Azerbaijan Journal of Mathematics
V. 15, No 2, 2025, July
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2025.2.37

Fredholm Property of a Nonlocal Boundary Value Problem for an Integro-Differential Equation of Elliptic Type in a Cube

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Abstract. This paper deals with a boundary value problem for a three-dimensional integro-differential equation of elliptic type with nonlocal boundary conditions in a unit cube. Singular necessary conditions of solvability are derived, the regularization of which is carried out according to a new original scheme. Based on the regularized necessary conditions, in combination with boundary conditions, the Fredholm property of the problem is proved.

Key Words and Phrases: three-dimensional integro-differential equation of elliptic type, nonlocal boundary conditions, unit cube, fundamental solution, necessary conditions, regularization, Fredholm property.

2010 Mathematics Subject Classifications: 35J25

1. Introduction

As is known, for an ordinary differential equation the number of additional conditions (Cauchy conditions or boundary conditions) always coincides with the order of the considered equation.

In the theory of equations of mathematical physics and partial differential equations, the canonical form of an equation of elliptic type is the Laplace equation (second-order equation) for which one local boundary condition (Dirichlet, Neumann or Poincare) is specified.

The non-local boundary conditions free us from the above mismatch between ordinary differential equations and partial differential equations. For nonlocal boundary value problems, the authors have found the possibility of proving Fredholm property with the help of so-called necessary conditions.

It should be noted that for an ordinary differential equation these necessary conditions, similar to nonlocal boundary conditions, are mentioned by A.A. Dezin [1, 2, 3] (who came to these conditions by an artificial way and so couldn't build them for partial differential equations).

The idea of necessary conditions for partial differential equations was first used by A.V. Bitsadze for the Laplace equation [4, p.185] both in two-dimensional and three-dimensional cases. But the regularization of the singularities in the necessary conditions was artificial, particularly in three-dimensional case, and contained some uncertainties.

Finally, Begehr derived these necessary conditions for Cauchy-Riemann equation [5, 6].

Some of the necessary conditions obtained for three-dimensional problem contain singular multiple integrals. But the regularization of these singularities doesn't subject to the conventional scheme [7, 8, 9].

As is known, the regularization of singular integral equations in general case is conducted by the method of successive substitutions: after the first substitution a double singular integral is obtained and, after changing the order of integration, the Poincare-Bertrand formula is applied to get a regular kernel and a jump which doesn't "eat" the external function. Thus, a Fredholm integral equation of the second kind with a regular kernel is obtained.

In the considered problem, the obtained necessary conditions, or integral equations, are in spectrum, so when they are regularized by the mentioned scheme we come to Fredholm integral equations of the first kind which is a "deadlock".

By the suggested new scheme, the singular necessary conditions are regularized using the given boundary conditions, which is principally new. As a result, the considered problem is reduced to a system of Fredholm integral equations of second kind with regular kernels [7, 8, 9].

2. Problem statement

Let us consider a second order integro-differential equation in a 3-dimensional domain – a unit cube $D = \left\{x \in R^3 : 0 < x_i < 1,\ i = \overline{1,3}\right\}$ (Fig.1):

$$Lu(x) = \Delta u(x) + \sum_{j=1}^{3} a_j(x)\frac{\partial u(x)}{\partial x_j} + a(x)u(x) +$$

$$+ \int_D \left\{ \sum_{j=1}^{3} K_j(x,\zeta)\frac{\partial u(\zeta)}{\partial \zeta_j} + K(x,\zeta)u(\zeta) \right\} d\zeta = 0, \quad x \in D, \tag{1}$$

with nonlocal boundary conditions binding boundary values of the desired function and its partial derivatives on the opposite facets $\Gamma_n,\ n=\overline{1,6}$, of the cube D $(\partial D=\Gamma=\bigcup_{n=1}^{6}\Gamma_n)$: Fig.1.

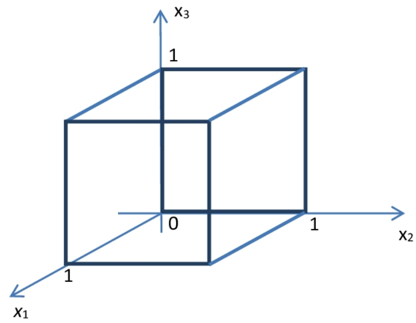

Figure 1:

$$\left[\sum_{j=1}^{3}\alpha_{1j}^{(1)}(x_2,x_3)\frac{\partial u(x)}{\partial x_j}+\alpha_1^{(1)}(x_2,x_3)u(x)\right]|_{x_1=0}+$$
$$+\left[\sum_{j=1}^{3}\alpha_{1j}^{(2)}(x_2,x_3)\frac{\partial u(x)}{\partial x_j}+\alpha_1^{(2)}(x_2,x_3)u(x)\right]|_{x_1=1}=$$
$$=\varphi_1(x_2,x_3),\ 0\le x_2\le 1,\ \ 0\le x_3\le 1,$$

$$\left[\sum_{j=1}^{3}\alpha_{2j}^{(1)}(x_1,x_3)\frac{\partial u(x)}{\partial x_j}+\alpha_2^{(1)}(x_1,x_3)u(x)\right]|_{x_2=0}+$$
$$+\left[\sum_{j=1}^{3}\alpha_{2j}^{(2)}(x_1,x_3)\frac{\partial u(x)}{\partial x_j}+\alpha_2^{(2)}(x_1,x_3)u(x)\right]|_{x_2=1}=$$
$$=\varphi_2(x_1,x_3),\ 0\le x_1\le 1,\ \ 0\le x_3\le 1,$$

$$\left[\sum_{j=1}^{3}\alpha_{3j}^{(1)}(x_1,x_2)\frac{\partial u(x)}{\partial x_j}+\alpha_3^{(1)}(x_1,x_2)u(x)\right]|_{x_3=0}+$$

$$+\left[\sum_{j=1}^{3}\alpha_{3j}^{(2)}(x_1,x_2)\frac{\partial u(x)}{\partial x_j}+\alpha_3^{(2)}(x_1,x_2)u(x)\right]|_{x_3=1}=$$

$$=\varphi_3(x_1,x_2),\ 0\le x_1\le 1,\ \ 0\le x_2\le 1, \tag{2}$$

where the functions $\alpha_{ij}^{(p)}(x_l,x_m),\ \alpha_i^{(p)}(x_l,x_m),\ \varphi_i(x_l,x_m),\ i,j=\overline{1,3},\ p=1,2; l,m=\overline{1,3}, l<m,\ l,m\neq i$, are continuous in the domains $0\le x_l\le 1,\ 0\le x_m\le 1$.

A fundamental solution of the principal part of equation (1) is [10]

$$U(x-\xi)=\frac{1}{4\pi\,|x-\xi|}. \tag{3}$$

3. Basic relationships and necessary conditions

Let us multiply (1) by (3) and integrate over the domain D:

$$\int_D Lu(x)U(x-\xi)dx =$$

$$= \int_D \left\{ \Delta u(x) + \sum_{j=1}^{3} a_j(x)\frac{\partial u(x)}{\partial x_j} + a(x)u(x) \right\} U(x-\xi)dx+$$

$$+\int_D \left(\int_D \left\{ \sum_{j=1}^{3} K_j(x,\zeta)\frac{\partial u(\zeta)}{\partial \zeta_j} + K(x,\zeta)u(\zeta) \right\} d\zeta \right) U(x-\xi)dx = 0, \;\; x \in D, \tag{4}$$

Integrating by parts and taking into account that $\Delta_x U(x-\xi) = \delta(x-\xi)$, where $\delta(x-\xi)$ is Dirac's function, we obtain **the first basic relationships**:

$$-\sum_{j=1}^{3} \int_\Gamma \left[\left(\frac{\partial u(x)}{\partial x_j} U(x-\xi) - u(x)\frac{\partial U(x-\xi)}{\partial x_j} \right) \cos(\nu, x_j)dx \right] -$$

$$-\sum_{k=1}^{3} \int_D a_k(x)\frac{\partial u(x)}{\partial x_k} U(x-\xi)dx - \int_D a(x)u(x)U(x-\xi)dx-$$

$$-\int_D \left(\int_D \left\{ \sum_{j=1}^{3} K_j(x,\zeta)\frac{\partial u(\zeta)}{\partial \zeta_j} + K(x,\zeta)u(\zeta) \right\} d\zeta \right) U(x-\xi)dx =$$

$$= \int_D u(x)\delta(x-\xi)dx = \begin{cases} u(\xi), & \xi \in D, \\ \frac{1}{2}u(\xi), & \xi \in \Gamma, \end{cases} \tag{5}$$

the second of which (on the boundary Γ) is called the **1-st necessary condition of solvability** of the problem (1)-(2). Taking into account that

$$\frac{\partial U(x-\xi)}{\partial x_j} = -\frac{(x_j - \xi_j)}{4\pi |x-\xi|^3} = -\frac{\cos(x-\xi, x_j)}{4\pi |x-\xi|^2},$$

and, therefore,

$$\sum_{j=1}^{3} \frac{\partial U(x-\xi)}{\partial x_j} \cos(\nu, x_j) = -\frac{\cos(x-\xi, \nu_x)}{4\pi |x-\xi|^2},$$

we get **the 1-st necessary condition** in the form

$$\frac{1}{2}u(\xi)\ |_{\xi\in\Gamma} = \frac{1}{4\pi}\int_\Gamma u(x)\frac{\cos(x-\xi,\nu_x)}{|x-\xi|^2}dx - \frac{1}{4\pi}\int_\Gamma \frac{1}{|x-\xi|}\frac{\partial u(x)}{\partial \nu_x}dx-$$
$$-\int_D \sum_{k=1}^3 a_k(x)\frac{\partial u(x)}{\partial x_k}\frac{1}{4\pi|x-\xi|}dx - \int_D \frac{a(x)u(x)}{4\pi|x-\xi|}dx-$$
$$-\int_D\left(\int_D\left\{\sum_{j=1}^3 K_j(x,\zeta)\frac{\partial u(\zeta)}{\partial \zeta_j}+K(x,\zeta)u(\zeta)\right\}d\zeta\right)\frac{dx}{4\pi|x-\xi|},$$

or, taking into account that the boundary Γ consists of 6 facets Γ_n, $n=\overrightarrow{1,6}$, we have

$$u(\xi)\ |_{\xi\in\Gamma_n} = \frac{1}{2\pi}\int_{\Gamma_n} u(x)\frac{\cos(x-\xi,\nu_x)}{|x-\xi|^2}\Bigg|_{\substack{x\in\Gamma_n,\\ \xi\in\Gamma_n}} dx+$$
$$+\sum_{\substack{g=1,\\ g\neq n}}^6 \frac{1}{2\pi}\int_{\Gamma_g} u(x)\frac{\cos(x-\xi,\nu_x)}{|x-\xi|^2}\Bigg|_{\substack{x\in\Gamma_g,\\ \xi\in\Gamma_n}} dx-$$
$$-\frac{1}{2\pi}\int_\Gamma \frac{1}{|x-\xi|}\frac{\partial u(x)}{\partial \nu_x}dx - \int_D \sum_{k=1}^3 a_k(x)\frac{\partial u(x)}{\partial x_k}\frac{1}{4\pi|x-\xi|}dx - \int_D \frac{a(x)u(x)}{4\pi|x-\xi|}dx-$$

$$-\int_D\left(\int_D\left\{\sum_{j=1}^3 K_j(x,\zeta)\frac{\partial u(\zeta)}{\partial \zeta_j}+K(x,\zeta)u(\zeta)\right\}d\zeta\right)\frac{dx}{4\pi\,|x-\xi|},\ n=\overline{1,6}. \quad (6)$$

Theorem 1. *The first necessary conditions* (6) *of the problem* (1)-(2) *are regular.*

Multiplying (1) by $\frac{\partial U(x-\xi)}{\partial x_i}$ and integrating over the domain D, and also taking the notation

$$K_{ij}(x,\xi) = \left(\cos(x-\xi,x_i)\cos(\nu_x,x_j) - \cos(x-\xi,x_j)\cos(\nu_x,x_i)\right),$$

we obtain **the second basic relationships**:

$$\int_\Gamma \frac{\partial u(x)}{\partial x_m}\frac{K_{im}(x,\xi)}{4\pi\,|x-\xi|^2}dx + \int_\Gamma \frac{\partial u(x)}{\partial x_l}\frac{K_{il}(x,\xi)}{4\pi\,|x-\xi|^2}dx-$$

$$-\int_\Gamma \frac{\partial u(x)}{\partial x_i}\frac{\partial U(x-\xi)}{\partial \nu_x}dx - \int_D \sum_{k=1}^3 a_k(x)\frac{\partial u(x)}{\partial x_k}\frac{\partial U(x-\xi)}{\partial x_i}dx-$$

$$-\int_D a(x)u(x)\frac{\partial U(x-\xi)}{\partial x_i}dx-$$

$$-\int_D\left(\int_D\left\{\sum_{j=1}^{3}K_j(x,\zeta)\frac{\partial u(\zeta)}{\partial\zeta_j}+K(x,\zeta)u(\zeta)\right\}d\zeta\right)U(x-\xi)dx=$$

$$=\begin{cases}\frac{\partial u(\xi)}{\partial\xi_i}, & \xi\in D,\\ \frac{1}{2}\frac{\partial u(\xi)}{\partial\xi_i}, & \xi\in\Gamma,\end{cases}\quad i=\overline{1,3}\,, \tag{7}$$

where $i,\ m,\ l$ are permutations of the numbers 1, 2, 3. The second of the relations (7) on the boundary Γ is called **the second necessary condition**:

$$\frac{1}{2}\frac{\partial u(\xi)}{\partial\xi_i}\,|_{\xi\in\Gamma}=\int_\Gamma\frac{\partial u(x)}{\partial x_m}\frac{K_{im}(x,\xi)}{4\pi\left|x-\xi\right|^2}dx+\int_\Gamma\frac{\partial u(x)}{\partial x_l}\frac{K_{il}(x,\xi)}{4\pi\left|x-\xi\right|^2}dx+$$

$$+\int_\Gamma\frac{\partial u(x)}{\partial x_i}\frac{\cos(x-\xi,\nu_x)}{4\pi\left|x-\xi\right|^2}dx+\int_D\sum_{k=1}^{3}a_k(x)\frac{\partial u(x)}{\partial x_k}\frac{\cos(x-\xi,x_i)}{4\pi\left|x-\xi\right|^2}dx-$$

$$+\int_D a(x)u(x)\frac{\cos(x-\xi,x_i)}{4\pi\left|x-\xi\right|^2}dx-$$

$$-\int_D\left(\int_D\left\{\sum_{j=1}^{3}K_j(x,\zeta)\frac{\partial u(\zeta)}{\partial\zeta_j}+K(x,\zeta)u(\zeta)\right\}d\zeta\right)\frac{1}{4\pi\left|x-\xi\right|}dx. \tag{8}$$

Obviously, the first two terms in (8) are singular, while the rest are weakly singular.

Theorem 2. *The second necessary conditions* (8) *of the problem* (1)-(2) *are singular.*

The boundary of D consists of six faces Γ_n, $n=\overrightarrow{1,6}$. Then, expanding the first two integrals in (8) over Γ_n, $n=\overrightarrow{1,6}$, we isolate the singular terms:

$$\frac{\partial u(\xi)}{\partial\xi_i}\,|_{\xi\in\Gamma_n}=\int_{\Gamma_n}\frac{\partial u(x)}{\partial x_m}\frac{K_{im}(x,\xi)}{2\pi|x-\xi|^2}\Bigg|_{\substack{x\in\Gamma_n,\\ \xi\in\Gamma_n}}dx+\int_{\Gamma_n}\frac{\partial u(x)}{\partial x_l}\frac{K_{il}(x,\xi)}{2\pi|x-\xi|^2}\Bigg|_{\substack{x\in\Gamma_n,\\ \xi\in\Gamma_n}}dx+$$
$$+\sum_{\substack{j=1,\\ j\neq n}}^{6}\left(\int_{\Gamma_j}\frac{\partial u(x)}{\partial x_m}\frac{K_{im}(x,\xi)}{2\pi|x-\xi|^2}dx+\int_{\Gamma_j}\frac{\partial u(x)}{\partial x_l}\frac{K_{il}(x,\xi)}{2\pi|x-\xi|^2}dx\right)+$$
$$+\int_\Gamma\frac{\partial u(x)}{\partial x_i}\frac{\cos(x-\xi,\nu_x)}{2\pi|x-\xi|^2}dx+\int_D\sum_{k=1}^{3}a_k(x)\frac{\partial u(x)}{\partial x_k}\frac{\cos(x-\xi,x_i)}{2\pi|x-\xi|^2}dx+$$
$$+\int_D a(x)u(x)\frac{\cos(x-\xi,x_i)}{2\pi|x-\xi|^2}dx-$$
$$-\int_D\left(\int_D\left\{\sum_{j=1}^{3}K_j(x,\zeta)\frac{\partial u(\zeta)}{\partial\zeta_j}+K(x,\zeta)u(\zeta)\right\}d\zeta\right)\frac{1}{2\pi|x-\xi|}dx,\ n=\overline{1,6}. \tag{9}$$

4. Regularization of necessary conditions

Let's create a linear combination of second necessary conditions (9):

$$\left[\sum_{j=1}^{3}\beta_{kj}^{(1)}(\xi)\frac{\partial u(\xi)}{\partial \xi_j}\right]|_{\xi_k=0}+\left[\sum_{j=1}^{3}\beta_{kj}^{(2)}(\xi)\frac{\partial u(\xi)}{\partial \xi_j}\right]|_{\xi_k=1}\ ,\quad k=\overline{1,\,3}, \tag{10}$$

where the unknown coefficients $\beta_{kj}^{(p)}(\xi)$, $p=1,\,2$, are chosen in order to obtain regular relations.

Remark 1. *To simplify the calculations, we make the following notations:*

$$\begin{array}{l}\beta_{1i}^{(p)}(x)=\beta_{1i}^{(p)}(x_2,\,x_3),\ x\in R^3,\,x_1=p-1,\,p=1,2,\\ \beta_{2i}^{(p)}(x)=\beta_{2i}^{(p)}(x_1,\,x_3),\ x\in R^3,\,x_2=p-1,\,p=1,2,\\ \beta_{3i}^{(p)}(x)=\beta_{3i}^{(p)}(x_1,\,x_2),\ x\in R^3,\,x_3=p-1,\,p=1,\,2.\end{array}$$

By putting $\beta_{kj}^{(p)}(\xi)$, $j=\overline{1,3}$; $p=1,2$, under the integral sign in the first two singular terms in (9) and subtracting and adding $\beta_{kj}^{(p)}(x)$, with the Hölder condition satisfied for $\beta_{kj}^{(p)}(x)$, we obtain weakly singular integrals with the expressions $\frac{\beta_{kj}^{(p)}(\xi)-\beta_{kj}^{(p)}(x)}{|x-\xi|^2}\Big|_{\xi_k=p-1}$. We are now only interested in singular integrals with $\beta_{kj}^{(p)}(x)$ under the integral sign over the face $\Gamma_n(\xi_k=p-1)$, associated with the indices and by the formula $n=2(k-1)+p, k=\overline{1,3}$; $p=1,2$:

$$\begin{aligned}&\left[\sum_{j=1}^{3}\beta_{kj}^{(1)}(\xi)\frac{\partial u(\xi)}{\partial \xi_j}\right]|_{\xi_k=0}+\left[\sum_{j=1}^{3}\beta_{kj}^{(2)}(\xi)\frac{\partial u(\xi)}{\partial \xi_j}\right]|_{\xi_k=1}=\\ &=\sum_{p=1}^{2}\sum_{i=1}^{3}\int\limits_{\Gamma_{2(k-1)+p}}\beta_{ki}^{(p)}(x)\left(\frac{\partial u(x)}{\partial x_m}\frac{K_{im}(x,\xi)}{2\pi|x-\xi|^2}+\frac{\partial u(x)}{\partial x_l}\frac{K_{il}(x,\xi)}{2\pi|x-\xi|^2}\right)|_{\xi_k=p-1}\,dx+\\ &+\sum_{p=1}^{2}\sum_{i=1}^{3}\left(\int\limits_{\Gamma_{2(k-1)+p}}\left[\frac{\beta_{kj}^{(p)}(\xi)-\beta_{kj}^{(p)}(x)}{|x-\xi|^2}\left(\frac{\partial u(x)}{\partial x_m}\frac{K_{im}(x,\xi)}{2\pi}+\frac{\partial u(x)}{\partial x_l}\frac{K_{il}(x,\xi)}{2\pi}\right)\right]|_{\xi_k=p-1}\,dx+\right.\\ &+\beta_{ki}^{(p)}(\xi)\sum_{\substack{j=1,\\ j\neq 2(k-1)+p}}^{6}\left(\int\limits_{\Gamma_j}\frac{\partial u(x)}{\partial x_m}\frac{K_{im}(x,\xi)}{2\pi|x-\xi|^2}dx+\int\limits_{\Gamma_j}\frac{\partial u(x)}{\partial x_l}\frac{K_{il}(x,\xi)}{2\pi|x-\xi|^2}dx\right)|_{\xi_k=p-1}+\end{aligned}$$

$$+\beta_{ki}^{(p)}(\xi)\int\limits_{\Gamma}\frac{\partial u(x)}{\partial x_i}\frac{\cos(x-\xi,\nu_x)}{2\pi|x-\xi|^2}\,|_{\xi_k=p-1}\,dx+$$
$$+\beta_{ki}^{(p)}(\xi)\int\limits_{D}\sum_{j=1}^{3}a_j(x)\frac{\partial u(x)}{\partial x_j}\frac{\cos(x-\xi,x_i)}{2\pi|x-\xi|^2}\,|_{\xi_k=p-1}\,dx+$$
$$+\beta_{ki}^{(p)}(\xi)\int\limits_{D}a(x)u(x)\frac{\cos(x-\xi,x_i)}{2\pi|x-\xi|^2}\,|_{\xi_k=p-1}\,dx-$$
$$-\beta_{ki}^{(p)}(\xi)\int_D\left(\int_D\left\{\sum_{j=1}^{3}K_j(x,\zeta)\frac{\partial u(\zeta)}{\partial\zeta_j}+K(x,\zeta)u(\zeta)\right\}d\zeta\right)\frac{1}{2\pi|x-\xi|}\,|_{\xi_k=p-1}\,dx. \tag{11}$$

Let us group the terms with $\frac{\partial u(x)}{\partial x_i}\,|_{x\in\Gamma_n}$, $n=\overline{1,6}$, in (11) under the integral. Then, equating them to the corresponding coefficients at $\frac{\partial u(x)}{\partial x_i}\,\big|_{x\in\Gamma_p}$ in the boundary condition (2), we obtain a system of equations for the unknowns $\beta_{kj}^{(p)}(x)$, $k,j=\overline{1,3}$; $p=1,\,2$:

$$\begin{cases}\beta_{12}^{(p)}(x_2,x_3)K_{21}^{(p)}(x,\xi)+\beta_{13}^{(p)}(x_2,x_3)K_{31}^{(p)}(x,\xi)=\alpha_{11}^{(p)}(x_2,x_3),\\ \beta_{11}^{(p)}(x_2,x_3)K_{12}^{(p)}(x,\xi)+\beta_{13}^{(p)}(x_2,x_3)K_{32}^{(p)}(x,\xi)=\alpha_{12}^{(p)}(x_2,x_3),\quad p=1,2\\ \beta_{11}^{(p)}(x_2,x_3)K_{13}^{(p)}(x,\xi)+\beta_{12}^{(p)}(x_2,x_3)K_{23}^{(p)}(x,\xi)=\alpha_{13}^{(p)}(x_2,x_3),\end{cases}$$

where $K_{ij}^{(p)}(x,\xi)=K_{ij}(x,\xi)\Bigg|_{\substack{x\in\Gamma_{2(k-1)+p},\\ \xi\in\Gamma_{2(k-1)+p}}}$

$$\begin{cases}\beta_{22}^{(p)}(x_1,x_3)K_{21}^{(p)}(x,\xi)+\beta_{23}^{(p)}(x_1,x_3)K_{31}^{(p)}(x,\xi)=\alpha_{21}^{(p)}(x_1,x_3),\\ \beta_{21}^{(p)}(x_1,x_3)K_{12}^{(p)}(x,\xi)+\beta_{23}^{(p)}(x_1,x_3)K_{32}^{(p)}(x,\xi)=\alpha_{22}^{(p)}(x_1,x_3),\quad p=1,2\\ \beta_{21}^{(p)}(x_1,x_3)K_{13}^{(p)}(x,\xi)+\beta_{22}^{(p)}(x_1,x_3)K_{23}^{(p)}(x,\xi)=\alpha_{23}^{(p)}(x_1,x_3),\end{cases}$$

$$\begin{cases}\beta_{32}^{(p)}(x_1,x_2)K_{21}^{(p)}(x,\xi)+\beta_{33}^{(p)}(x_1,x_2)K_{31}^{(p)}(x,\xi)=\alpha_{31}^{(p)}(x_1,x_2),\\ \beta_{31}^{(p)}(x_1,x_2)K_{12}^{(p)}(x,\xi)+\beta_{33}^{(p)}(x_1,x_2)K_{32}^{(p)}(x,\xi)=\alpha_{32}^{(p)}(x_1,x_2),\quad p=1,2.\\ \beta_{31}^{(p)}(x_1,x_2)K_{13}^{(p)}(x,\xi)+\beta_{32}^{(p)}(x_1,x_2)K_{23}^{(p)}(x,\xi)=\alpha_{33}^{(p)}(x_1,x_2),\end{cases} \tag{12}$$

If system (12) has a unique solution, then, substituting (12) into (11), we obtain:

$$\left[\sum_{j=1}^{3}\beta_{kj}^{(1)}(\xi)\frac{\partial u(\xi)}{\partial\xi_j}\right]|_{\xi_k=0}+\left[\sum_{j=1}^{3}\beta_{kj}^{(2)}(\xi)\frac{\partial u(\xi)}{\partial\xi_j}\right]|_{\xi_k=1}=$$
$$=\sum_{p=1}^{2}\sum_{i=1}^{3}\int\limits_{\Gamma_{2(k-1)+p}}\left(\frac{1}{2\pi|x-\xi|^2}\alpha_{ki}^{(p)}(x)\frac{\partial u(x)}{\partial x_i}\right)\Bigg|_{\substack{\xi_k=p-1,\\ x_k=p-1}}dx+$$

$$+\sum_{p=1}^{2}\sum_{i=1}^{3}\left(\int_{\Gamma_{2(k-1)+p}}\left[\frac{\beta_{kj}^{(p)}(\xi)-\beta_{kj}^{(p)}(x)}{|x-\xi|^2}\left(\frac{\partial u(x)}{\partial x_m}\frac{K_{im}(x,\xi)}{2\pi}+\frac{\partial u(x)}{\partial x_l}\frac{K_{il}(x,\xi)}{2\pi}\right)\right]|_{\xi_k=p-1}\,dx+\right.$$
$$+\beta_{ki}^{(p)}(\xi)\sum_{\substack{j=1,\\ j\neq 2(k-1)+p}}^{6}\left(\int_{\Gamma_j}\frac{\partial u(x)}{\partial x_m}\frac{K_{im}(x,\xi)}{2\pi|x-\xi|^2}dx+\int_{\Gamma_j}\frac{\partial u(x)}{\partial x_l}\frac{K_{il}(x,\xi)}{2\pi|x-\xi|^2}dx\right)|_{\xi_k=p-1}+$$
$$+\beta_{ki}^{(p)}(\xi)\int_{\Gamma}\frac{\partial u(x)}{\partial x_i}\frac{\cos(x-\xi,\nu_x)}{2\pi|x-\xi|^2}|_{\xi_k=p-1}\,dx+$$
$$+\beta_{ki}^{(p)}(\xi)\int_{D}\sum_{j=1}^{3}a_j(x)\frac{\partial u(x)}{\partial x_j}\frac{\cos(x-\xi,x_i)}{2\pi|x-\xi|^2}|_{\xi_k=p-1}\,dx+$$
$$+\beta_{ki}^{(p)}(\xi)\int_{D}a(x)u(x)\frac{\cos(x-\xi,x_i)}{2\pi|x-\xi|^2}|_{\xi_k=p-1}\,dx-$$
$$-\beta_{ki}^{(p)}(\xi)\int_{D}\left(\int_{D}\left\{\sum_{j=1}^{3}K_j(x,\zeta)\frac{\partial u(\zeta)}{\partial\zeta_j}+K(x,\zeta)u(\zeta)\right\}d\zeta\right)\frac{1}{2\pi|x-\xi|}|_{\xi_k=p-1}\,dx,\ k=\overline{1,3} \tag{13}$$

Expressing $\sum_{p=1}^{2}\sum_{i=1}^{3}\alpha_{ki}^{(p)}(x)\frac{\partial u(x)}{\partial x_i}|_{\xi_k=p-1}$ in (13), from the boundary conditions (2) we obtain:

$$\left[\sum_{j=1}^{3}\beta_{kj}^{(1)}(\xi)\frac{\partial u(\xi)}{\partial\xi_j}\right]|_{\xi_k=0}+\left[\sum_{j=1}^{3}\beta_{kj}^{(2)}(\xi)\frac{\partial u(\xi)}{\partial\xi_j}\right]|_{\xi_k=1}=$$
$$=\sum_{p=1}^{2}\int_{\Gamma_{p+k}}\frac{\varphi_k(x)}{2\pi|x-\xi|^2}\Bigg|_{\substack{\xi_k=p-1,\\ x_k=p-1}}dx-\sum_{p=1}^{2}\left\{\int_{\Gamma_{p+k}}\left(\frac{1}{2\pi|x-\xi|^2}\alpha_k^{(p)}(x)u(x)\right)\Bigg|_{\substack{\xi_k=p-1,\\ x_k=p-1}}dx+\right.$$

$$+\sum_{p=1}^{2}\sum_{i=1}^{3}\left(\int_{\Gamma_{2(k-1)+p}}\left[\frac{\beta_{kj}^{(p)}(\xi)-\beta_{kj}^{(p)}(x)}{|x-\xi|^2}\left(\frac{\partial u(x)}{\partial x_m}\frac{K_{im}(x,\xi)}{2\pi}+\frac{\partial u(x)}{\partial x_l}\frac{K_{il}(x,\xi)}{2\pi}\right)\right]|_{\xi_k=p-1}\,dx+\right.$$
$$+\beta_{ki}^{(p)}(\xi)\sum_{\substack{j=1,\\ j\neq 2(k-1)+p}}^{6}\left(\int_{\Gamma_j}\frac{\partial u(x)}{\partial x_m}\frac{K_{im}(x,\xi)}{2\pi|x-\xi|^2}dx+\int_{\Gamma_j}\frac{\partial u(x)}{\partial x_l}\frac{K_{il}(x,\xi)}{2\pi|x-\xi|^2}dx\right)|_{\xi_k=p-1}+$$
$$+\beta_{ki}^{(p)}(\xi)\int_{\Gamma}\frac{\partial u(x)}{\partial x_i}\frac{\cos(x-\xi,\nu_x)}{2\pi|x-\xi|^2}|_{\xi_k=p-1}\,dx+\beta_{ki}^{(p)}(\xi)\int_{D}\sum_{j=1}^{3}a_j(x)\frac{\partial u(x)}{\partial x_j}\frac{\cos(x-\xi,x_i)}{2\pi|x-\xi|^2}|_{\xi_k=p-1}\,dx+$$
$$+\beta_{ki}^{(p)}(\xi)\int_{D}a(x)u(x)\frac{\cos(x-\xi,x_i)}{2\pi|x-\xi|^2}|_{\xi_k=p-1}\,dx-$$
$$-\beta_{ki}^{(p)}(\xi)\int_{D}\left(\int_{D}\left\{\sum_{j=1}^{3}K_j(x,\zeta)\frac{\partial u(\zeta)}{\partial\zeta_j}+K(x,\zeta)u(\zeta)\right\}d\zeta\right)\frac{1}{2\pi|x-\xi|}|_{\xi_k=p-1}\,dx \tag{14}$$

where the functions $\alpha_k^{(p)}(x)$ and $\varphi_k(x)$, $k=\overline{1,3}$, are defined as follows:

$$\alpha_1^{(p)}(x)=\alpha_1^{(p)}(x_2,\,x_3),\ \alpha_2^{(p)}(x)=\alpha_2^{(p)}(x_1,\,x_3),\ \alpha_3^{(p)}(x)=\alpha_1^{(p)}(x_1,\,x_2),$$

$$\varphi_1(x) = \varphi_1(x_2,\, x_3),\ \varphi_2(x) = \varphi_2(x_1,\, x_3),\ \varphi_3(x) = \varphi_3(x_1,\, x_2).$$

$$S_1 = \{(x_2,\, x_3) : 0 < x_2 < 1,\ 0 < x_3 < 1\},$$

$$S_2 = \{(x_1,\, x_3) : 0 < x_1 < 1,\ 0 < x_3 < 1\},$$

$$S_3 = \{(x_1,\, x_2) : 0 < x_1 < 1,\ 0 < x_2 < 1\}. \tag{15}$$

Assuming that the functions $\varphi_k(x)$, $k = \overline{1,3}$, satisfy the Hölder condition, we obtain the convergence of the first integral on the right-hand side of (14) in the sense of Cauchy. Substituting the first necessary condition (6) into the second term on the right-hand side of (14), we obtain:

$$\begin{aligned}
&\int\limits_{\Gamma_{2(k-1)+p}} \left(\frac{1}{2\pi|x-\xi|^2}\alpha_k^{(p)}(x)u(x)\right)\Bigg|_{\substack{\xi_k\in\Gamma_{p+k},\\ x_k\in\Gamma_{p+k}}} dx = \\
&= \int\limits_{\Gamma_{2(k-1)+p}} \frac{1}{2\pi|x-\xi|^2}\alpha_k^{(p)}(x)\left\{\frac{1}{2\pi}\int\limits_{\Gamma_{2(k-1)+p}} u(\zeta)\frac{\cos(\zeta-x,\nu_\zeta)}{|\zeta-\xi|^2}\Bigg|_{\substack{\zeta\in\Gamma_{p+k},\\ x\in\Gamma_{p+k}}} d\zeta + \right. \\
&+ \sum_{\substack{g=1,\\ g\neq 2(k-1)+p}}^{6} \frac{1}{2\pi}\int\limits_{\Gamma_g} u(\zeta)\frac{\cos(\zeta-x,\nu_\zeta)}{|\zeta-x|^2}\Bigg|_{\substack{\zeta\in\Gamma_g,\\ x\in\Gamma_{2(k-1)+p}}} d\zeta - \frac{1}{2\pi}\int\limits_{\Gamma}\frac{1}{|\zeta-x|}\frac{\partial u(\zeta)}{\partial\nu_\zeta}d\zeta - \\
&\left. -\int\limits_D \sum_{k=1}^{3} a_k(\zeta)\frac{\partial u(\zeta)}{\partial\zeta_k}\frac{1}{4\pi|\zeta-x|}d\zeta - \int\limits_D \frac{a(\zeta)u(\zeta)}{4\pi|\zeta-x|}d\zeta\right\}\Bigg|_{\substack{\xi\in\Gamma_{2(k-1)+p},\\ x\in\Gamma_{2(k-1)+p}}} dx,\ k = \overline{1,3}
\end{aligned} \tag{16}$$

Changing the order of integration in the first term on the right-hand side of (16), we obtain:

$$\begin{aligned}
&\int\limits_{\Gamma_{2(k-1)+p}} \frac{dx}{2\pi|x-\xi|^2}\alpha_k^{(p)}(x)\Bigg|_{\substack{\xi\in\Gamma_{2(k-1)+p},\\ x\in\Gamma_{2(k-1)+p}}} \frac{1}{2\pi}\int\limits_{\Gamma_{2(k-1)+p}} u(\zeta)\,\frac{\cos(\zeta-x,\nu_\zeta)}{|\zeta-\xi|^2}\Bigg|_{\substack{\zeta\in\Gamma_{2(k-1)+p},\\ x\in\Gamma_{2(k-1)+p}}} d\zeta = \\
&= \int\limits_{\Gamma_{2(k-1)+p}} \frac{1}{2\pi}u(\zeta)d\zeta \int\limits_{\Gamma_{2(k-1)+p}} \alpha_k^{(p)}(x)\frac{1}{2\pi}\frac{\cos(\zeta-x,\nu_\zeta)}{|\zeta-\xi|^2|x-\xi|^2}\Bigg|_{\substack{\zeta\in\Gamma_{2(k-1)+p},\\ x\in\Gamma_{2(k-1)+p},\\ \xi\in\Gamma_{2(k-1)+p}.}} dx,\ k = \overline{1,3},
\end{aligned} \tag{17}$$

Taking into account that $\lim\limits_{x\to\zeta}\cos(\zeta - x, \nu_\zeta) = 0$, as well as the fact that the faces of the cube D are the Lyapunov boundary since the extreme position of the vector $\zeta - x$ at $x \to \zeta$ with respect to the normal ν_ζ is perpendicular, the inner integral in (17) has a singularity of order $2 - \alpha$ that is less than the order of the outer integral: $2 - \alpha < 2$, which ensures convergence of the integral (17). Thus, taking into account (16) and (17) in (14), we obtain the regularized relations:

$$\left[\sum_{j=1}^{3}\beta_{kj}^{(1)}(\xi)\frac{\partial u(\xi)}{\partial\xi_j}\right]|_{\xi_k=0}+\left[\sum_{j=1}^{3}\beta_{kj}^{(2)}(\xi)\frac{\partial u(\xi)}{\partial\xi_j}\right]|_{\xi_k=1}=$$
$$=\sum_{p=1}^{2}\int\limits_{\Gamma_{2(k-1)+p}}\frac{\varphi_k(x)}{2\pi|x-\xi|^2}\Bigg|_{\substack{\xi_k=p-1,\\ x_k=p-1}}dx-$$
$$-\sum_{p=1}^{2}\left(\int\limits_{\Gamma_{2(k-1)+p}}\frac{1}{2\pi}u(\zeta)d\zeta\int\limits_{\Gamma_{2(k-1)+p}}\alpha_k^{(p)}(x)\frac{1}{2\pi}\frac{\cos(\zeta-x,\nu_\zeta)}{|\zeta-\xi|^2|x-\xi|^2}\right|_{\substack{\zeta\in\Gamma_{2(k-1)+p},\\ x\in\Gamma_{2(k-1)+p},\\ \xi\in\Gamma_{2(k-1)+p}.}}dx+$$

$$+\int\limits_{\Gamma_{2(k-1)+p}}\frac{1}{2\pi|x-\xi|^2}\alpha_k^{(p)}(x)\left(\sum_{\substack{g=1,\\ g\neq 2(k-1)+p}}\frac{1}{2\pi}\int\limits_{\Gamma_g}u(\zeta)\frac{\cos(\zeta-x,\nu_\zeta)}{|\zeta-x|^2}\right|_{\substack{\zeta\in\Gamma_g,\\ x\in\Gamma_{2(k-1)+p}}}d\zeta-$$
$$\left.\frac{1}{2\pi}\int\limits_{\Gamma}\frac{1}{|\zeta-x|}\frac{\partial u(\zeta)}{\partial\nu_\zeta}d\zeta-\int\limits_{D}\sum_{k=1}^{3}a_k(\zeta)\frac{\partial u(\zeta)}{\partial\zeta_k}\frac{1}{4\pi|\zeta-x|}d\zeta-\int\limits_{D}\frac{a(\zeta)u(\zeta)}{4\pi|\zeta-x|}d\zeta\right\}\Bigg|_{\substack{\xi\in\Gamma_{2(k-1)+p},\\ x\in\Gamma_{2(k-1)+p}}}\Bigg)dx+$$
$$+\sum_{p=1}^{2}\sum_{i=1}^{3}\Bigg\{\int\limits_{\Gamma_{2(k-1)+p}}\left[\frac{\beta_{kj}^{(p)}(\xi)-\beta_{kj}^{(p)}(x)}{|x-\xi|^2}\left(\frac{\partial u(x)}{\partial x_m}\frac{K_{im}(x,\xi)}{2\pi}+\frac{\partial u(x)}{\partial x_l}\frac{K_{il}(x,\xi)}{2\pi}\right)\right]|_{\xi_k=p-1}dx+$$
$$+\beta_{ki}^{(p)}(\xi)\sum_{\substack{j=1,\\ j\neq 2(k-1)+p}}^{6}\left(\int\limits_{\Gamma_j}\frac{\partial u(x)}{\partial x_m}\frac{K_{im}(x,\xi)}{2\pi|x-\xi|^2}dx+\int\limits_{\Gamma_j}\frac{\partial u(x)}{\partial x_l}\frac{K_{il}(x,\xi)}{2\pi|x-\xi|^2}dx\right)|_{\xi_k=p-1}+$$
$$+\beta_{ki}^{(p)}(\xi)\int\limits_{\Gamma}\frac{\partial u(x)}{\partial x_i}\frac{\cos(x-\xi,\nu_x)}{2\pi|x-\xi|^2}|_{\xi_k=p-1}dx+$$
$$+\beta_{ki}^{(p)}(\xi)\int\limits_{D}\sum_{j=1}^{3}a_j(x)\frac{\partial u(x)}{\partial x_j}\frac{\cos(x-\xi,x_i)}{2\pi|x-\xi|^2}|_{\xi_k=p-1}dx+$$
$$\beta_{ki}^{(p)}(\xi)\int\limits_{\Gamma}\frac{\partial u(x)}{\partial x_i}\frac{\cos(x-\xi,\nu_x)}{2\pi|x-\xi|^2}|_{\xi_k=p-1}dx+$$
$$+\beta_{ki}^{(p)}(\xi)\int\limits_{D}\sum_{j=1}^{3}a_j(x)\frac{\partial u(x)}{\partial x_j}\frac{\cos(x-\xi,x_i)}{2\pi|x-\xi|^2}|_{\xi_k=p-1}dx+$$
$$\beta_{ki}^{(p)}(\xi)\int\limits_{D}a(x)u(x)\frac{\cos(x-\xi,x_i)}{2\pi|x-\xi|^2}|_{\xi_k=p-1}dx-$$
$$+\beta_{ki}^{(p)}(\xi)\int\limits_{D}a(x)u(x)\frac{\cos(x-\xi,x_i)}{2\pi|x-\xi|^2}|_{\xi_k=p-1}dx-$$
$$-\beta_{ki}^{(p)}(\xi)\int\limits_{D}\left(\int\limits_{D}\left\{\sum_{j=1}^{3}K_j(x,\zeta)\frac{\partial u(\zeta)}{\partial\zeta_j}+K(x,\zeta)u(\zeta)\right\}d\zeta\right)\frac{1}{2\pi|x-\xi|}|_{\xi_k=p-1}dx,k=\overline{1,3} \tag{18}$$

Remark 2. *Obviously, the right-hand sides of the regular relations* (18) *contain the values of the desired function and its partial derivatives under the integral both* *over the boundary* Γ *and over the domain* D*, which will be taken into account when proving the Fredholm property of our problem.*

Obviously, if $\varphi_i(x)$, $i=\overline{1,3}$, in the boundary conditions (2) are continuously differentiable in the domains S_i (15) and are equal to zero on the boundary of S_i, then the integrals $\int_{\Gamma_{2(k-1)+p}} \frac{\varphi_k(x)}{2\pi|x-\xi|^2}\Bigg|_{\substack{\xi_k=p-1,\\ x_k=p-1}} dx$ converge in the usual sense.

Thus, the following assertion is proven:

Theorem 3. *If* $\alpha_{ij}^{(p)}(x)$, $i,j=\overline{1,3}$, $p=1,2$; $x\in\Gamma$*, belong to the Hölder class with index* $\mu>0$ *and* $\alpha_i^{(p)}(x)\in C(S_i)$, $i=\overline{1,3}$, $p=1,2$*, and* $\varphi_i(x)\in C^{(1)}(S_i)$, $\varphi_i(x)|_{\partial S_i}=0$, $i=\overline{1,3}$*, then, under the conditions of Theorems 2 and 3, the relations* (18) *are regular.*

5. Fredholm property

There are 24 unknowns: $u(\xi)\,|_{\xi\in\Gamma_n}, \frac{\partial u(\xi)}{\partial\xi_i}\,|_{\xi\in\Gamma_n}$, $i=\overline{1,3}\,, n=\overline{1,6}$, in problem (1)-(2). From the theory of analysis it is known that

$$\frac{\partial u(\xi)}{\partial\xi_1}\,|_{\xi_2=0}=\lim_{\xi_2\to0}\frac{\partial u(\xi)}{\partial\xi_1}=\frac{\partial}{\partial\xi_1}\lim_{\xi_2\to0}u(\xi)=\frac{\partial u(\xi_1,\,0,\,\xi_3)}{\partial\xi_1}=\frac{\partial u(\xi)\,|_{\xi_2=0}}{\partial\xi_1}.$$

Similarly we obtain:

$$\frac{\partial u(\xi)}{\partial\xi_i}\,|_{\xi_k=p-1}=\frac{\partial u(\xi)\,|_{\xi_k=p}}{\partial\xi_i},\ i,k=\overline{1,3},\ k\neq i,\ p=0,\,1. \tag{19}$$

Therefore, 12 unknowns $\frac{\partial u(\xi)}{\partial\xi_i}\,|_{\xi_k=p}$, $i,k=\overline{1,3}$, $k\neq i$, $p=0,1$, are expressed through the derivatives of the boundary values of the desired function $u(\xi)\,|_{\xi_k=p}$, $k=\overline{1,3}$, $p=0,1$. Then we have only 12 unknowns: $u(\xi)\,|_{\xi_i=p}$, $\frac{\partial u(\xi)}{\partial\xi_i}\,|_{\xi_i=p}$, $i=\overline{1,3}$, $p=0,1$, and 12 equations consisting of boundary conditions (2), necessary conditions (6) and regular relations (18). The resulting system (2), (6), (18) is a system of Fredholm integro-differential equations of the 2nd kind.

The solution $u(\xi)\,|_{\xi_i=p}$, $\frac{\partial u(\xi)}{\partial\xi_i}\,|_{\xi_i=p}$, $i=\overline{1,3}$, $p=0,1$, to the system (2), (6), (18), and, therefore, according to (19), all the 24 boundary values $u(\xi)\,|_{\xi_i=p}$, $\frac{\partial u(\xi)}{\partial\xi_j}\,|_{\xi_i=p}$, $i,j=\overline{1,3}$, $p=0,1$, are expressed through integrals over the domain D which are present in (18):

$$u(\xi)\,|_{\xi_i=p}=\int_D a_{ip}(\zeta)u(\zeta)d\zeta+\sum_{m=1}^{3}\int_D b_{ipm}(\zeta)\frac{\partial u(\zeta)}{\partial\zeta_m}d\zeta,$$

$$\frac{\partial u(\xi)}{\partial \xi_j} |_{\xi_i = p} = \int_D c_{jip}(\zeta) u(\zeta) d\zeta + \sum_{m=1}^{3} \int_D d_{jipm}(\zeta) \frac{\partial u(\zeta)}{\partial \zeta_m} d\zeta,$$

$$i, j = \overline{1,3}, \; p = 0, 1. \tag{20}$$

Substituting (20) into the 1st basic relationship (5) and the three 2nd basic relationships (7), we obtain a system of Fredholm equations of the 2nd kind over the domain D with respect to $u(\xi)$, $\frac{\partial u(\xi)}{\partial \xi_j}$, $j = \overline{1,3}$, with regular kernels. So we have proved

Theorem 4. *If the conditions of Theorem 3 are satisfied, the problem* (1)-(2) *has a Fredholm property.*

References

[1] A.A. Dezin, *Multidimensional analysis and discrete models*, Nauka, Moscow, 1990.

[2] A.A. Dezin, *General questions of the theory of boundary value problems*, Nauka, Moscow, 1980.

[3] A.A. Dezin, *Operators with first order derivatives with respect to "time" and nonlocal boundary conditions*, Mathematics of the USSR-Izvestiya, **31(1)**, 1967, 61-86.

[4] A.V. Bitsadze, *Boundary value problems for elliptic equations of second order*, Nauka, Moscow, 1966.

[5] H. Begehr, Y. Wang, *A Dirichlet problem for polyharmonic functions*, Ann Mat. Pura Appl., **187**, 2008, 435-437.

[6] H. Begehr, *Basic Boundary Value Problems for the Cauchy-Riemann and Poisson Equation in a Quarter Disc*, Master thesis, Bucharest, Sch. Normala, Dep. of math., 2009.

[7] Y.Y. Mustafayeva, N.A. Aliyev, *New Method of Solvability of a Three-dimensional Laplace Equation with Nonlocal Boundary Conditions*, Journal of Mathematical Physics, Analysis, Geometry, **12(3)**, 2016, 185-204.

[8] Y.Y. Mustafayeva, N.A. Aliyev, *On one method of investigation of Steklov problem for 3-dimensional Laplace equation with nonlocal boundary conditions*, J. of Tomsk State University, **6(44)**, 2016, 19-33.

[9] Y.Y. Mustafayeva, N.A. Aliyev, *New method of investigation of the Fredholm property of three-dimensional Helmholtz equation with nonlocal boundary value conditions*, Filomat, **32(12)**, 2018, 4445-4460.

[10] V.S. Vladimirov, *Equations of Mathematical Physics*, Mir, Moscow, 1981.

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Received 10 July 2023
Accepted 25 September 2024