2(y_1) & U_2(y_2) \end{vmatrix}.$$

Using (8), we have

$$U_i(y_j) = A_{ij}(\lambda) + B_{ij}(\lambda) + O_{ij}(\lambda), i,j = 1,2, \tag{9}$$

where

$$A_{i1}(\lambda)=\sum_{k=0}^{n}\sum_{\mu=-m_i}^{m}\lambda^{\mu}\left\{\gamma_{k0}^{i\mu}\frac{d^k}{dx^k}y_1^0\Big|_{x=0}+\int_0^h\gamma_k^{i\mu}(x)\frac{d^k}{dx^k}y_1^0dx\right\},$$

$$B_{i1}(\lambda)=\sum_{k=0}^{n}\sum_{\mu=-m_i}^{m}\lambda^{\mu}\left\{\gamma_{k0}^{i\mu}\frac{d^k}{dx^k}q_1(x,\lambda)\Big|_{x=0}+\int_0^h\gamma_k^{i\mu}(x)\frac{d^k}{dx^k}q_1(x,\lambda)dx\right\};$$

$$O_{i_1}(\lambda)=\sum_{k=0}^{n}\sum_{\mu=-m_i}^{m_i}\lambda^{\mu}\left\{\sum_{s=1}^{q}\gamma_{ks}^{i\mu}\frac{d^k}{dx^k}y_1\Big|_{x=v_s}+\int_h^1\gamma_k^{i\mu}(x)\frac{d^k}{dx^k}y_1dx\right\};$$

$$A_{i_2}(\lambda)=\sum_{k=0}^{n}\sum_{\mu=-m_i}^{m_i}\lambda^{\mu}\left\{\gamma_{kq}^{i\mu}\frac{d^k}{dx^k}y_2^0\Big|_{x=1}+\int_{1-h}^1\gamma_k^{i\mu}(x)\frac{d^k}{dx^k}y_2^0dx\right\};$$

$$B_{i_2}(\lambda)=\sum_{k=0}^{n}\sum_{\mu=-m_i}^{m_i}\lambda^{\mu}\left\{\gamma_{kq}^{i\mu}\frac{d^k}{dx^k}q_2(x,\lambda)\Big|_{x=1}+\int_{1-h}^1\gamma_k^{i\mu}(x)\frac{d^k}{dx^k}q_2(x,\lambda)dx\right\};$$

$$O_{i_2}(\lambda)=\sum_{k=0}^{n}\sum_{\mu=-m_i}^{m_i}\lambda^{\mu}\left\{\sum_{s=0}^{q-1}\gamma_{ks}^{i\mu}\frac{d^k}{dx^k}y_2\Big|_{x=v_s}+\int_0^{1-h}\gamma_k^{i\mu}(x)\frac{d^k}{dx^k}y_2dx\right\};$$

$h(0<h<1)$ is any fixed number.

3°. Let the functions $\gamma_k^{i\mu}(x)$ in the segment [h, 1-h] be piecewise continuous and $\gamma_k^{j\mu}(x)\in C^l([0,h]\cup[1-h,1])$, where $h(0<h<1), l(l\geq 0)$ are some numbers.

The integrals

$$I_1\equiv\int_0^h\gamma_k^{i_1\mu}(x)\frac{d^k}{dx^k}y_1^0dx,$$

$$I_2\equiv\int_{1-h}^1\gamma_k^{i_2\mu}(x)\frac{d^k}{dx^k}y_2^0dx,$$

contained in the expressions $A_{i1}(\lambda)$ and $A_{i2}(\lambda)$, can be transformed into the form

$$I_1=\lambda^k\int_0^h g_1(x)\exp\left(-\lambda\int_0^x\theta(\xi)d\xi\right)dx=\lambda^{k-1}\frac{g_1(0)}{\theta(0)}+I_1^{(1)}+I_1^{(2)},$$

where

$$I_1^{(1)}=\lambda^{k-1}\int_0^h g_2(x)\exp\left(-\lambda\int_0^x\theta(\xi)d\xi\right)dx,\quad g_2(x)=\frac{d}{dx}\left(\frac{g_1(x)}{\theta(x)}\right),$$

$$I_1^{(2)}=-\lambda^{k-1}\frac{g_1(h)}{\theta(h)}\exp\left(-\lambda\int_0^h\theta(\xi)d\xi\right).$$

For $\lambda \in R_\varphi$, taking into account the estimate

$$\left|\exp\left(-\lambda\int_0^h \theta(\xi)d\xi\right)\right| \le C\exp(-\varepsilon|\lambda|h) \le \frac{C_{\varepsilon hp}}{|\lambda|^p}$$

(p is any real number), we have the asymptote

$$I_1 = \lambda^{k-1}\frac{g_1(0)}{\theta(0)} + O(\lambda^{k-2}), \lambda \in R_\varphi. \tag{10}$$

If the functions $g_1(x)$ and $\theta(x)$ are sufficiently smooth, then repeating the above procedures for $I_1^{(1)}$, the asymptotes for I_1 can be further refined. Similar procedures are performed for I_2.

Expanding the determinant

$$\Delta_0(\lambda) = \begin{vmatrix} A_{11}(\lambda) & A_{12}(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) \end{vmatrix},$$

we have

$$\Delta_0(\lambda) = b_M\lambda^M + b_{M-1}\lambda^{M-1} + \cdots + b_{M-s}\lambda^{M-s} + O(\lambda^{M-s-1}), \text{ for } \lambda \in R_\varphi, \tag{11}$$

where $M(M = m_1 + 2n + m_2), S$ are some integers, and $b_M, \ldots, b_{M-s}$ are some numbers.

Note that the number S, included in (11), can be taken sufficiently large (i.e., for $\Delta_0(\lambda)$ we can obtain more accurate asymptotes) if the functions $a_i(x)$ for $x \in [0,1]$ and $\gamma_k^{j\mu}(x)$ for $x \in [0,h] \cup [1-h,1]$ are sufficiently smooth, i.e., the numbers m, n are l sufficiently large.

4°. Let at least one of the numbers

$$b_M, b_{M-1}, \ldots, b_{M-s} \text{ (from (11))} \tag{12}$$

be non-zero.

Let in the sequence (12) the first non-zero number be b_v, i.e., $b_j = 0$ for $M \ge j > v, \quad b_v \ne 0$. Then for $\Delta_0(\lambda)$, with$\lambda \in R_\varphi$ we have the asymptotes

$$\Delta_0(\lambda) = b_v\lambda^v + O(\lambda^{v-1}). \tag{13}$$

Note that

$$|B_{i_1}(\lambda)| \le \frac{C}{|\lambda|^{m+1-n-m_i}} \sum_{k=0}^{n} \sum_{\mu=-m_i}^{m_i} \left\{|\gamma_{k0}^{i\mu}| + \int_0^h |\gamma_k^{i\mu}(x)|dx\right\};$$

$$|O_{i_1}(\lambda)| \leq C|\lambda|^{m_i+n}[e^{-\varepsilon|\lambda|v_1}+e^{-\varepsilon|\lambda|h}]\sum_{k=0}^{n}\sum_{\mu=-m_i}^{m_i}\left\{\sum_{s=1}^{q}|\gamma_{ks}^{i\mu}|+\int_h^1|\gamma_k^{i\mu}(x)|dx\right\};$$

$$|B_{i_2}(\lambda)| \leq \frac{C}{|\lambda|^{m+1-n-m_i}}\left\{|\gamma_{kq}^{i\mu}|+\int_{1-h}^1|\gamma_k^{i\mu}(x)|dx\right\};$$

$$|O_{i_2}(\lambda)| \leq C|\lambda|^{m_i+n}[e^{-\varepsilon|\lambda|v_i}+e^{-\varepsilon|\lambda|h}]\sum_{k=0}^{n}\sum_{\mu=-m_i}^{m_i}\left\{\sum_{s=0}^{q-1}|\gamma_{ks}^{i_2\mu}|+\int_0^{1-h}|\gamma_k^{i_2\mu}(x)|dx\right\}; \tag{14}$$

$$l=q-1.$$

Assume

$$\Delta(\lambda)=\begin{vmatrix}U_1(y_1) & U_1(y_2)\\ U_2(y_1) & U_2(y_2)\end{vmatrix}. \tag{15}$$

Taking into account equalities (9), (13), and estimate (14), we have

$$\Delta(\lambda)=b_v\lambda^v+O(\lambda^{v-1}), \quad \lambda\in R_\varphi. \tag{16}$$

Definition 1. *If constraints 1°, 2°, 3°, and 4° are fulfilled, we say that conditions (6) for equation (5) are well-posed.*

Remark 1. *To avoid imposing unnecessary restrictions on the functions $a_j(x)$ and $\gamma_k^{i\mu}$, the numbers m, N and l should, as far as possible, be taken sufficiently small to obtain the asymptotes (16).*

Remark 2. *For the equation (5), the well-posedness of integro-differential conditions (6) is affected only by:*

1. *the coefficients of derivatives $\frac{d^k y}{dx^k}$, contained in (6), only for $x=0$ and for $x=1$, i.e. only the numbers $\gamma_{k0}^{j\mu}$ and $\gamma_{kq}^{j\mu}$;*
2. *the values of the functions $\gamma_k^{j\mu}(x)$ and their derivatives at the extreme points for $x=0$ and for $x=1$;*
3. *the values of the functions $a_i(x)$ $(i=0,1,2)$ and their derivatives at the extreme points for $x=0$ and for $x=1$.*

For the equation (5), the well-posedness of the integro-differential conditions (6) is not affected by:

i) the coefficients of the derivatives $\frac{d^k}{dx^k}y$, contained in (6), for $x=v_s$, where $0<v_s<1$, i.e. the numbers $\gamma_{ks}^{j\mu}$ for $0<s<q$;

ii) the values of the function $\gamma_k^{j\mu}(x)$ *and their derivatives for* $0 < x < 1$.

Assume

$$Q_0(x,\psi,\lambda^2) \equiv 0, \quad F_0^{(0)}(x,\lambda^2) \equiv 1, \quad F_0^{(1)}(x,\lambda^2) \equiv 0,$$

$$Q_1(x,\psi,\lambda^2) \equiv 0, \qquad F_1^{(0)}(x,\lambda^2) \equiv 0, \qquad F_1^{(1)}(x,\lambda^2) \equiv 1,$$

$$Q_k(x,\psi,\lambda^2) = \frac{\partial}{\partial x} Q_{k-1}(x,\psi,\lambda^2) + F_{k-1}^{(1)}(x,\lambda^2)\frac{\psi(x)}{a_2(x)},$$

$$F_k^{(0)}(x,\lambda^2) = \frac{\partial}{\partial x} F_{k-1}^{(0)}(x,\lambda^2) + F_{k-1}^{(1)}(x,\lambda^2)\left(-\frac{a_0(x)}{a_2(x)} + \lambda^2\frac{1}{a_2(x)}\right),$$

$$F_k^{(1)}(x,\lambda^2) = F_{k-1}^{(0)}(x,\lambda^2) - \frac{\partial}{\partial x} F_{k-1}^{(1)}(x,\lambda^2) - F_{k-1}^{(1)}(x,\lambda^2)\left(-\frac{a_1(x)}{a_2(x)}\right). \qquad (17)$$

Then for the function

$$y_0(x,\lambda) = \int_0^1 P(x,\xi,\lambda)\psi(\xi)d\xi,$$

we have

$$\frac{d^k}{dx^k} y_0(x,\lambda) = Q_k(x,\psi,\lambda^2) + F_k^{(0)}(x,\lambda^2)\int_0^1 P(x,\xi,\lambda)\psi(\xi)d\xi +$$

$$+F_k^{(1)}(x,\lambda^2)\int_0^1 \frac{\partial P(x,\xi,\lambda)}{\partial x}\psi(\xi)d\xi$$

Assume

$$g_i(\xi,\lambda) = \sum_{\mu=-m_i}^{m} \lambda^\mu \sum_{k=0}^{n} \left\{ \int_0^1 \gamma_k^{i\mu}(x)\left[F_k^{(0)}(x,\lambda^2)P(x,\xi,\lambda) + F_k^{(1)}(x,\lambda^2)\frac{\partial P(x,\xi,\lambda)}{\partial x}\right] dx \right.$$

$$\left. + \sum_{s=0}^{q} \gamma_{ks}^{i\mu}\left[F_k^{(0)}(x,\lambda^2)P(x,\xi,\lambda) + F_k^{(1)}(x,\lambda^2)\frac{\partial P(x,\xi,\lambda)}{\partial x}\right]_{x=v_s} \right\}$$

$$\tilde{\beta}_j^{(i)}(x,\lambda) = \sum_{\mu=-m}^{m} \lambda^\mu \sum_{k=j+2}^{n} \gamma_k^{j\mu}(x)\tilde{Q}_k(x,\lambda^2);$$

$$\beta_{js}^{(1)}(\lambda) = \sum_{\mu=-m}^{m} \lambda^\mu \sum_{k=j+2}^{n} \gamma_{ks}^{j\mu}\tilde{Q}_k(x,\lambda^2)\Big|_{x=v_s};$$

here the functions $\tilde{Q}_k(x,\lambda^2)$ are defined from the identity

$$Q_k(x,\psi,\lambda^2)=\sum_{j=0}^{k-2}\tilde{Q}_{kj}(x,\lambda^2)\frac{d^j\psi(x)}{dx^j}.$$

5°. Let $\psi(x)\in C([0,1])$ for $n\le 2$,while $\psi(x)\in C^{n-2}([0,1])$ for $n>2$.

Theorem 4. *Under the constraints 1°–5° for $\lambda\in R_\varphi$, the problem (5)–(6) has a unique solution, and this solution is represented by the formula*

$$\begin{aligned} y(x,\lambda)=&\delta(x,\lambda,\gamma_1,\gamma_2)+\int_0^1 G(x,\xi,\lambda)\psi(\xi)d\xi+\sum_{j=0}^{n-2}\int_0^1 G_j(x,\xi,\lambda)\frac{d^j\psi(\xi)}{d\xi^j}d\xi\\ &+\sum_{j=0}^{n-2}\sum_{s=0}^{q}\delta_{js}(x,\lambda)\frac{d^j\psi(\xi)}{d\xi^j}\Big|_{\xi=v_s}, \end{aligned} \tag{18}$$

where

$$\delta(x,\lambda,\gamma_1,\gamma_2)=\frac{1}{\Delta(\lambda)}\begin{vmatrix} 0 & y_1(x,\lambda) & y_2(x,\lambda)\\ -\gamma_1 & U_1(y_1) & U_1(y_2)\\ -\gamma_2 & U_2(y_1) & U_2(y_2)\end{vmatrix},$$

$$G(x,\xi,\lambda)=\frac{1}{\Delta(\lambda)}\begin{vmatrix} P(x,\xi,\lambda) & y_1(x,\lambda) & y_2(x,\lambda)\\ g_1(\xi,\lambda) & U_1(y_1) & U_1(y_2)\\ g_2(\xi,\lambda) & U_2(y_1) & U_2(y_2)\end{vmatrix},$$

$$G_j(x,\xi,\lambda)=\frac{1}{\Delta(\lambda)}\begin{vmatrix} 0 & y_1(x,\lambda) & y_2(x,\lambda)\\ \tilde{\beta}_j^{(1)}(\xi,\lambda) & U_1(y_1) & U_1(y_2)\\ \tilde{\beta}_j^{(2)}(\xi,\lambda) & U_2(y_1) & U_2(y_2)\end{vmatrix},$$

$$\delta_{js}(x,\lambda)=\frac{1}{\Delta(\lambda)}\begin{vmatrix} 0 & y_1(x,\lambda) & y_2(x,\lambda)\\ \beta_{js}^{(1)}(\lambda) & U_1(y_1) & U_1(y_2)\\ \beta_{js}^{(2)}(\lambda) & U_2(y_1) & U_2(y_2)\end{vmatrix}.$$

Proof. The general solution of the equation (5) is defined by the formula

$$y(x,\lambda)=c_1y_1(x,\lambda)+c_2y_2(x,\lambda)+\int_0^1 P(x,\xi,\lambda)\psi(\xi)d\xi.$$

Here, c_1 and c_2 are arbitrary constants. By using this function in (6), we obtain a system of linear algebraic equations to determine c_1 and c_2, and the main determinant of this system is $\Delta(x)$. When $\Delta(x)\neq 0$, we can solve this system for c_1 and c_2 and obtain the formula (19) for the solution to the problem (5)–(6). ◀

We have

Theorem 5. *Under the constraints 1° and 2°, if the function $\psi(x)$ is piecewise absolutely continuous in the segment $[0,1]$ and $\psi'(x)| \in L_p(0,1)$, $(p>1)$, then for $0 \le x \le 1$, we have the following inversion formulas:*

$$\int_L \lambda^k d\lambda \int_0^1 P(x,\xi,\lambda)\psi(\xi)d\xi = \begin{cases} 0, & \text{for } k=0, \\ -\sqrt{-1}\left(\frac{\pi}{2}+2\varphi\right)\frac{\psi(x-0)+\psi(x+0)}{2}, & \text{for } k=1, \end{cases} \tag{19}$$

where L is an infinite smooth line, the distant part of which coincides with the continuations of the rays $\arg\lambda = \pm\left(\frac{\pi}{4}+\varphi\right) R_\varphi$, and in (16) the line integral over L is understood in the sense of principal value.

Proof. From (7), we have

$$P(x\xi,\lambda) = Q_0(x,\xi,\lambda) + Q_1(x,\xi,\lambda),$$

where

$$Q_0(x,\xi,\lambda) = -\frac{g(x)g(\xi)}{2\lambda a_2(\xi)} \exp\left(\int_0^\xi \frac{a_1(\tau)}{a_2(\tau)}d\tau\right) \times$$

$$\begin{cases} \exp\left(-\lambda\int_\xi^x \theta(\tau)d\tau\right), & \text{for } 0 \le \xi \le x \le 1, \\ \exp\left(-\lambda\int_x^\xi \theta(\tau)d\tau\right), & \text{for } 0 \le x \le \xi \le 1; \end{cases}$$

$$g(x) = \frac{1}{\sqrt{\theta(x)}} \exp\left(-\frac{1}{2}\int_0^x \frac{a_1(\tau)}{a_2(\tau)}d\tau\right).$$

$G_1(x,\xi,\lambda)$ is some function for which we have the inequality

$$|G_j(x,\xi,\lambda)| \le \frac{C}{|\lambda|^2}\exp(-\varepsilon|\lambda||x-\xi|), x,\xi \in [0,1], \quad \lambda \in R_\varphi.$$

Let $x(0<x<1)$ be an arbitrarily fixed point, and $\delta = \delta_x$ be some positive number for which $0 \le x-\delta < x < x+\delta < 1$. Then we have

$$\int_0^1 Q_0(x,\xi,\lambda)\psi(\xi)d\xi = I_1 + I_2,$$

where

$$I_1 = \int_{x-\delta}^{x+\delta} Q_0(x,\xi,\lambda)\psi(\xi)d\xi, \quad I_2 = I_2 = \int_0^{x-\delta} + \int_{x+\delta}^1.$$

Consequently,

$$I_1 = -\frac{1}{2\lambda^2}[\psi(x-0)+\psi(x+0)] + I_3,$$

$$I_3 = \frac{1}{2\lambda^2} g(x)\tilde{g}(\xi)\Big|_{\xi=x-\delta}\left[\exp\left(-\lambda\int_{x-\delta}^{x}\theta(\tau)\,d\tau\right)\right] +$$
$$+\frac{1}{2\lambda^2} g(x)\tilde{g}(\xi)\Big|_{\xi=x+\delta}\left[\exp\left(-\lambda\int_{x}^{x+\delta}\theta(\tau)\,d\tau\right)\right] +$$
$$+\frac{1}{2\lambda^2} g(x)\int_{x-\delta}^{x}\left[\exp\left(-\lambda\int_{\xi}^{x}\theta(\tau)\,d\tau\right)\right]\frac{d}{d\xi}\tilde{g}(\xi)\,d\xi -$$
$$-\frac{1}{2\lambda^2} g(x)\int_{x}^{x+\delta}\left[\exp\left(-\lambda\int_{x}^{\xi}\theta(\tau)\,d\tau\right)\right]\frac{d}{d\xi}\tilde{g}(\xi)\,d\xi,$$

$$\tilde{g}(\xi) = \frac{g(\xi)\psi(\xi)}{\theta(\xi)a_2(\xi)}\exp\left(\int_0^{\xi}\frac{a_1(\tau)}{a_2(\tau)}d\tau\right).$$

For $\lambda \in R_\varphi$, using the inequality

$$\left|\int_{x-\delta}^{x}\exp\left(-\lambda\int_{\xi}^{x}\theta(\tau)d\tau\right)\psi'(\xi)d\xi\right| \le \int_{x-\delta}^{x}\exp(-\varepsilon|\lambda||x-\xi|)|\psi'(\xi)|d\xi$$

$$\le \left(\int_{x-\delta}^{x}|\psi'(\xi)|^p\right)^{\frac{1}{p}}\left(\int_{x-\delta}^{x}\exp(-\varepsilon q|\lambda||x-\xi|)d\xi\right)^{\frac{1}{q}} \le \frac{C}{|\lambda|^{\frac{1}{q}}}\left(\frac{1}{p}+\frac{1}{q}=1\right).$$

we have

$$|I_3| \le \frac{C}{|\lambda|^{2+\frac{1}{q}}}$$

for $\lambda \in R_\varphi$.

And also we have

$$|I_2| \le \frac{C}{|\lambda|}e^{-\varepsilon\delta|\lambda|} \le \frac{C_{\varepsilon\delta p}}{|\lambda|^p}$$

for $\lambda \in R_\varphi$. p is any positive number.

$$\left|\int_0^1 Q_1(x,\xi,\lambda)\psi(\xi)d\xi\right| \le \frac{C}{|\lambda|^3}.$$

Let $R < \tilde{R} < r_1 < r_2 < \cdots < r_n < \ldots$ be a sequence of such positive numbers that $\lim_{n\to\infty} r_n = \infty$ and let $\Gamma_n = \left\{\lambda : |\lambda| = r_n, |\arg\lambda| \le \frac{\pi}{4}+\varphi\right\}$. Let L_n be a part of Γ_R remaining inside the circle $|\lambda| = r_n \ge \tilde{R}$. $\tilde{R} \gg R$ is such a positive number that for $|\lambda| \ge \tilde{R}$, a part of L coincides with the half-straight lines

$\left\{\lambda : |\lambda| \geq \tilde{R}, \arg\lambda = \pm\left(\frac{\pi}{4} + \varphi\right)\right\}$. By the analyticity of the integrand function in the domain R_φ, we have

$$\int_{L_n} \lambda^k d\lambda \int_0^1 P(x,\xi,\lambda)\psi(\xi)d\xi = \int_{\Gamma_n} = I_1 + I_2. \tag{20}$$

where

$$I_1 = -\frac{\psi(x-0)+\psi(x+0)}{2}\int_{\Gamma_n}\frac{1}{\lambda^{2-k}}d\lambda,$$

$$|I_2| \leq \int_{\Gamma_n}\frac{|d\lambda|}{|\lambda|^{2-k+\frac{1}{q}}} \leq \frac{C\cdot 2\pi}{r_n^{1-k+\frac{1}{q}}}. \tag{21}$$

The validity of (20) follows from (21) and (22). The theorem is proved. ◄

We have

Theorem 6. *Under the constraints 1°–4°, if the function $\psi(x)$ is piecewise absolutely continuous in the segment $[0,1]$ and $|\psi(x)| \in L_p(0,1)$, $(p > 1)$, then for $0 < x < 1$, we have the following inversion formulas:*

$$\int_L \lambda^k d\lambda \int_0^1 G(x,\xi,\lambda)\psi(\xi)d\xi = \begin{cases} 0, & \text{for } k=0, \\ -\sqrt{-1}\left(\frac{\pi}{2}+2\varphi\right)\frac{\psi(x-0)+\psi(x+0)}{2}, & \text{for } k=1. \end{cases} \tag{22}$$

Proof. From (19), we have

$$G(x,\xi,\lambda) = P(x,\xi,\lambda) - \tilde{G}(x,\xi,\lambda). \tag{23}$$

For $x \in R_\varphi$, using the inequalities

$$\begin{aligned} &|y_1(x,\lambda)| \leq C\exp(-\varepsilon|\lambda||x|), \quad \lambda \in R_\varphi, \\ &|y_2(x,\lambda)| \leq C\exp(-\varepsilon|\lambda||1-x|), \end{aligned} \tag{24}$$

we obtain

$$|\tilde{G}(x,\xi,\lambda)| \leq C|\lambda|^N\left(\exp(-\varepsilon|\lambda||x|) + \exp(-\varepsilon|\lambda||1-x|)\right),$$

$x,\xi \in [0,1]$, $\lambda \in R_\varphi$, N is some number.

Consequently,

$$\int_L \lambda^n d\lambda \int_0^1 \tilde{G}(x,\xi,\lambda)\psi(\xi)d\xi = 0, \quad \text{as } 0 < x < 1, \tag{25}$$

where n is any integer.

Taking into account (24), (26), and (20, we obtain the validity of (23).

The theorem is proved. ◄

Using the inequality (25) and following the proof schemes of Theorems 5 and 6, the following Theorems 7 and 8 are proven.

Theorem 7. *Under the constraints 1°–4°, if $\psi(x)$ is an absolutely integrable function in the segment $[0,1]$, we have*

$$\int_L \lambda^s d\lambda \int_0^1 G_j(x,\xi,\lambda)\psi(\xi)d\xi = 0, \quad \text{as } 0 < x < 1, \quad j = 0,1,\ldots,n-2, \tag{26}$$

where s is any integer.

Theorem 8. *Under the constraints 1°–4° and for $0 < x < 1$, we have*

$$\begin{gathered} \int_L \lambda^s \delta(x,\lambda,\gamma_1,\gamma_2)d\lambda = 0, \\ \int_L \lambda^s \delta_{ji}(x,\lambda)d\lambda = 0, \quad j = \overline{0,n-2}, \quad i = \overline{0,q}, \end{gathered} \tag{27}$$

where s is any integer.

Theorem 9. *Under the constraints 1°–4° and in the case $n \le 2$, if the function $\psi(x)$ is absolutely continuous in the segment $[0,1]$ and $|\psi'(x)| \in L_p(0,1), (p>1)$, and in the case $n > 2$, if the constraint 5° is fulfilled, then for $x \in (0,1)$, we have the following inversion formulas:*

$$\int_L \lambda^k y(x,\lambda)d\lambda = \begin{cases} 0, & \text{for } k = 0, \\ -\sqrt{-1}\left(\frac{\pi}{2} + 2\varphi\right)\frac{\psi(x-0)+\psi(x+0)}{2}, & \text{for } k = 1, \end{cases} \tag{28}$$

where $y(x,\lambda)$ is defined by (18).

Proof. Using the analytical formula for $y(x,\lambda)$ in (18) and the statements of Theorems 7 and 8, the correctness of the transformation formula (28) is proven in accordance with the method discussed in [3, 5]. ◄

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Received 25 February 2025
Accepted 05 May 2025