

On the Completeness of Classes of Correcting Functions of Heuristic Algorithms

A. Kabulov, I. Normatov, I. Saymanov*, A. Baizhumanov

Abstract. We study classes of monotone k -valued corrective functions that preserve values when heuristic algorithms make the correct decision, defined on a certain subset of sets of n -dimensional k -valued lattice S^n and corrective sets of heuristic algorithms for solving problems from poorly formalized areas. To completely solve the problem of constructing the simplest continuation in classes of corrective functions, theorems on the completeness of classes of monotone k -valued functions and functions preserving sets of S^n for $k = 3$ are proved, and also bases are created in the given classes.

Key Words and Phrases: k -valued, corrective function, heuristic algorithm, completeness, monotone.

2010 Mathematics Subject Classifications: 03B15, 03B50

1. Introduction

The use of mathematical methods in poorly formalized areas has led to the emergence of a large number of heuristic methods for studying algorithms [1, 2]. Such methods and algorithms are formed on the basis of substantive principles and then written in the language accepted in mathematics. At the same time, there is no mathematical justification for the proposed methods, and their verification is carried out by solving a sufficient number of control problems [3, 4]. Since the number of incorrect decisions is relatively small, methods called heuristic or incorrect are accepted and used. Their wide distribution is explained, firstly, by the fact that in many areas of science and practice (for example, geological forecasting) accuracy close to absolute is not required; secondly, a rigorous justification at the modern mathematical level requires the construction of formal models in the field under study (which is often impossible) or the imposition

*Corresponding author.

of strong restrictions, which significantly reduces the value of the method (in practice, such restrictions are usually not met) [5, 6]. The largest number of heuristic methods have been proposed in the study of applied problems such as the methods of forecasting, recognition, classification, and searching for absolute extrema of a function of many variables [7, 8]. Heuristic methods have significant drawbacks. When solving new problems, it is a priori unknown what accuracy such heuristics will provide. Therefore, the problem of increasing their accuracy is of particular importance [9].

When solving a wide class of practical problems, algorithms are often used that allow errors in the calculation of elementary properties or failures to solve problems. In such cases, several incorrect algorithms are usually used to solve the same problem, and then a corrective function is constructed [10, 11]. Since the result of calculating an elementary property can be either 0 - refusal to calculate, or 1 - the property is fulfilled, or 2 - the property is not fulfilled, the corrective function is a function of three-valued logic. For substantive reasons, not many corrective functions should be subject to restrictions [12, 13].

In this paper, special classes of correction functions are considered. Theorems on the completeness of classes of monotone k -valued functions and functions preserving sets of S^n for $k = 3$ are proved, and bases in the given classes are also created [14, 15].

Definition 1. *A family of functions preserving the set L^* , is called the class σ_1 .*

It is easy to see that the class σ_i is functionally closed.

It is natural to take only functions from σ_1 as correcting functions. Indeed, let, for example, when calculating the property $P(S)$ of object S according to the algorithms, A_1, A_2, \dots, A_m , the values $\beta_1, \beta_2, \dots, \beta_m$ be formed on the set $0, 2$. This means that some algorithms "refused" to calculate $P(S)$ and with the help of others it was established that $P(S)$ was not executed [16, 17]. Then it is reasonable to assume that $f(\beta_1, \beta_2, \dots, \beta_m) \neq 1$ [18, 19].

Let the set $\{0, 1, 2\}$ have a partial order

$$0 < 1, \quad 0 < 2. \quad (1)$$

This order induces a partial order in the set of sets

$$\begin{aligned} \tilde{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n), \quad \alpha_i \in \{0, 1, 2\} \quad (i = \overline{1, n}) : \dots \\ \tilde{\beta} = (\beta_1, \beta_2, \dots, \beta_n) \leq \tilde{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_n), \dots \\ \text{if } \alpha_i \leq \beta_i \quad (i = \overline{1, n}). \end{aligned} \quad (2)$$

Definition 2. *The class σ_2 of functions f of three-valued logic is called monotone in order (2) (for brevity we will call σ_2 the class of monotone functions) if from condition $\beta \leq \alpha$ by (2) it follows that $f(\tilde{\beta}) \leq f(\tilde{\alpha})$ according to the order (1).*

The class of L^* preserving and monotone functions is denoted below by σ_3 .

It is obvious that $\sigma_3 = \sigma_1 \cap \sigma_2$, and in this article we prove that the classes $\sigma_1, \sigma_2, \sigma_3$ are functionally closed.

In many applied problems, it is natural to consider correction functions only from σ_2 or σ_3 .

In practice, we usually deal with correction functions defined on some subset of sets of the structure S^n . Consequently, it is often necessary to solve the problem of finding the simplest extension of corrective functions from this subset to S^n so that the restrictions imposed on the function are not violated [19]. To solve this problem, it is advisable to introduce bases in the classes σ_1 and σ_3 . Next, for each function from these classes, it is necessary to construct its expression in terms of basis functions (canonical form), which is an analogue of the d.n.f. Boolean functions. Having constructed an algorithm for finding the minimal canonical forms of functions of the classes σ_2 and σ_3 , it is necessary to move on to the problem of the simplest continuation of functions that are not defined everywhere.

In this article, to completely solve the problem of constructing the simplest continuation in the classes σ_2 and σ_3 , theorems on the completeness of the classes σ_1 and σ_3 are proved for $k = 3$ and bases are constructed in the given classes.

2. Bases in the classes σ_1 and σ_3 for $k = 3$

Bases in the classes σ_1 and σ_3 are constructed. For each function from these classes, an expression is found in terms of the basis functions.

2.1. Basis in class σ_1

Let

$$\mathfrak{M}_1 = [(\max(x_1, x_2), \min(x_1, x_2), m_{11}(x_1, x_2), m_{100}(x_1, x_2, x_3), m_{200}(x_1, x_2, x_3), m_{201}(x_1, x_2, x_3)]$$

be a system of functions in the class σ_1 , where

$$m_{11}(x_1, x_2) = \begin{cases} \alpha, & \text{if } x_1 = x_2 = \alpha, \alpha \in [0, 1, 2], \\ 0, & \text{in other cases;} \end{cases}$$

$$m_{100}(x_1, x_2, x_3) = \begin{cases} \alpha, & \text{if } x_1 = x_2 = x_3 = \alpha, \\ 1, & \text{if } x_1 = 1, x_2 = x_3 = 0, \\ 0, & \text{in other cases;} \end{cases}$$

$$m_{200}(x_1, x_2, x_3) = \begin{cases} \alpha, & \text{if } x_1 = x_2 = x_3 = \alpha, \\ 2, & \text{if } x_1 = 2, \quad x_2 = x_3 = 0, \\ 0, & \text{in other cases;} \end{cases}$$

$$m_{201}(x_1, x_2, x_3) = \begin{cases} \alpha, & \text{if } x_1 = x_2 = x_3 = \alpha, \\ 2, & \text{if } x_1 = 2, \quad x_2 = 0, \quad x_3 = 1, \\ 0, & \text{in other cases.} \end{cases}$$

Further, based on the works by S.V. Yablonsky [22, p. 3-10], we will denote the functions $\min(x_1, x_2)$ and $\max(x_1, x_2)$ by $(x_1 \wedge x_2)$ and $(x_1 \vee x_2)$, respectively and we will use expressions obtained by omitting some parentheses to write formulas.

Theorem 1. *The system of functions \mathfrak{M}_1 is complete in the class σ_1 .*

Proof. Let an arbitrary function $f(x_1, x_2, \dots, x_n)$ of class σ_1 be specified by the table. Let us construct a formula f^* from functions of the system \mathfrak{M}_1 in such a way that the function corresponding to f^* was identical to the function $f(x_1, x_2, \dots, x_n)$.

For each set $\tilde{\alpha} \in E^n(0, 1)$, where $\tilde{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$, such that $f(\tilde{\alpha}) = 1$ and $\alpha_{i_j} = \begin{cases} 1, & \text{if } j = 1, 2, \dots, t \\ 0, & \text{if } j = t + 1, 2, \dots, n \end{cases}$ $0 < t < n$, let's write out the formula.

$$U_{\tilde{\alpha}}(\tilde{x}) = m_{11}(x_{i_1}, x_{i_2}) \cdot \dots \cdot m_{11}(x_{i_1}, x_{i_t}) m_{11}(x_{i_1}, x_{i_1}) \times \\ \times m_{100}(x_{i_1}, x_{i_{t+1}}, x_{i_{t+2}}) \cdot \dots \cdot m_{100}(x_{i_1}, x_{i_{t+1}}, x_{i_n}) \times \\ \times m_{100}(x_{i_1}, x_{i_{t+1}}, x_{i_{t+1}}).$$

$$\text{It's obvious that } U_{\tilde{\alpha}}(\tilde{x}) = \begin{cases} 1, & \text{if } (\tilde{x} = \tilde{\alpha}) \vee (\tilde{x} = \tilde{1}), \\ 2, & \text{if } \tilde{x} = 2, \\ 0, & \text{in other cases.} \end{cases}$$

For each set $\tilde{\beta} \in E^n(0, 2)$, where $\tilde{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$, such that $f(\tilde{\beta}) = 2$ and $\beta_{i_j} = \begin{cases} 2, & \text{if } j = 1, 2, \dots, h, \\ 0, & \text{if } j = h + 1, 2, \dots, n, \end{cases}$ $0 < h < n$, let's write out the formula:

$$U_{\tilde{\beta}}(\tilde{x}) = m_{11}(x_{i_1}, x_{i_2}) \cdot \dots \cdot m_{11}(x_{i_1}, x_{i_h}) m_{11}(x_{i_1}, x_{i_1}) \times \\ \times m_{200}(x_{i_1}, x_{i_{h+1}}, x_{i_{h+2}}) \cdot \dots \cdot m_{200}(x_{i_1}, x_{i_{h+1}}, x_{i_n}) \times \\ \times m_{200}(x_{i_1}, x_{i_{h+1}}, x_{i_{h+1}}).$$

$$\text{It's easy to see that } U_{\tilde{\beta}}(\tilde{x}) = \begin{cases} 2, & \text{if } (\tilde{x} = \tilde{\beta}) \vee (\tilde{x} = \tilde{2}), \\ 1, & \text{if } \tilde{x} = \tilde{1}, \\ 0, & \text{in other cases.} \end{cases}$$

For all sets $\tilde{\gamma}' \in S_1^n$ and $\tilde{\gamma}' = (\gamma'_1, \gamma'_2, \dots, \gamma'_n)$ such that $f(\tilde{\gamma}') = 2$ and

$$\gamma_{ij} = \begin{cases} 1, & \text{if } j = 1, 2, \dots, h, \\ 0, & \text{if } j = h + 1, \dots, l, \quad 0 < h < l < n, \\ 2, & \text{if } j = l + 1, \dots, n, \end{cases}$$

write out a formula:

$$U_{\tilde{\gamma}}(\tilde{x}) = m_{11}(x_{i_1}, x_{i_2}) \cdot \dots \cdot m_{11}(x_{i_1}, x_{i_h}) m_{11}(x_{i_1}, x_{i_1}) \times \\ \times m_{100}(x_{i_1}, x_{i_{h+1}}, x_{i_{h+2}}) \cdot \dots \cdot m_{100}(x_{i_1}, x_{i_{h+1}}, x_{i_l}) \times \\ \times m_{100}(x_{i_1}, x_{i_{h+1}}, x_{i_{h+1}}) m_{11}(x_{i_{l+1}}, x_{i_{l+2}}) \cdot \dots \cdot m_{11}(x_{i_{l+1}}, x_{i_n}) \times \\ \times m_{200}(x_{i_{l+1}}, x_{i_{l+1}}, x_{i_{h+1}}) m_{11}(x_{i_{l+1}}, x_{i_{l+1}}).$$

It's easy to see that $U_{\tilde{\gamma}}(\tilde{x}) = \begin{cases} 1, & \text{if } (\tilde{x} = \tilde{\gamma}) \vee (\tilde{x} = \tilde{1}), \\ 2, & \text{if } \tilde{x} = \tilde{2}, \\ 0, & \text{in other cases.} \end{cases}$

For all sets $\tilde{\gamma}' \in S_1^n$ and $\tilde{\gamma}' = \gamma'_1, \gamma'_2, \dots, \gamma'_n$ such that $f(\tilde{\gamma}') = 2$ and

$$\gamma'_{ij} = \begin{cases} 1, & \text{if } j = 1, 2, \dots, p, \\ 0, & \text{if } j = p + 1, \dots, q, \quad 0 < p < q < n, \\ 2, & \text{if } j = q + 1, \dots, n, \end{cases}$$

write out a formula:

$$U_{\tilde{\gamma}}(\tilde{x}) = m_{201}(U_1, x_{i_{p+1}}, U_2) \cdot \dots \cdot m_{201}(U_1, x_{i_q}, U_2) \times \\ \times m_{200}(x_{i_{q+1}}, x_{i_{p+1}}, x_{i_{p+2}}) \cdot \dots \cdot m_{200}(x_{i_{q+2}}, x_{i_{p+1}}, x_{i_q}),$$

where

$$U_1 = m_{11}(x_{i_{q+1}}, x_{i_{q+2}}) \cdot \dots \cdot m_{11}(x_{i_{q+1}}, x_{i_n}) m_{11}(x_{i_{q+1}}, x_{i_{q+1}}), \\ U_2 = m_{11}(x_{i_1}, x_{i_2}) \cdot \dots \cdot m_{11}(x_{i_1}, x_{i_p}) m_{11}(x_{i_1}, x_{i_1}).$$

It's obvious that

$$U_{\tilde{\gamma}'}(\tilde{x}) = \begin{cases} 2, & \text{if } (\tilde{x} = \tilde{\gamma}') \vee (\tilde{x} = \tilde{2}), \\ 1, & \text{if } \tilde{x} = \tilde{1}, \\ 0, & \text{in other cases.} \end{cases}$$

For all sets $\tilde{\delta} = (\delta_1, \delta_2, \dots, \delta_n)$, all coordinates of which belong to the set $[1, 2]$,

such that $f(\tilde{\delta}) = 1$ and $\delta_{ij} = \begin{cases} 1, & \text{if } j = 1, 2, \dots, t, \quad t < n, \\ 2, & \text{in other cases.} \end{cases}$ let's write out

the formula

$$U_{\tilde{\delta}}(\tilde{x}) = m_{100}(x_{i_1}, U, U) \cdot \dots \cdot m_{100}(x_{i_t}, U, U) \times \\ \times m_{200}(x_{i_{t+1}}, U, U) \cdot \dots \cdot m_{200}(x_{i_n}, U, U),$$

where

$$U = m_{11}(x_{i_1}, x_{i_2}) \cdot \dots \cdot m_{11}(x_{i_1}, x_{i_n}) m_{11}(x_{i_1}, x_{i_1}).$$

It's easy to see that

$$U_{\tilde{\delta}}(\tilde{x}) = \begin{cases} 1, & \text{if } (\tilde{x} = \tilde{\delta}) \vee (\tilde{x} = 1), \\ 2, & \text{if } \tilde{x} = 2, \\ 0, & \text{in other cases.} \end{cases}$$

For each set $\tilde{\delta}' = (\delta'_1, \delta'_2, \dots, \delta'_n)$, all coordinates of which belong to the set $[1, 2]$, such that $f(\tilde{\delta}') = 2$ and $\delta_{i_j} = \begin{cases} 1, & \text{if } j = 1, 2, \dots, p, \\ 2, & \text{if } j = p + 1, \dots, n, \end{cases} \quad 0 < p < n$, let's write out the expressions

$$\begin{aligned} U_{\tilde{\delta}'}(\tilde{x}) &= m_{201}(U_2, m_{11}(U_1, U_2)U_1), \\ U_1 &= m_{11}(x_{i_1}, x_{i_2}) \cdot \dots \cdot m_{11}(x_{i_1}, x_{i_p}) m_{11}(x_{i_1}, x_{i_1}), \\ U_2 &= m_{11}(x_{i_{p+1}}, x_{i_{p+2}}) \cdot \dots \cdot m_{11}(x_{i_{p+1}}, x_{i_n}) m_{11}(x_{i_{p+1}}, x_{i_{p+1}}). \end{aligned}$$

It's obvious that

$$U_{\tilde{\delta}}(\tilde{x}) = \begin{cases} 2, & \text{if } (\tilde{x} = \tilde{\delta}') \vee (\tilde{x} = \tilde{1}), \\ 1, & \text{if } \tilde{x} = 1, \\ 0, & \text{in other cases.} \end{cases}$$

From the construction of formulas $U_{\tilde{\alpha}}$, $U_{\tilde{\beta}}$, $U_{\tilde{\gamma}}$, $U_{\tilde{\delta}}$, $U_{\tilde{\delta}'}$ it is easy to see that the function

$$\begin{aligned} &\left(\bigvee_{\tilde{\alpha}:f(\tilde{\alpha})=1} U_{\tilde{\alpha}} \right) \vee \left(\bigvee_{\tilde{\beta}:f(\tilde{\beta})=2} U_{\tilde{\beta}} \right) \vee \left(\bigvee_{\tilde{\gamma}:f(\tilde{\gamma})=1} U_{\tilde{\gamma}} \right) \vee \\ &\vee \left(\bigvee_{\tilde{\gamma}':f(\tilde{\gamma}')=2} U_{\tilde{\gamma}'} \right) \vee \left(\bigvee_{\tilde{\delta}:f(\tilde{\delta})=1} U_{\tilde{\delta}} \right) \vee \left(\bigvee_{\tilde{\delta}':f(\tilde{\delta}')=2} U_{\tilde{\delta}'} \right), \end{aligned}$$

corresponding to the formula $f^*(\tilde{x})$, is identical to the function $f(\tilde{x})$.

The theorem is proved. ◀

Let $R = \begin{pmatrix} a_{11} & \dots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{p1} & \dots & a_{ps} \end{pmatrix}$ be a matrix whose elements take values from the set. We will say that the three-valued logic function $f(x_1, x_2, \dots, x_n)$

preserves relation if from the equalities

$$\begin{aligned} f(\alpha_{1j_1}, \alpha_{1j_2}, \dots, \alpha_{1j_n}) &= \beta_1, \\ f(\alpha_{2j_1}, \alpha_{2j_2}, \dots, \alpha_{2j_n}) &= \beta_2, \\ &\dots\dots\dots \\ f(\alpha_{pj_1}, \alpha_{pj_2}, \dots, \alpha_{pj_n}) &= \beta_n, \end{aligned}$$

where $(\alpha_{1j_r}, \alpha_{2j_r}, \dots, \alpha_{pj_r})$, $r = \overline{1, n}$, are arbitrary columns of the matrix R , it follows that $(\beta_1, \beta_2, \dots, \beta_p)$ is a column of the matrix R .

The set of all functions preserving the relation R will be called the class σ_R . It is easy to see that the class σ_R is functionally closed.

Theorem 2. *System of functions $\mathfrak{M}_2 = [\max(x_1, x_2), m_{100}(x_1, x_2, x_3), m_{200}(x_1, x_2, x_3)]$ forms a basis in the class σ_1 of all functions of three-valued logic and n variables.*

Proof. Let's write out the formulas $U_1(x_1, x_2, x_3) = m_{200}(m_{200}(x_1, x_2, x_3), m_{100}(x_1, x_2, x_3), m_{200}(x_1, x_2, x_3))$ and $U_2(x_1, x_2, x_3) = m_{200}(x_1, m_{11}(x_1, x_2, x_3), m_{11}(x_1, x_2))$, where

$$U_1(x_1, x_2, x_3) = \begin{cases} \alpha, & \text{if } x_1 = x_2 = x_3 = \alpha, \quad \alpha \in [1, 2], \\ 2, & \text{if } x_1 = 2, \quad x_2 = 0, \quad x_3 = 1, \\ 1, & \text{if } x_1 = x_2 = 2, \quad x_3 = 1, \\ 0, & \text{in other cases,} \end{cases}$$

and

$$U_2(x_1, x_2, x_3) = \begin{cases} \alpha, & \text{if } x_1 = x_2 = x_3 = \alpha, \\ 2, & \text{if } x_1 = 2, \quad x_2, x_3 \in [0, 1], \\ 0, & \text{in other cases.} \end{cases}$$

It's easy to see that:

$$\begin{aligned} m_{201}(x_1, x_2, x_3) &= U_1(x_1, x_2, x_3) U_2(x_1, x_2, x_3), \\ m_{11}(x_1, x_2) &= m_{100}(x_1, x_2, x_2) m_{100}(x_2, x_1, x_1) \end{aligned}$$

and

$$\begin{aligned} \min(x_1, x_2) &= m_{100}(x_1, m_{100}(x_1, x_2, x_2), m_{11}(x_1, x_2)) \vee \\ &\vee m_{100}(x_2, m_{100}(x_2, x_1, x_2), m_{11}(x_1, x_2)). \end{aligned}$$

Consider the system \mathfrak{M}_2 . For any subsystem $\tilde{\mathfrak{M}} = \mathfrak{M}_2 \setminus f$ of the system \mathfrak{M}_2 , where $f \in \mathfrak{M}_2$, we construct the relation R in such a way that $\mathfrak{M} \subseteq \sigma_R$, and the

function f does not preserve this relation. From this fact it follows that none of the functions of \mathfrak{M}_2 can be expressed in terms of the others.

Let $\mathfrak{M}_3 = [\max(x_1, x_2), m_{100}(x_1, x_2, x_3)]$ and $R = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$.

It is easy to see that $\max(x_1, x_2)$ and $m_{100}(x_1, x_2, x_3)$ belong to the class σ_R .

The function $m_{200}(x_1, x_2, x_3)$ satisfies the equalities $\begin{cases} m_{200}(2, 0, 0) = 2, \\ m_{200}(1, 2, 0) = 0. \end{cases}$

This shows that m_{200} does not belong to the class σ_R .

Let now $\mathfrak{M}_4 = [\max, m_{200}]$ and $R = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$. It is easy to see that the func-

tions \max and m_{200} belong to the class σ_R . For m_{100} we have $\begin{cases} m_{100}(1, 0, 0) = 1, \\ m_{100}(1, 1, 0) = 0. \end{cases}$

Therefore, $m_{100} \notin \sigma_R$.

Finally, consider the system $\mathfrak{M}_5 = [m_{200}, m_{100}]$. Let $R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

It is obvious that the functions m_{100} and m_{200} are functions of the class σ_R .

For $\max(x_1, x_2)$ we have $\begin{cases} \max(1, 0) = 1, \\ \max(0, 1) = 1 \end{cases}$. Therefore, $\max(x_1, x_2) \notin \sigma_R$.

So, we have shown that none of the functions of \mathfrak{M}_2 can be expressed in terms of the others.

Since the functions $m_{11}(x_1, x_2)$, $m_{201}(x_1, x_2, x_3)$ and $\min(x_1, x_2)$ are expressed by the formulas through the system \mathfrak{M}_2 , from Theorem 1 it easily follows that the system \mathfrak{M}_2 forms a basis in the class σ_1 .

The theorem is proved. \blacktriangleleft

3. Basis in class σ_3 for $k = 3$

Bases for the classes σ_2 and σ_3 are constructed. For each function from these classes, an expression is found in terms of the basis functions.

3.1. Basis in class σ_3

Let three-valued logic functions $f(x_1, x_2, \dots, x_n)$ be given that preserves the set L^* and is monotone in the introduced order. The set of such functions forms the class $\sigma_3 = \sigma_1 \cap \sigma_2$. Let a system of functions of class σ_3 be given: $\mathfrak{M}_{11} = [\max(x_1, x_2), \min(x_1, x_2), m_{11}(x_1, x_2), m_{12}(x_1, x_2), m_{21}(x_1, x_2), m_{120}(x_1, x_2, x_3)]$, where

$$m_{11}(x_1, x_2) = \begin{cases} \alpha, & \text{if } x_1 = x_2 = \alpha, \quad \alpha \in [1, 2], \\ 0, & \text{in other cases;} \end{cases}$$

$$m_{12}(x_1, x_2) = \begin{cases} \alpha, & \text{if } x_1 = x_2 = \alpha, \\ 1, & \text{if } x_1 = 1, x_2 \in [0, 2], \\ 0, & \text{in other cases;} \end{cases}$$

$$m_{21}(x_1, x_2) = \begin{cases} \alpha, & \text{if } x_1 = x_2 = \alpha, \\ 2, & \text{if } x_1 = 2, x_2 \in [0, 1], \\ 0, & \text{in other cases;} \end{cases}$$

$$m_{120}(x_1, x_2, x_3) = \begin{cases} \alpha, & \text{if } x_1 = x_2 = x_3 = \alpha, \\ 2, & \text{if } x_1 = 1, x_2 = 2, x_3 \in [0, 1, 2], \\ 0, & \text{in other cases.} \end{cases}$$

Theorem 3. *The system of functions \mathfrak{M}_{11} is complete in the class σ_3 .*

Proof. Let an arbitrary function $f(x_1, x_2, \dots, x_n)$ of class σ_3 be given by the table above. Let us construct a formula $f^*(\tilde{x})$ from the functions of the system \mathfrak{M}_{11} in such a way that the function corresponding to f^* was identical to the function $f(\tilde{x})$.

For each set $\tilde{\alpha} \in E^n(0, 1)$, where $\tilde{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$, such that $f(\tilde{\alpha}) = 1$ and $\alpha_{i_j} = \begin{cases} 1, & \text{if } j = 1, 2, \dots, t, \\ 0, & \text{if } j = t + 1, \dots, n, \end{cases}$ let's write out the formula $U_{\tilde{\alpha}}(\tilde{x}) = m_{12}(x_{i_2}, U) \cdot \dots \cdot m_{12}(x_{i_1}, U)$, where $U = m_{11}(x_1, x_2) \cdot m_{11}(x_1, x_3) \cdot \dots \cdot m_{11}(x_1, x_n)$.

It's obvious that $U_{\tilde{\alpha}}(x) = \begin{cases} 1, & \text{if } \tilde{x} \geq \tilde{\alpha}, \\ 2, & \text{if } \tilde{x} = \tilde{\alpha}, \\ 0, & \text{in other cases.} \end{cases}$

For each set $\tilde{\beta} \in E^n(0, 2)$, where $\tilde{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$, such that $f(\tilde{\beta}) = 2$ and $\beta_{i_j} = \begin{cases} 2, & \text{if } j = 1, 2, \dots, h, \\ 0, & \text{if } j = h + 1, \dots, n, \end{cases}$ let's write out the expression $U_{\tilde{\beta}}(\tilde{x}) = m_{21}(x_{i_1}, U) \cdot m_{21}(x_{i_2}, U) \cdot \dots \cdot m_{21}(x_{i_h}, U)$.

It's easy to see that

$$U_{\tilde{\beta}}(\tilde{x}) = m_{21}(x_{i_1}, U) \cdot m_{21}(x_{i_2}, U) \cdot \dots \cdot m_{21}(x_{i_h}, U)$$

and

$$U_{\tilde{\beta}}(x) = \begin{cases} 2, & \text{if } \tilde{x} \geq \tilde{\beta}, \\ 1, & \text{if } \tilde{x} = \tilde{\beta}, \\ 0, & \text{in other cases.} \end{cases}$$

For all sets $\tilde{\gamma} \in S_1^n$ and $\tilde{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_n)$ such that $f(\tilde{\gamma}) = 1$ and

$$\gamma_{i_j} = \begin{cases} 1, & \text{if } j = 1, 2, \dots, h, \\ 0, & \text{if } j = h + 1, \dots, l, \\ 2, & \text{if } j = l + 1, \dots, n, \end{cases}$$

let's build a formula

$$U_{\tilde{\gamma}}(\tilde{x}) = m_{12}(x_{i_1}, U) \cdot \dots \cdot m_{12}(x_{i_h}, U) \times m_{21}(x_{i_{l+1}}, U) \cdot \dots \cdot m_{21}(x_{i_n}, U).$$

It's easy to see that

$$U_{\tilde{\gamma}}(\tilde{x}) = \begin{cases} 1, & \text{if } (\tilde{x} \geq \tilde{\gamma}) \vee (\tilde{x} = \tilde{1}), \\ 2, & \text{if } \tilde{x} = \tilde{2}, \\ 0, & \text{in other cases.} \end{cases}$$

For all sets $\tilde{\delta} \in S_1^n$ and $\tilde{\delta} = (\delta_1, \delta_2, \dots, \delta_n)$ such that $f(\tilde{\delta}) = 2$ and

$$\delta_{i_j} = \begin{cases} 1, & \text{if } j = 1, 2, \dots, p, \\ 0, & \text{if } j = p + 1, \dots, q, \\ 2, & \text{if } j = q + 1, \dots, n, \end{cases}$$

let's write out the expression

$$U_{\tilde{\delta}}(\tilde{x}) = m_{120}(x_{i_1}, x_{i_{q+1}}, U) \cdot \dots \cdot m_{120}(x_{i_p}, x_{i_{q+1}}, U) \cdot m_{21}(x_{i_{q+1}}, U) \cdot \dots \cdot m_{21}(x_{i_n}, U).$$

Obviously,

$$U_{\tilde{\delta}}(\tilde{x}) = \begin{cases} 2, & \text{if } (\tilde{x} \geq \tilde{\delta}) \vee \text{left}(\tilde{x} = \tilde{2}), \\ 1, & \text{if } \tilde{x} = \tilde{1}, \\ 0, & \text{in other cases.} \end{cases}$$

From the construction of formulas $U_{\tilde{\alpha}}$, $U_{\tilde{\beta}}$, $U_{\tilde{\gamma}}$, $U_{\tilde{\delta}}$ it is easy to see that the function, $\left(\bigvee_{\tilde{\alpha}: f(\tilde{\alpha})=1} U_{\tilde{\alpha}} \right) \vee \left(\bigvee_{\tilde{\beta}: f(\tilde{\beta})=2} U_{\tilde{\beta}} \right) \vee \left(\bigvee_{\tilde{\gamma}: f(\tilde{\gamma})=1} U_{\tilde{\gamma}} \right) \vee \left(\bigvee_{\tilde{\delta}: f(\tilde{\delta})=2} U_{\tilde{\delta}} \right)$ corresponding to the formula $f^*(\tilde{x})$, is identical to the function $f(\tilde{x})$.

The theorem is proved. \blacktriangleleft

Let a system of functions of class σ_3 be given:

$$\mathfrak{M}_{12} = [\max(x_1, x_2), \min \text{left}(x_1, x_2), m_{12}(x_1, x_2), m_{21}(x_1, x_2), m_{120}(x_1, x_2, x_3)].$$

Theorem 4. *The system of functions \mathfrak{M}_{12} is a basis in the class σ_3 .*

Proof. It's easy to see that $m_{11}(x_1, x_2) = m_{12}(x_1, x_2) m_{21}(x_1, x_2)$.

Consider the system of functions \mathfrak{M}_{12} . As in Theorem 1 and Theorem 3, we will show that none of the functions of this system can be expressed in terms of the others.

Let $\mathfrak{M}_{12} = [\max, m_{12}, m_{21}, m_{120}]$ and $R = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 & 2 \end{pmatrix}$. It is easy to see that the functions of the system \mathfrak{M}_{13} preserve the relation R . For the function $\min(x_1, x_2)$, the following relations are valid:
$$\begin{cases} \min(0, 1) = 0, \\ \min(2, 1) = 1. \end{cases}$$

It is easy to see that the functions of the system \mathfrak{M}_{13} preserve the relation R . For the function $\min(x_1, x_2)$, the following relations are valid:
$$\begin{cases} \min(0, 1) = 0, \\ \min(2, 1) = 1. \end{cases}$$

This shows that $\min(x_1, x_2)$ does not belong to the class σ_R .

Let $\mathfrak{M}_{14} = [\min, m_{12}, m_{21}, m_{120}]$ and $R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

It is obvious that $\mathfrak{M}_{14} \subseteq \sigma_R$. For the function $\max(x_1, x_2)$, the following equalities hold:
$$\begin{cases} \max(0, 1) = 1, \\ \max(1, 0) = 1. \end{cases}$$

Therefore, $\max \bar{\in} \sigma_R$.

Let us now consider the system $\mathfrak{M}_{15} = [\max, \min, m_{21}, m_{120}]$.

Suppose $R = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

It is easy to see that $\mathfrak{M}_{15} \subseteq \sigma_R$. For the function m_{12} , the following relations are valid:
$$\begin{cases} m_{12}(2, 0) = 0, \\ m_{12}(1, 0) = 1. \end{cases}$$

This shows that $m_{12}(x_1, x_2) \bar{\in} \sigma_R$.

Let $\mathfrak{M}_{16} \subseteq \sigma_R$. The function m_{21} satisfies the equalities
$$\begin{cases} m_{21}(1, 0) = 0, \\ m_{21}(2, 0) = 2. \end{cases}$$

Therefore, $m_{21}(x_1, x_2) \bar{\in} \sigma_R$.

Finally, consider the system $\mathfrak{M}_{17} = [\max, \min, m_{12}, m_{21}]$.

Let $R = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.

It is easy to see that $\mathfrak{M}_{17} \subseteq \sigma_R$. For the function m_{120} , the following relations are valid:
$$\begin{cases} m_{120}(1, 2, 0) = 2, \\ m_{120}(0, 2, 0) = 0. \end{cases}$$

This indicates that $m_{120}(x_1, x_2, x_3) \bar{\in} \sigma_R$.

Thus, we have shown that none of the functions of \mathfrak{M}_{17} can be expressed in terms of the others.

Since the function $m_{11}(x_1, x_2)$ is expressed through the functions of the system \mathfrak{M}_{17} , from Theorem 4 it easily follows that the system \mathfrak{M}_{17} forms a basis in the class σ_3 .

The theorem is proved. ◀

4. Conclusion

The classes σ_1 , σ_2 and σ_3 of the system of functions defined on a subset of the structure of sets S^n and the corrective sets of the heuristics are studied, which allows us to determine the completeness of the system of functions \mathfrak{M}_1 , \mathfrak{M}_2 and \mathfrak{M}_{11} respectively. Also, to completely solve the problem of constructing a simple continuation of these classes σ_1 , σ_2 and σ_3 three theorems on the completeness of the system of functions \mathfrak{M}_1 , \mathfrak{M}_2 and \mathfrak{M}_{11} , when $k = 3$ are presented and they are fully proven, and their bases are constructed in these given classes. Algorithms for solving problems from weakly formalized areas are presented.

References

- [1] S.V. Ablameyko, A.S. Biryukov, A.A. Dokukin, A.G. D'yakonov, Yu.I. Zhuravlev, V.V. Krasnoproshin, V.A. Obratsov, M.Yu. Romanov, V.V. Ryazanov, *Practical algorithms for algebraic and logical correction in precedent-based recognition problems*, Comput. Math. and Math. Phys., **54(12)**, 2014, 1915-1928. <https://doi.org/10.1134/S0965542514120033>
- [2] Yu.I. Zhuravlev et al., *The Evolution of fuzzy rules in two-player games*, Southern Economic Journal, **69(3)**, 2003, 705-717.
- [3] A.N. Nurlibaev, *On normal forms k-valued logic*, Computational Mathematics and Cybernetics, **29(1)**, 1976, 16-28.
- [4] A. Kabulov, I. Saymanov, A. Babadjanov, A. Babadzhanov, *Algebraic recognition approach in IoT ecosystem*, Mathematics, **12(7)**, 1086, 2024, 1–26. <https://doi.org/10.3390/math12071086>.
- [5] A. Kabulov, I. Yarashov, A. Otakhonov, *Algorithmic analysis of the system based on the functioning table and information security*, 2022 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), 2022, 1-5. doi: 10.1109/IEMTRONICS55184.2022.9795746.

- [6] A. Kabulov, I. Saymanov, I. Yarashov, A. Karimov, *Using algorithmic modeling to control user access based on functioning table*, 2022 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), 2022, 1-5. doi: 10.1109/IEMTRONICS55184.2022.9795850.
- [7] E. Navruzov, A. Kabulov, *Detection and analysis types of DDoS attack*, 2022 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), 2022, 1-7. doi: 10.1109/IEMTRONICS55184.2022.9795729.
- [8] A. Kabulov, I. Saymanov, I. Yarashov, F. Muxammadiev, *Algorithmic method of security of the Internet of Things based on steganographic coding*, 2021 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), 2021, 1-5, doi: 10.1109/IEMTRONICS52119.2021.9422588.
- [9] A. Kabulov, I. Normatov, E. Urunbaev, F. Muxammadiev, *Invariant Continuation of Discrete Multi-Valued Functions and Their Implementation*, 2021 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), 2021, 1-6, doi: 10.1109/IEMTRONICS52119.2021.9422486.
- [10] S.V. Yablonsky, *Functional constructions of the k -valued logic*, Proceedings of Mat. Inst. Steklov, **51**, 1958, 5-142.
- [11] E.V. Dyukova, Yu.I. Zhuravlev, P.A. Prokofjev, *Logical correctors in problem of classification by precedents*, Comput. Math. Math. Phys., **57(11)**, 2017, 1866-1886.
- [12] N.N. Bondarenko, Yu.I. Zhuravlev, *Algorithm for choosing conjunctions for logical recognition methods*, Comput. Math. Math. Phys., **52(4)**, 2012, 649-652.
- [13] A. Kabulov, I. Normatov, A. Seytov, A. Kudaybergenov, *Optimal management of water resources in large main canals with cascade pumping stations*, 2020 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), 2020, 1-4, doi: 10.1109/IEMTRONICS51293.2020.9216402.
- [14] A.V. Kabulov, I.H. Normatov, *About problems of decoding and searching for the maximum upper zero of discrete monotone functions*, Journal of Physics: Conference Series, **1260(10)**, 102006, 2019, doi:10.1088/1742-6596/1260/10/102006.

- [15] A.V. Kabulov, I.H. Normatov, A.O. Ashurov, *Computational methods of minimization of multiple functions*, Journal of Physics: Conference Series, **1260(10)**, 102007, 2019, doi:10.1088/1742-6596/1260/10/102007.
- [16] A. Kabulov, A. Baizhumanov, M. Berdimurodov, *On the minimization of k-valued logic functions in the class of disjunctive normal forms*, Journal of Mathematics, Mechanics and Computer Science, **121(1)**, 2024, 37-45. <https://doi.org/10.26577/JMMCS202412114>.
- [17] I. Saymanov, *Logical automatic implementation of steganographic coding algorithms*, Journal of Mathematics, Mechanics and Computer Science, **121(1)**, 2024, 122-131. <https://doi.org/10.26577/JMMCS2024121112>.
- [18] I. Saymanov, *Logical recognition method for solving the problem of identification in the Internet of Things*, arXiv 2024, arXiv:2402.04338.
- [19] A. Kabulov, I. Normatov, I. Saymanov, A. Baizhumanov, I. Yarashov, *Models, methods and algorithms for monitoring environmental impact on agricultural production*, arXiv 2024, arXiv:2402.03346.

Anvar Kabulov

New Uzbekistan University, Mustaqillik Ave. 54, Tashkent, 100007, Uzbekistan
National University of Uzbekistan, Almazar district 4, Tashkent 100174, Uzbekistan
E-mail: kabulov_a@nuu.uz

Ibrokhimali Normatov

National University of Uzbekistan, Almazar district 4, Tashkent 100174, Uzbekistan
E-mail: ibragim_normatov@mail.ru

Islambek Saymanov

New Uzbekistan University, Mustaqillik Ave. 54, Tashkent, 100007, Uzbekistan
National University of Uzbekistan, Almazar district 4, Tashkent 100174, Uzbekistan
E-mail: islambeksaymanov@gmail.com

Abdussattar Baizhumanov

O. Zhanibekov South Kazakhstan State Pedagogical University, A. Baitursynov Street No.13, Shymkent City 160012, Kazakhstan
E-mail: baizhumanov.abdussattar@okmpu.kz

Received 05 March 2024

Accepted 28 September 2024