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Integration of the Hierarchy of the Kaup-Boussinesq System Via Inverse Scattering Method

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Abstract. In this study we derive a rich hierarchy for the Kaup system with time-dependent coefficients in the class of rapidly decreasing functions. We show that a hierarchy of the Kaup-Boussinesq system with time-dependent coefficients is also an important theoretical model, since it is a completely integrable system. For solving the problem under consideration, we use the direct and inverse scattering problem of the Sturm-Liouville equation with an energy-dependent potential. The time evolution of the scattering data for the Sturm-Liouville equation with an energy-dependent potential associated with the solution of the hierarchy of the Kaup-Boussinesq system with time-dependent coefficients is determined. Using the solution of the inverse scattering problem with respect to the time-dependent scattering data we give an algorithm of constructing the solution of the hierarchy of the Kaup-Boussinesq system with time-dependent coefficients.

Key Words and Phrases: monlinear soliton equation, hierarchy of the Kaup-Boussinesq system, inverse scattering method, Sturm-Liouville equation with an energy-dependent potential, time evolution of scattering data.

2010 Mathematics Subject Classifications: 34A34, 34B15, 35Q51, 35J60, 35J66, 35L05

1. Introduction

The study of soliton equations is one of the most prominent subjects in the field of nonlinear science. Nonlinear behavior is observed in various branches of science including fluid mechanics, solid-state physics, plasma physics, biology, optics, earthquakes, and our focus: surface water waves. Completely integrable nonlinear partial differential equations are used to describe such phenomena because exact solutions can be produced using the inverse scattering transform.

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This method, which can be thought of as the nonlinear analogue to the Fourier transform, was first introduced in 1967 by Gardner, Greene, Kruskal, Miura [1].

Our goal is to construct a hierarchy for the Kaup-Boussinesq system with variable coefficients that can be integrated via the inverse scattering transform method.

The Kaup-Boussinesq system describing wave propagation in shallow water was first derived by Boussinesq [2]. In [3], D.J. Kaup, using inverse scattering transform method, proved that this system is completely integrable. Different techniques were used to construct its solution [4]-[23].

In [24], the Inverse Scattering Transform Method is used to solve a class of nonlinear equations associated with the inverse problem for the one-dimensional Schrodinger equation with the energy-dependent potential. The corresponding inverse scattering problems were solved in [25]-[32].

In this paper, we show that a hierarchy of the Kaup-Boussinesq system with time-dependent coefficients is also an important theoretical model, since it is a completely integrable system. Namely, we find the time evolution of scattering data for a quadratic pencil of Sturm–Liouville operators associated with the solution of the Kaup-Boussinesq system with time-dependent coefficients in the class of rapidly decreasing functions. The resulting equalities completely determine the scattering data at any t, which allows applying the inverse scattering method to solve the Cauchy problem for the Kaup-Boussinesq system with time-dependent coefficients. Using the solution of the inverse scattering problem with respect to the time-dependent scattering data we give an algorithm of constructing the solution of the hierarchy of the Kaup-Boussinesq system with time-dependent coefficients in the class of rapidly decreasing functions.

2. Formulation of the problem

We consider the Kaup-Boussinesq system with a self-consistent source

$$U_t + \Omega(L^*)U_x = G \tag{1}$$

and the initial condition

$$v(x,t)|_{t=0} = v_0(x), \;\; u(x,t)|_{t=0} = u_0(x), \;\; x \in \mathbb{R}, \tag{2}$$

where

$$U = \begin{pmatrix} v(x,t) \\ u(x,t) \end{pmatrix}, \; G = \begin{pmatrix} G_1(x,t) \\ G_2(x,t) \end{pmatrix},$$
$$G_1(x,t) = \mu(t)v_x, \quad G_2(x,t) = \mu(t)u_x,$$

$$L^* = \begin{pmatrix} 0 & -\frac{\partial^2}{\partial x^2} + 4v - 2v_x \int_x^\infty d\tau \\ 1 & 4u - 2u_x \int_x^\infty d\tau \end{pmatrix}, \tag{3}$$

$\Omega(s)$ is any polynomial function of s (whose coefficients may depend on time) and $\mu(t)$ is an arbitrary prescribed continuous function. The functions $v_0(x)$, $u_0(x)$ satisfy the following conditions:

(i) $u_0(x)$ is absolutely continuous on each finite closed interval $[\alpha, \beta] \subset (-\infty, \infty)$ and the following inequalities hold:

$$\int_{-\infty}^{\infty} |u_0(x)| dx < \infty, \quad \int_{-\infty}^{\infty} (1 + |x|)[|v_0(x)| + |u_0'(x)|] dx < \infty; \tag{4}$$

(ii) the pencil of operators

$$T(0,\, k) := -\frac{d^2}{dx^2} + v_0(x) + 2ku_0(x) - k^2$$

has precisely $2N$ simple eigenvalues $k_1(0), k_2(0), ..., k_{2N}(0)$, where N is a positive integer and k is a spectral parameter.

Let us give two particular simple examples contained in rich family of equations (1). We get the classical Kaup-Boussinesq system with time-dependent coefficients for $\Omega(s) = s$:

$$\begin{cases} v_t = u_{xxx} - 4vu_x - 2uv_x + \mu(t)v_x, \\ u_t = -6uu_x - v_x + \mu(t)u_x, \;\; x \in \mathbb{R}, \;\; t > 0, \end{cases}$$

for $\Omega(s) = s^2$:

$$\begin{cases} v_t = v_{xxx} + 6uu_{xxx} + 18u_x u_{xx} - 6vv_x - 24vuu_x - 6v_x u^2 + \mu(t)v_x, \\ u_t = u_{xxx} - 6vu_x - 6v_x u - 30u_x u^2 + \mu(t)u_x, \;\; x \in \mathbb{R}, \;\; t > 0. \end{cases}$$

Note that for $u(x,t) = 0$ the last equation reduces to the Korteweg-de Vries equation with time-dependent coefficients.

The main aim of this work is to derive representations for the solutions

$$\{v(x,t), \; u(x,t), \; \phi_1(x,t), \; \phi_2(x,t), ..., \phi_N(x,t)\}$$

of the problem (1)-(4) with the use of inverse scattering method for the operator $T(t,k)$:

$$T(t,k)y \equiv -y'' + v(x,t)y + 2ku(x,t)y - k^2 y = 0, \quad x \in \mathbb{R}. \tag{5}$$

3. Preliminaries

In this section, we give the basic information about the scattering theory for the quadratic pencil of Sturm-Liouville operators [27]. For convenience, we temporarily omit the variable t in the functions $v(x,t)$ and $u(x,t)$.

We consider the quadratic pencil of Sturm-Liouville equations

$$T(0,k)y \equiv -y'' + v(x)y + 2ku(x)y - k^2y = 0, \ x \in \mathbb{R}, \tag{6}$$

where $u(x)$ and $v(x)$ are real functions satisfying condition (3). Under condition (3), Eq. (5) for all k from the half-plane $Imk \geq 0$ has solutions $f_+(x,k), \ f_-(x,k)$, which can be represented as

$$f_+(x,k) = e^{i\alpha_+(x)} \cdot e^{ikx} + \int_x^\infty K_+(x,\tau)e^{ik\tau}d\tau, \tag{7}$$

$$f_-(x,k) = e^{i\alpha_-(x)} \cdot e^{-ikx} + \int_{-\infty}^x K_-(x,\tau)e^{-ik\tau}d\tau, \tag{8}$$

where

$$\alpha_+(x) \ = \int_x^\infty u(\tau)d\tau, \quad \alpha_-(x) \ = \int_{-\infty}^x u(\tau)d\tau$$

and the following relations are satisfied for the kernels $K_\pm(x,\tau)$:

$$|K_+(x,\tau)| \leq \frac{1}{2}\sigma^+(\frac{x+\tau}{2})e^{\sigma_1^+(x)},$$

$$\sigma^+(x) = \int_x^\infty (|v(\tau)| + |u'(\tau)|)d\tau,$$

$$\sigma_1^+(x) = \int_x^\infty [(\tau - x)|v(\tau)| + 2|u(\tau)|]d\tau,$$

$$K_+(x,x) = \frac{1}{2}\int_x^\infty v(\tau)e^{i\alpha_+(\tau)}d\tau - \frac{i}{2}u(x)e^{i\alpha_+(x)} + i\int_x^\infty u(\tau)K_+(\tau,\tau)d\tau,$$

$$K_-(x,x) = \frac{1}{2}\int_{-\infty}^x v(\tau)e^{i\alpha_-(\tau)}d\tau - \frac{i}{2}u(x)e^{i\alpha_-(x)} + i\int_{-\infty}^x u(\tau)K_-(\tau,\tau)d\tau.$$

Obviously, for each $x \in (-\infty, \infty)$, the functions $f_+(x,k), \ f_-(x,k)$ are regular in the half-plane $Imk > 0$, and the following asymptotic formulas are true:

$$f_+(x,k) = e^{ikx}[1 + o(1)], \ \ x \to +\infty, \tag{9}$$

$$f_-(x,k) = e^{-ikx}[1 + o(1)], \ \ x \to -\infty. \tag{10}$$

For real $k \neq 0$, the pairs $f_+(x,k),\ f_-(x,k)$ and $f_-(x,k),\ \bar{f}_-(x,k)$ (the bar over the function here and below denotes complex conjugation) form two fundamental systems of solutions to equation (6). The following relations hold

$$f_+(x,k) = b(k)f_-(x,k) + a(k)\bar{f}_-(x,k), \tag{11}$$

$$f_-(x,k) = -\bar{b}(k)f_+(x,k) + a(k)\bar{f}_+(x,k). \tag{12}$$

The functions $a(k)$ and $b(k)$ are defined for all $k \in R^* = (-\infty,\infty)\backslash\{0\}$ and the following equality is fulfilled

$$a(k) = -\frac{1}{2ik}W\{f_+, f_-\}, \quad b(k) = \frac{1}{2ik}W\{f_+, \bar{f}_-\}. \tag{13}$$

Moreover, the function $a(k)$ admits an analytic continuation to the half-plane $Imk > 0$ and can have at most a finite number of zeros $k_1, k_2, ..., k_N$. Besides, at $k = k_n,\ n = 1,2,...,N$ the following equality holds:

$$f_\mp(x,k_n) = B_n^\pm f_\pm(x,k_n).$$

where the quantities $B_n^\pm$ are independent of x. The corresponding functions $f_\pm(x,k_n)$ are the only $L^2(R)$ solutions of (6) for $Imk > 0$ and are the 'bound states'.

The set of the quantities

$$\left\{ r_-(k) = \frac{b(k)}{a(k)},\ k \in R,\ k_1, k_2, ..., k_N,\ \gamma_1^-, \gamma_2^-, ..., \gamma_N^- \right\} \tag{14}$$

and

$$\left\{ r_+(k) = -\frac{\bar{b}(k)}{a(k)},\ k \in R,\ k_1, k_2, ..., k_N,\ \gamma_1^+, \gamma_2^+, ..., \gamma_N^+ \right\} \tag{15}$$

are called the left and right scattering data of Eq. (6), respectively, where

$$\gamma_n^\pm = B_n^\pm \left(\left.\frac{da(k)}{dk}\right|_{k=k_n} \right)^{-1},\ n = 1,2,...,N.$$

Functions $r_-(k)$ and $r_+(k)$ are called the left and right reflection coefficients, respectively. The quantities $\alpha_\pm(x),\ K_\pm(x,t)$ from the formulas (7), (8) satisfy the equations

$$e^{i\alpha_+(x)}F_+(x+y) + \overline{K_+(x,y)} + \int_x^\infty K_+(x,\tau)F_+(\tau+y)d\tau = 0,\ x \le y < \infty, \tag{16}$$

$$e^{i\alpha_-(x)}F_-(x+y)+\overline{K_-(x,y)}+\int_{-\infty}^{x}K_-(x,\tau)F_-(\tau+y)d\tau=0,\ \ -\infty<y\le x, \tag{17}$$

where

$$F_\pm(x)=-i\sum_{n=1}^{N}\gamma_n^\pm e^{\pm ik_nx}+\frac{1}{2\pi}\int_{-\infty}^{\infty}r_\pm(k)e^{\pm ikx}dk. \tag{18}$$

To ensure that Eqs. (16) and (17) have a unique solution for any $t\ge 0$, we add the condition

$$\begin{cases}\overline{a^+(y,t)}=\int_x^\infty F_+(y+\tau,t)a^+(\tau,t)d\tau,\ \ x\le y<\infty,\\ \overline{a^-(y,t)}=\int_{-\infty}^{x} F_-(y+\tau,t)a^-(\tau,t)d\tau,\ \ -\infty<y\le x,\end{cases}\Rightarrow$$

$$\Rightarrow\begin{pmatrix}\overline{a^+(y,t)}\\ \overline{a^-(y,t)}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

That is, we require that a homogeneous system of integral equations has only a zero solution. If this condition is satisfied, then the problem is uniquely solvable. For example, this condition is automatically satisfied in the following two cases:

(i) the discrete spectrum is absent;

(ii) the spectrum of the pencil consists only of the discrete spectrum, i.e., $r_+(k)=0$.

In the general case, i.e., without an additional condition, we could not solve the problem under consideration.

We now turn to the question of constructing $u(x)$ and $v(x)$ from scattering data (14) or (15). Note that scattering data (14), (15) and $F_\pm(x)$ are bijectively related via transforms (18). To restore the coefficient functions $u(x)$ and $v(x)$ in equation (6) from the right reflection coefficient $r_+(k)$, we proceed as follows:

1) It is necessary to find the function $F_+(x)$ by formula (18) and solve with respect to $K_+^0(x,y)\in L_1(x,\infty)$, $K_+^1(x,y)\in L_1(x,\infty)$ the integral equations

$$F_+(x+y)+\overline{K_+^{(0)}(x,y)}+\int_x^\infty K_+^{(0)}(x,\tau)F_+(\tau+y)d\tau=0,\ \ x\le y<\infty, \tag{19}$$

$$iF_+(x+y)+\overline{K_+^{(1)}(x,y)}+\int_x^\infty K_+^{(1)}(x,\tau)F_+(\tau+y)d\tau=0,\ \ x\le y<\infty. \tag{20}$$

2) Next, we have to define the function $\alpha_+(x)$ as a solution to a nonlinear integral equation of the Volterra type

$$\alpha_+(x) = \int_x^\infty \Phi(s,\, \alpha_+(s))ds, \;\; -\infty < x < \infty, \tag{21}$$

where

$$\Phi(s,z) = \left[\mathrm{Re}\, K_+^{(0)}(s,s) - \mathrm{Im}\, K_+^{(1)}(s,s)\right] \sin 2z +$$

$$+2\left[\mathrm{Re}\, K_+^{(1)}(s,s)\right]\sin^2 z - 2\left[\mathrm{Im}\, K_+^{(0)}(s,s)\right]\cos^2 z \tag{22}$$

and

$$K_+(x,y) = K_+^{(0)}(x,y)\cos\alpha_+(x) + K_+^{(1)}(x,y)\sin\alpha_+(x). \tag{23}$$

3) Find the coefficients $u(x)$, $v(x)$ of Eq. (5) from the formulas

$$u(x) = -\alpha'_+(x), \tag{24}$$

$$v(x) = -u^2(x) - 2\frac{d}{dx}\left\{[\mathrm{Re}\, K_+(x,x)]\cos\alpha_+(x) + [\mathrm{Im}\, K_+(x,x)]\sin\alpha_+(x)\right\}. \tag{25}$$

Note that the functions

$$h_n(x) = \left(\left.\frac{da(k)}{dk}\right|_{k=k_n}\right)^{-1} \left.\frac{d}{dk}[f_-(x,k) - B_n^+ f_+(x,k)]\right|_{k=k_n}, \;\; n = 1,2,...,N$$

are the solutions of the equations $T(0,k_n)y = k_n^2 y$, $n = 1,2,...,N$. For $\mathrm{Im}k > 0$, using (9) and (10), we obtain the asymptotics

$$h_n(x) \to e^{-ik_n x}, \;\; x \to +\infty, \tag{26}$$

$$h_n(x) \to -B_n^+ e^{ik_n x}, \;\; x \to -\infty. \tag{27}$$

From the asymptotics in (9), (10), (26) and (27), we obtain

$$W\{h_n(x),\, f_+(x,k_n)\} = 2ik_n, \;\; W\{h_n(x),\, f_-(x,k_n)\} = 2ik_n B_n^+.$$

Lemma 1. *If $y(x,\lambda)$ and $z(x,\mu)$ are the solutions of the equations $T(\lambda)y = \lambda^2 y$ and $T(\mu)z = \mu^2 z$, then the identity*

$$(\lambda + \mu - 2u)yz = \frac{(yz' - y'z)'}{\lambda - \mu} \tag{28}$$

holds.

Lemma 1 is proved by direct verification.

4. Time evolution of the scattering data

In this section, we derive the time evolution of the scattering data, which allows us to present an algorithm for solving the problem (1)–(4).

We now introduce the "scalar product"

$$\langle V(x),\ W(x)\rangle = \int_{-\infty}^{\infty} [V_1(x)W_1(x) + V_2(x)W_2(x)]dx$$

for $V(x) = (V_1(x), V_2(x))^T$, and the vector functions

$$\Phi_1(x,t,k) = (f_+(x,t,k)\, f_-(x,t,k),\ 2kf_+(x,t,k)\, f_-(x,t,k))^T, \qquad (29)$$

$$\Phi_2(x,t,k) = \left(f_+(x,t,k)\, \bar{f}_-(x,t,k),\ 2kf_+(x,t,k)\, \bar{f}_-(x,t,k)\right)^T, \qquad (30)$$

$$\Phi_3(x,t,k_n) = (h_n(x,t)\, f_-(x,t,k_n),\ 2k_n h_n(x,t)\, f_-(x,t,k_n))^T. \qquad (31)$$

The following theorem contains the main results of this paper.

Theorem 1. *If the functions $v = v(x,t)$ and $u = u(x,t)$ are the solutions of the problem (1)-(4), then the scattering data of the operator $T(t,\ k)$ evolve in t as follows:*

$$\dot{r}_+(t,k) = -2ik\,[\Omega(2k) - \mu(t)]\, r_+(t,k), \qquad (32)$$

$$\dot{k}_n(t) = 0, \qquad (33)$$

$$\dot{\gamma}_n^+(t) = 2ik_n\,[\Omega(2k_n) + \mu(t)]\,\gamma_n^+(t). \qquad (34)$$

Proof. It readily follows from (5), (9), (10), (12), (26), (27) and the notations (29), (30) and (31) that

$$\dot{a}(k,t) = -(2ik)^{-1}\langle U_t + \Omega(L^*)U_x,\ \Phi_1\rangle,\ \ (\mathrm{Im}k \ge 0, k \ne 0), \qquad (35)$$

$$\dot{b}(k,t) - 2ik\Omega(2k)b(k,t) = (2ik)^{-1}\langle U_t + \Omega(L^*)U_x,\ \Phi_2\rangle,\ \ (k \in \mathbb{R}^*), \qquad (36)$$

$$\dot{B}_n^+(t) - 2ik_n\Omega(2k_n)B_n^+(t) = (2ik_n)^{-1}\langle U_t + \Omega(L^*)U_x,\ \Phi_3\rangle. \qquad (37)$$

If $U(x,t)$ satisfies (1), then Eqs. (35), (36) and (37) become

$$\dot{a}(k,t) = -(2ik)^{-1}\langle G,\ \Phi_1\rangle,\ \ (\mathrm{Im}k \ge 0, k \ne 0), \qquad (38)$$

$$\dot{b}(k,t) - 2ik\Omega(2k)b(k,t) = (2ik)^{-1}\langle G,\ \Phi_2\rangle,\ \ (k \in \mathbb{R}^*), \qquad (39)$$

$$\dot{B}_n^+(t) - 2ik_n\Omega(2k_n)B_n^+(t) = (2ik_n)^{-1}\langle G,\ \Phi_3\rangle. \qquad (40)$$

Let's calculate the right-hand sides of (38)-(40).

$$\langle G,\ \Phi_1\rangle = \int_{-\infty}^{\infty}(G_1 f_+ f_- + 2kG_2 f_+ f_-)\,dx = \mu(t)\int_{-\infty}^{\infty}(v_x + 2ku_x)\, f_+ f_- dx =$$

$$= \mu(t)(v+2ku) f_+ f_-|_{-\infty}^{\infty} - \mu(t)\int_{-\infty}^{\infty}(v+2ku)(f_+f_-)' dx =$$

$$= -\mu(t)\int_{-\infty}^{\infty}\left[\left(k^2 f_- + f_-''\right) f_+' + \left(k^2 f_+ + f_+''\right) f_-'\right] dx =$$

$$= -\mu(t)\, k^2 (f_- f_+)\big|_{-\infty}^{\infty} - -\mu(t)\, k^2 \left(f_-' f_+'\right)\big|_{-\infty}^{\infty} =$$

$$= -\mu(t)\, k^2\left(-\bar{b}(k) f_+ + a(k)\bar{f}_+\right) f_+\big|^{\infty} + \mu(t)\, k^2\left(b(k) f_- + a(k)\bar{f}_-\right) f_-\big|^{-\infty} -$$

$$-\mu(t)\, k^2\left(-\bar{b}(k) f_+' + a(k)\bar{f}_+'\right) f_+'\big|^{\infty} + \mu(t)\, k^2\left(b(k) f_-' + a(k)\bar{f}_-'\right) f_-'\big|^{-\infty} = 0.$$

From the last relation, taking (38) into account, we obtain

$$\dot{a}(k,t) = 0, \tag{41}$$

whence it follows that the zeros $k_n = k_n(t),\ \ n = 1, 2, ..., N$ of the function $a(k,t)$ are also independent of time, which means that the relation (33) is true.

Now we calculate the right-hand side of (39). As in the foregoing, we then derive

$$\langle G,\ \Phi_2\rangle = \int_{-\infty}^{\infty}\left(G_1 f_+ \bar{f}_- + 2kG_2 f_+ \bar{f}_-\right) dx = 4\mu(t)\, k^2 b(k,t).$$

This implies that

$$\dot{b}(t,k) = 2ik\left[\Omega(2k) - \mu(t)\right] b(t,k). \tag{42}$$

From (41) and (42) and the form for the function $r_+(t,k)$, we obtain (32).

Further, we calculate the dependence of $B_n^+(t),\ \ n = 1, 2, ..., N$.

$$\langle G,\ \Phi_3\rangle = \int_{-\infty}^{\infty}\left(G_1 h_n f_-(k_n) + 2k_n G_2 h_n f_-(k_n)\right) dx =$$

$$= \mu(t)\int_{-\infty}^{\infty}(v+2k_n u)_x\, h_n f_-(k_n) dx =$$

$$= -\mu(t)\int_{-\infty}^{\infty}(v+2k_n u)\left(h_n f_-(k_n)\right)' dx =$$

$$= -\mu(t)\int_{-\infty}^{\infty}\left[\left(k_n^2 f_-(k_n) + f_-''(k_n)\right) h_n' + \left(k_n^2 h_n + h_n''\right) f_-'\right] dx =$$

$$= -\mu(t)\, k_n^2\left(h_n f_-(k_n)\right)\big|_{-\infty}^{\infty} - \mu(t)\left(h_n' f_-'(k_n)\right)\big|_{-\infty}^{\infty} = -4\mu(t)\, k_n^2 B_n^+(t)$$

By (40), we have

$$\dot{B}_n^+(t) = 2ik_n\left[\Omega(2k_n) + \mu(t)\right] B_n^+(t). \tag{43}$$

From (43) and the form of γ_n^+, we obtain (34). ◄

Remark 1. *The obtained results completely determine the time evolution of spectral data, which allows solving the problem* (1)-(4) *by the following algorithm. Assume that* $v_0(x)$ *and* $u_0(x)$ *are given.*

1. For given functions $v_0(x)$ *and* $u_0(x)$*, find the scattering data*

$$\{r_+(k),\ k \in R,\ k_1, k_2, ..., k_N,\ \gamma_1^+, \gamma_2^+, ..., \gamma_N^+\}$$

for $T(0, k)$*;*

2. By Theorem 1, derive the time evolution of the scattering data

$$\{r_+(t,k),\ k \in R,\ k_1(t), k_2(t), ..., k_N(t),\ \gamma_1^+(t), \gamma_2^+(t), ..., \gamma_N^+(t)\}$$

for $T(t, k)$;

3. With this scattering data, uniquely determine the function $F_+(x,t)$ *from relation* (18)*;*

4. Substituting $F_+(x,t)$ *in Gelfand-Levitan-Marchenko integral equations* (19), (20) *and solving this system, obtain* $K_+^{(0)}(x,y,t)$ *and* $K_+^{(1)}(x,y,t)$*;*

5. Further, from (21), (22) *and* (23) *derive* $K_+(x,y,t)$*, and then find the potentials* $v(x,t)$ *and* $u(x,t)$ *by formulas* (24) *and* (25)*;*

5. Conclusion

In this paper, we derive a rich hierarchy for the Kaup system with time-dependent coefficients in the class of rapidly decreasing functions. We discuss the complete integrability of the constructed systems that is based on the transformation to the scattering data of an associated quadratic pencil of Sturm-Liouville equations with rapidly decreasing coefficients. It is shown that the Kaup-Boussinesq system with time-dependent coefficients is also an important theoretical model, since it is a fully integrable system. Namely, the time evolution of scattering data is found for a quadratic pencil of Sturm-Liouville operators associated with the solution of the Kaup-Boussinesq system with time-dependent coefficients. This allows us to find the solution to the problem (1)-(4) in the class of rapidly decreasing functions using the inverse scattering problem method.

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