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Hardy Inequalities on Cones of Monotone Functions on a Measure Space

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Abstract. Consider a Hausdorff topological space X with Borel measure ν. Let $\{\Omega(t)\}$ be a totally ordered family of open subsets of an open set Ω in X parameterized by $t \in [0,\infty)$. This family generates a partial order in the set of functions defined on Ω. We obtain Sawyer-type bounds for a linear functional on cones of decreasing and increasing functions in weighted spaces. These results are further applied to obtain criteria of boundedness of averaging operators in those weighted spaces.

Key Words and Phrases: Hardy inequality, inequality on decreasing and increasing functions, Steklov operator, Hausdorff space.

2010 Mathematics Subject Classifications: 26D10, 26D15

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1. Introduction

In the seminal paper [1] Sawyer characterized a linear functional $L(f) = \int_0^\infty fg$ on a nonlinear set (cone) of decreasing functions and applied his result to the study of Lorentz spaces. Variations and generalizations of his results in the one-dimensional case have been obtained in [2, 7, 3]. [5] completed those results by investigating spaces of sequences. A series of general statements with relatively simple proofs is given in [11]. [6] has order-sharp three-weight estimates. [8] have developed further three-weight estimates. [4] extended results for monotone functions to R^n and [9] applied them to study potential operators on homogeneous groups.

We concentrate on the original Sawyer's results and obtain their very general versions by employing the theorems on the boundedness of the Hardy operator in a topological Hausdorff space with measures from the recent paper [10].

Assumption on $\Omega(t)$

Let Ω be an open subset of a Hausdorff topological space X with σ-additive measures μ, ν. The measures are defined on a σ-algebra $\mathfrak{M}$ that contains the Borel-measurable sets.

Assumption 1. a) $\{\Omega(t) : t \geq 0\}$ is a one-parametric family of open subsets of Ω which satisfy monotonicity condition:

$$\text{for } t_1 < t_2,\ \Omega(t_1) \text{ is a proper subset of } \Omega(t_2).$$

b) $\Omega(t)$ start at the empty set and eventually cover almost all Ω: $\Omega(0) = \cap_{t>0}\Omega(t) = \varnothing,\ \nu\left(\Omega \backslash \cup_{t>0} \Omega(t)\right) = 0$.

c) Further, denote by $\omega(t) = \overline{\Omega(t)} \cap \overline{(\Omega \backslash \Omega(t))}$ the boundary of $\Omega(t)$ in the relative topology. We require the boundaries to be disjoint and cover almost all Ω:

$$\omega(t_1) \cap \omega(t_2) = \varnothing,\ t_1 \neq t_2,\ \nu(\Omega \backslash \cup_{t>0} \omega(t)) = 0. \tag{1}$$

d) Passing to a different parametrization, if necessary, we can assume that

$$\nu\left(\Omega \backslash \cup_{t \leq N} \omega(t)\right) > 0 \text{ for any } N < \infty.$$

e) Finally, we assume that boundaries are thin in the sense that

$$\nu(\omega(t)) = 0 \text{ for all } t > 0. \tag{2}$$

(1) implies that for ν-almost each $y \in \Omega$ there exists a unique $\tau(y) > 0$ such that $y \in \omega(\tau(y))$. The Hardy operator is defined by

$$Pf(x) = \int_{\Omega(\tau(x))} f d\nu,\ x \in \Omega,$$

for any non-negative $\mathfrak{M}$-measurable f. On the set $\Omega_0 \subset \Omega$ of those y for which $\tau(y)$ is not defined we can put $\tau(\Omega_0) = \varnothing$. (2) allows us to define segments $\Omega[s,t] = \Omega(t) \backslash \Omega(s)$ without having to distinguish between variations such as $\overline{\Omega}(t) \backslash \Omega(s)$ etc. The averaging operator generalizes $\frac{1}{x}\int_0^x f$ and is defined as

$$Tf(x) = \frac{1}{\nu(\tau(x))} \int_{\Omega(\tau(x))} f d\nu,\ x \in \Omega. \tag{3}$$

Notation

$L_p(vd\nu)$ denotes the space with the norm $\|f\|_{p,vd\nu} = \left(\int_\Omega |f|^p v d\nu\right)^{1/p}$, where v is a weight function. The weights v, w are assumed positive and finite almost everywhere. Our main task is to find necessary and sufficient conditions for

$$\left[\int_\Omega \left(\frac{1}{\nu\left(\tau\left(x\right)\right)}\int_{\Omega(\tau(x))} f d\nu\right)^q w(x)d\nu(x)\right]^{1/q} \leq C\left(\int_\Omega f^p v d\nu\right)^{1/p}, \quad f \geq 0, \tag{4}$$

where C denotes the least constant. Everywhere we can assume that f is integrable, the general case following by passing to the limit. We write $A \asymp B$ to mean that $c_1 A \leq B \leq c_2 A$ with positive constants c_1, c_2 that do not depend on weights and measures.

We use the usual convention $1/\infty = 0$ and $0 \cdot \infty = 0$.

Auxiliary results on Hardy inequality

We need results on validity of

$$\left[\int_\Omega \left(\int_{\Omega(\tau(x))} f d\nu\right)^q u(x)d\mu(x)\right]^{1/q} \leq C\left(\int_\Omega f^p v d\nu\right)^{1/p}$$

from [10]. Denote

$$\Psi(t) = \left(\int_{\Omega\backslash\Omega(t)} u d\mu\right)^{1/q}\left(\int_{\Omega(t)} v^{-p'/p} d\nu\right)^{1/p'}, \ t \geq 0,$$

$$\Phi(t) = \left(\int_{\Omega\backslash\Omega(\tau(t))} u d\mu\right)^{1/p}\left(\int_{\Omega(\tau(t))} v^{-p'/p} d\nu\right)^{1/p'}, \ t \in \Omega.$$

Theorem A. *a) If $1 < p \leq q < \infty$, then $C \asymp \sup_{t>0} \Psi(t)$.*

b) If $0 < q < p$, $1 < p < \infty$, put $1/r = 1/q - 1/p$. Then $C \asymp \left(\int_\Omega \Phi^r u d\mu\right)^{1/r}$.

The corresponding result for the adjoint operator $P^* f\left(y\right) = \int_{\Omega\backslash\Omega(\tau(x))} f d\nu$ is stated in terms of the functions

$$\Psi^*(t) = \left(\int_{\Omega(t)} u d\mu\right)^{1/q}\left(\int_{\Omega\backslash\Omega(t)} v^{-p'/p} d\nu\right)^{1/p'}, \ t \geq 0,$$

$$\Phi^*(t) = \left(\int_{\Omega(\tau(t))} u d\mu\right)^{1/p}\left(\int_{\Omega\backslash\Omega(\tau(t))} v^{-p'/p} d\nu\right)^{1/p'}, \ t \in \Omega.$$

Denote by C^* the least constant in the inequality

$$\left[\int_\Omega \left(\int_{\Omega\backslash\Omega(\tau(x))} f d\nu\right)^q u(x)d\mu(x)\right]^{1/q} \leq C^*\left(\int_\Omega f^p v d\nu\right)^{1/p}.$$

Theorem B. *a) If* $1 < p \le q < \infty$, *then* $C^* \asymp \sup_{t>0} \Psi^*(t)$.

b) If $0 < q < p$, $1 < p < \infty$, *then* $C^* \asymp \left(\int_\Omega (\Phi^*)^r \, ud\mu\right)^{1/r}$.

2. Main results

The function $f : \Omega \to R_+ = [0, \infty)$ is called $\mathcal{A}$-decreasing (notation: $f \in \mathcal{AD}$) if $\tau(x) = \tau(y)$ implies $f(x) = f(y)$ and $\tau(x) < \tau(y)$ implies $f(x) \ge f(y)$. Symmetric decreasing rearrangements on R^n generated by the family of balls in R^n centered at zero are a special case and have numerous applications [12]. The function $f : \Omega \to R_+$ is called $\mathcal{A}$-increasing (notation: $f \in \mathcal{AI}$) if $\tau(x) = \tau(y)$ implies $f(x) = f(y)$ and $\tau(x) < \tau(y)$ implies $f(x) \le f(y)$.

We obtain Sawyer-type bounds for the functional

$$C_{\mathcal{AD}}(g) = \sup_{f \in \mathcal{AD}} \int_\Omega fg d\nu \left(\int_\Omega f^p v d\nu\right)^{-1/p}$$

and a corresponding result for $f \in \mathcal{AI}$. Applications to averaging operators are given.

Denote

$$V(t) = \int_{\Omega(t)} v d\nu, \; G(t) = \int_{\Omega(t)} g d\nu, \; t \in [0, \infty].$$

Theorem 1. *Let* $1 < p < \infty$. *If* $V(\infty) < \infty$, *then*

$$C_{\mathcal{AD}}(g) \asymp G(\infty) V(\infty)^{-1/p} + \left(\int_\Omega [G(\tau(x)) / V(\tau(x))]^{p'} v(x) \, d\nu(x)\right)^{1/p'}. \quad (5)$$

If $V(\infty) = \infty$, *then* (5) *holds without the term* $G(\infty) V(\infty)^{-1/p}$.

Let

$$\begin{aligned} C_{\mathcal{AI}}(g) &= \sup_{f \in \mathcal{AI}} \int_\Omega fg d\nu \left(\int_\Omega f^p v d\nu\right)^{-1/p}, \\ V^*(t) &= \int_{\Omega \setminus \Omega(t)} v d\nu, \; G^*(t) = \int_{\Omega \setminus \Omega(t)} g d\nu, \; t \in [0, \infty]. \end{aligned}$$

Theorem 2. *Let* $1 < p < \infty$. *If* $V(\infty) < \infty$, *then*

$$C_{\mathcal{AI}}(g) \asymp G(\infty) V(\infty)^{-1/p} + \left(\int_\Omega [G^*(\tau(x)) / V^*(\tau(x))]^{p'} v(x) \, d\nu(x)\right)^{1/p'}. \quad (6)$$

If $V(\infty) = \infty$, *then* (6) *holds without the term* $G(\infty) V(\infty)^{-1/p}$.

Instead of the usual notation for composite functions $V \circ \tau$, $G \circ \tau$, $V^* \circ \tau$ and $G^* \circ \tau$, it will be convenient to use V_τ, G_τ, V_τ^* and G_τ^*, resp.

Consider an integral operator

$$Kf(x) = \int_\Omega k(x,y) f(y) \, d\nu(y)$$

with a non-negative kernel k. The next corollary reduces a non-linear problem to a linear one.

Corollary 1. *Let $1 < p, q < \infty$. a) The bound*

$$\left(\int_\Omega (Kf)^q w d\nu \right)^{1/q} \le C \left(\int_\Omega f^p v d\nu \right)^{1/p} \tag{7}$$

holds for all $f \in \mathcal{AD}$ if and only if the bound

$$\left\{ \int_\Omega K^* h d\nu \left(\int_\Omega v d\nu \right)^{-1/p} + \left[\int_\Omega \left(\int_{\Omega(\tau(x))} K^* h d\nu / V_\tau(x) \right)^{p'} v(x) \, d\nu(x) \right]^{1/p'} \right\}$$
$$\le C \left(\int_\Omega h^{q'} w^{1-q'} d\nu \right)^{1/q'} \tag{8}$$

holds for all $h \ge 0$.

b) The bound (7) *holds for all $f \in \mathcal{AI}$ if and only if the bound* (8) *holds with $\Omega(\tau(x))$ in the inner integrals replaced by $\Omega \backslash \Omega(\tau(x))$ (V_τ gets replaced by V_τ^*).*

For the averaging operator (3) with the notation

$$m(y) = \frac{1}{\nu(\Omega(\tau(y)))}, \ P^* f(y) = \int_{\Omega \backslash \Omega(\tau(y))} f d\nu$$

we see that $T^* f = P^*(mf)$ and

$$\begin{aligned}
(PT^* f)(x) &= \int_{\Omega(\tau(x))} \left(\int_{\Omega[\tau(y),\tau(x)]} mf d\nu + \int_{\Omega[\tau(x),\infty]} mf d\nu \right) d\nu(y) \\
&= \int_{\Omega(\tau(x))} (mf)(z) \left(\int_{\Omega(\tau(z))} d\nu \right) d\nu(z) + \int_{\Omega[\tau(x),\infty]} mf d\nu \int_{\Omega(\tau(x))} d\nu \\
&= (Pf)(x) + \nu(\Omega(\tau(x))) \left[P^*(mf) \right](x). \qquad (9)
\end{aligned}$$

Theorem 3. *Let C be the best constant in* (4) *and denote $1/r = 1/q - 1/p$ for $q > p$. a) If $1 < p \le q < \infty$, then $C \asymp \alpha + \beta$, where*

$$\alpha = \sup_{x \in \Omega} \left(\int_{\Omega(\tau(x))} w d\nu \right)^{1/q} \left(\int_{\Omega(\tau(x))} v d\nu \right)^{-1/p}, \tag{10}$$

$$\beta = \sup_{x \in \Omega} \left(\int_{\Omega(\tau(x))} (V_\tau/m)^{-p'} v d\nu \right)^{1/p'} \left(\int_{\Omega \setminus \Omega(\tau(x))} m^q w d\nu \right)^{1/q}, \tag{11}$$

resp.

b) If $1 < q < p < \infty$, then $C \asymp A + B$, where

$$A = \left\{ \int_\Omega \left(\int_{\Omega \setminus \Omega(\tau(x))} V_\tau^{-p'} v d\nu \right)^{r/q'} \left(\int_{\Omega(\tau(x))} w d\nu \right)^{r/q} V_\tau^{-p'}(x)\, v(x)\, d\nu(x) \right\}^{1/r}, \tag{12}$$

$$\begin{aligned} B \;=\; & \left\{ \int_\Omega \left(\int_{\Omega(\tau(x))} (V_\tau/m)^{-p'} v d\nu \right)^{r/q'} \left(\int_{\Omega \setminus \Omega(\tau(x))} m^q w d\nu \right)^{r/q} \right. \\ & \left. \times \left(V_\tau(x) / m(x) \right)^{-p'} v(x)\, d\nu(x) \right\}^{1/r}. \end{aligned} \tag{13}$$

3. Proofs

All long proofs use binary partitions. In the proof of Lemma 1 we give such derivations in full and later indicate shortened proofs with enough detail for the interested reader to be able to restore complete proofs.

Lemma 1. *Let $1 < p < \infty$. Suppose that $0 < V(\infty) < \infty$. Then for all $0 < G(\infty) \le \infty$ the following statements a)-c) are true: a) we have with some $c_1, c_2 > 0$*

$$c_1 V_\tau(x)^{1-p'} \le \int_{\Omega[\tau(x),\infty]} V_\tau^{-p'} v d\nu \le c_2 V_\tau(x)^{1-p'}, \tag{14}$$

where the right inequality is true for all x and the left one for x such that $V_\tau(x) < \frac{1}{2} V(\infty)$. When $\tau(x) \to \infty$, $\int_{\Omega[\tau(x),\infty]} V_\tau^{-p'} v d\nu \to 0$ because if $\frac{1}{2} V(\infty) \le V_\tau(x) \le V(\infty)$, then

$$1 \le \int_{\Omega[\tau(x),\infty]} V_\tau^{-p'} v d\nu \left(V(\infty)^{-p'} \int_{\Omega[\tau(x),\infty]} v d\nu \right)^{-1} \le 2^{p'}. \tag{15}$$

Thus the left side of (14) *cannot be true for large* $\tau(x)$.

b) A Hardy type inequality

$$\int_{\Omega} (P^* f)^p \, v d\nu \leq c \int_{\Omega} f^p V_\tau^p v^{1-p} d\nu$$

holds with c independent of f, v, V.

c) For all $y \in \Omega$ *we have*

$$\left(P^* \frac{g}{V_\tau}\right)(y) = \int_{\Omega \setminus \Omega(\tau(y))} g/V_\tau d\nu \leq 8 \left(G(\infty)/V(\infty) + \int_{\Omega \setminus \Omega(\tau(y))} G_\tau V_\tau^{-2} v d\nu\right).$$

Parts a) and b) hold also with p replaced by p'.

Proof. a) Denote $t_0 = \infty$ and define successively $t_0 > t_1 > ... > t_n \to 0$ such that

$$v_n \equiv \int_{\Omega[t_n, t_{n-1}]} v d\nu = 2^{-n} V(\infty), \; n = 1, 2, ... \tag{16}$$

This is possible because of (2). For $s \in (0, \infty)$, define $n(s)$ by the condition $s \in [t_{n(s)}, t_{n(s)-1})$, $n(s) \geq 1$.

Since $t_{n(\tau(x))} \leq \tau(x)$ and $[t_{n(\tau(x))}, \infty) = \cup_{j=1}^{n(\tau(x))} [t_j, t_{j-1})$, we can discretize the integral in the middle of (14):

$$\begin{aligned} I_1(x) &\equiv \int_{\Omega[\tau(x), \infty]} V_\tau^{-p'} v d\nu \leq \int_{\Omega\left[t_{n(\tau(x))}, \infty\right]} V_\tau^{-p'} v d\nu \\ &= \sum_{j=1}^{n(\tau(x))} \int_{\Omega[t_j, t_{j-1}]} V_\tau^{-p'} v d\nu \leq \sum_{j=1}^{n(\tau(x))} V(t_j)^{-p'} v_j. \end{aligned} \tag{17}$$

By (16), $v_k = 2^m v_{k+m}$, for $k \geq 1$ and $k + m \geq 1$, so

$$V(t_j) = \sum_{k \geq j+1} v_k = 2^m \sum_{k \geq j+1} v_{k+m} = 2^m V(t_{j+m}), \; j, m \geq 0, \tag{18}$$

which implies $V(t_j) = 2^{n(\tau(x))-j-1} V\left(t_{n(\tau(x))-1}\right) \geq 2^{n(\tau(x))-j-1} V_\tau(x)$ for $1 \leq j \leq n(\tau(x)) - 1$. For $j = n(\tau(x))$, obviously, $V\left(t_{n(\tau(x))}\right) = \frac{1}{2} V\left(t_{n(\tau(x))-1}\right) \geq \frac{1}{2} V_\tau(x)$. Therefore, the right inequality in (14) is true:

$$I_1(x) \leq c_1 V_\tau(x)^{-p'} \sum_{j=1}^{n(\tau(x))} 2^{(1-p')[n(\tau(x))-j]} v_{j+n(\tau(x))+1-j}$$

$$= c_1 V_\tau(x)^{-p'} v_{n(\tau(x))+1} \sum_{j=1}^{n(\tau(x))} 2^{(1-p')[n(\tau(x))-j]} \le c_2 V_\tau(x)^{1-p'}. \qquad (19)$$

When proving the lower bound in (14) we can use $n(\tau(x)) \ge 2$ because the inequality $V_\tau(x) < \frac{1}{2}V(\infty)$ is equivalent to $\tau(x) < t_1$ and to $n(\tau(x)) \ge 2$.

$$\begin{aligned} I_1(x) &= \int_{\Omega[\tau(x),\infty]} V_\tau^{-p'} v d\nu \ge \int_{\Omega\left[t_{n(\tau(x))-1},\infty\right]} V_\tau^{-p'} v d\nu \\ &= \sum_{j=1}^{n(\tau(x))-1} \int_{\Omega[t_j,t_{j-1}]} V_\tau^{-p'} v d\nu \ge \sum_{j=1}^{n(\tau(x))-1} V(t_{j-1})^{-p'} v_j. \end{aligned}$$

Choosing $m = n(\tau(x)) - j + 1$ in (18) gives $V(t_{j-1}) \le 2^{n(\tau(x))-j+1} V_\tau(x)$ for $1 \le j \le n(\tau(x)) - 1$. Using also $v_j = 2^{n(\tau(x))-j-1} V\left(t_{n(\tau(x))-1}\right)$, we get

$$I_1(x) \ge c_3 V_\tau(x)^{-p'} \sum_{j=1}^{n(\tau(x))-1} 2^{(1-p')[n(\tau(x))-j]} V\left(t_{n(\tau(x))-1}\right) \ge c_4 V_\tau(x)^{1-p'}.$$

In the case $\frac{1}{2}V(\infty) \le V_\tau(x) \le V(\infty)$, (15) is evident.

Statement b) will follow from Theorem B if we prove that

$$\begin{aligned} A(x) &= \left(\int_{\Omega(\tau(x))} v d\nu\right)^{1/p} \left(\int_{\Omega\setminus\Omega(\tau(x))} \left(V_\tau^p v^{1-p}\right)^{1-p'} d\nu\right)^{1/p'} \\ &= V_\tau(x)^{1/p} \left(\int_{\Omega\setminus\Omega(\tau(x))} V_\tau^{-p'} v d\nu\right)^{1/p'} \le c. \end{aligned}$$

But this follows from (14).

To avoid triviality in the proof of c), assume that $G(\infty) < \infty$. Denote

$$g_n = \int_{\Omega[t_n,t_{n-1}]} g d\nu,\ G(t_{n-1}) = \sum_{i=n}^\infty g_i,\ V(t_{n-1}) = \sum_{i=n}^\infty v_i,\ n \ge 1.$$

The first step is discretization. Since $\tau(z) \ge t_{n(\tau(z))}$, we have

$$V_\tau(z) \ge \int_{\Omega\left(t_{n(\tau(z))}\right)} v d\nu = \sum_{i=t_{n(\tau(z))+1}}^\infty v_i = V\left(t_{n(\tau(z))}\right), \qquad (20)$$

so

$$I(y) \equiv \int_{\Omega[\tau(y),\infty]} g/V_\tau d\nu \le \int_{\Omega\left[t_{n(\tau(y))},\infty\right]} g(z)/V\left(t_{n(\tau(z))}\right) d\nu(z)$$

$$= \sum_{i=1}^{n(\tau(y))} \int_{\Omega[t_i,t_{i-1}]} g(z) / V\left(t_{n(\tau(z))}\right) d\nu(z).$$

Here in the integral $\tau(z) \in [t_i, t_{i-1})$ implies $n(\tau(z)) = i$. Therefore,

$$I(y) \leq \sum_{i=1}^{n(\tau(y))} \int_{\Omega[t_i,t_{i-1}]} g/V(t_i)\, d\nu = \sum_{i=1}^{n(\tau(y))} \frac{g_i}{V(t_i)} \equiv J(y). \tag{21}$$

The next step is essentially a change of the summation order.

$$\begin{aligned} J(y) &= \sum_{i=1}^{n(\tau(y))} \frac{G(t_{i-1}) - G(t_i)}{V(t_i)} = \left(\frac{G(\infty)}{V(t_1)} + \sum_{i=2}^{n(\tau(y))} \frac{G(t_{i-1})}{V(t_i)} \right) \\ &\quad - \left(\sum_{i=1}^{n(\tau(y))-1} \frac{G(t_i)}{V(t_i)} + \frac{G\left(t_{n(\tau(y))}\right)}{V\left(t_{n(\tau(y))}\right)} \right) \\ &\leq \frac{G(\infty)}{V(t_1)} + \sum_{i=2}^{n(\tau(y))} \frac{G(t_{i-1})\, v_i}{V(t_{i-1})\, V(t_i)} \leq \frac{G(\infty)}{V(t_1)} + \sum_{i=2}^{n(\tau(y))} \frac{G(t_{i-1})\, v_i}{V(t_i)^2}. \end{aligned} \tag{22}$$

The last step is bounding this expression by an integral. Note that for $\tau(s) \geq t_{i-1}$, one has $G(t_{i-1}) \leq G_\tau(s)$, and $v_i = \frac{1}{2} v_{i-1},\ i \geq 2$, so

$$G(t_{i-1})\, v_i = \frac{1}{2} G(t_{i-1}) \int_{\Omega[t_{i-1},t_{i-2}]} v\, d\nu \leq \frac{1}{2} \int_{\Omega[t_{i-1},t_{i-2}]} G_\tau v\, d\nu,\ i \geq 2. \tag{23}$$

Further, for $\tau(s) \leq t_{i-2}$ with $i \geq 2$

$$V(t_i) = \sum_{j \geq i+1} v_j = \frac{1}{4} \sum_{j \geq i-1} v_j = \frac{1}{4} \int_{\Omega(t_{i-2})} v\, d\nu \geq \frac{1}{4} V_\tau(s). \tag{24}$$

Combining (23) and (24) and using $\tau(y) < t_{n(\tau(y))-1}$, we see that

$$\begin{aligned} \sum_{i=2}^{n(\tau(y))} \frac{G(t_{i-1})\, v_i}{V(t_i)^2} &\leq \frac{1}{2} \sum_{i=2}^{n(\tau(y))} \frac{1}{V(t_i)^2} \int_{\Omega[t_{i-1},t_{i-2}]} G_\tau v\, d\nu \\ &\leq 8 \sum_{i=2}^{n(\tau(y))} \int_{\Omega[t_{i-1},t_{i-2}]} G_\tau V_\tau^{-2} v\, d\nu \leq 8 \int_{\Omega[\tau(y),\infty]} G_\tau V_\tau^{-2} v\, d\nu. \end{aligned}$$

The remainder at the right of (22) is, obviously, $2G(\infty)/V(\infty)$. (21), (22) and the last displayed equation prove c).

Lemma 2. *Let* $1 < p < \infty$ *and* $V(\infty) = \infty$. *Then*

a) the right side of (14) *holds for all* $x \in \Omega$,

b) part b) of Lemma 1 holds and

c) part c) of Lemma 1 is true without the term $G(\infty)/V(\infty)$.

Proof. a) Define the segments $[t_n, t_{n+1})$ by the condition $v_n = \int_{\Omega[t_n,t_{n+1})} v d\nu = 2^n$, $n \in Z$. Then,

$$V(t_n) = \int_{\Omega(t_n)} v d\nu = \sum_{j=-\infty}^{n-1} v_j = 2^n,\ V(t_n) = 2^{n-m} V(t_m),\ v_n = 2^{n-m} v_m.$$

Define $n(\tau(x))$ by the condition $t_{n(\tau(x))} \le \tau(x) < t_{n(\tau(x))+1}$. Then, instead of (17) we have

$$\int_{\Omega[\tau(x),\infty]} V_\tau^{-p'} v d\nu \le \sum_{j=n(\tau(x))}^{\infty} V(t_j)^{-p'} v_j \tag{25}$$

and (19) can be changed to bound the last expression by $c_2 V_\tau(x)^{1-p'}$. As a result, we get the right inequality in (14). As in Lemma 1, it implies statement b).

c) We can prove the statement for $g\chi_{\Omega(N)}$ instead of g and then let $N \to \infty$. Thus, we can assume $G(\infty) < \infty$. With the notation $g_n = \int_{\Omega[t_n,t_{n+1}]} g d\nu$, we have

$$G(t_n) = \int_{\Omega(t_n)} g d\nu = \sum_{i=-\infty}^{n-1} g_i,\ V(t_n) = \sum_{i=-\infty}^{n-1} v_i.$$

With this notation, (20) holds and with obvious changes in calculation, instead of (21) we get

$$I(y) \equiv \int_{\Omega[\tau(y),\infty]} g/V_\tau d\nu \le \sum_{i=n(\tau(y))}^{\infty} \frac{g_i}{V(t_i)} \equiv J(y).$$

The analog of (22) is

$$J(y) \le \sum_{i=n(\tau(y))+1}^{\infty} \frac{G(t_i) v_{i-1}}{V(t_{i-1})^2}.$$

Instead of (23) we have $G(t_i) v_{i-1} \le \frac{1}{2} \int_{\Omega[t_i,t_{i+1}]} G_\tau v d\nu$, while (24) does not change. Putting all those inequalities together, we prove c).

Proof. [Proof of Theorem 1] **Part 1**. Here we assume that $V(\infty) < \infty$. For $C_{\mathcal{AD}}(g)$ to be meaningful, we exclude the case $v = 0$ a.e. Then $V(\infty) > 0$ and

$G(\infty) > 0$. If $G(\infty) = \infty$, we can prove (5) for $g\chi_{\Omega(N)}$ instead of g and then let $N \to \infty$. Thus, we can assume $G(\infty) < \infty$.

Bound from below. Since $f \equiv c > 0$ belongs to $\mathcal{AD}$, obviously,

$$C_{\mathcal{AD}}(g) \geq \int_\Omega g d\nu \left(\int_\Omega v d\nu\right)^{-1/p}. \tag{26}$$

Define $f(x) = \int_{\Omega[\tau(x),\infty]} h d\nu$ with $h(t) \geq 0$. Then $f \in \mathcal{AD}$ and

$$\int_\Omega f g d\nu = \int_\Omega g P^* h d\nu = \int_\Omega (Pg)\, h d\nu = \int_\Omega G_\tau h d\nu. \tag{27}$$

By Lemma 1b),

$$\int_\Omega f^p v d\nu = \int_\Omega (P^* h)^p\, v d\nu \leq c_1 \int_\Omega h^p V_\tau^p v^{1-p} d\nu. \tag{28}$$

(27) and (28) imply

$$\begin{aligned} C_{\mathcal{AD}}(g) &\geq c_2 \sup_{h\geq 0} \frac{\int_\Omega G_\tau h d\nu}{\left(\int_\Omega h^p V_\tau^p v^{1-p} d\nu\right)^{1/p}} \text{ (replacing throughout } u = hV_\tau/v) \\ &= c_2 \sup_{u\geq 0} \frac{\int u G_\tau / V_\tau v d\nu}{\left(\int_\Omega u^p v d\nu\right)^{1/p}} = c_2 \left(\int_\Omega (G_\tau/V_\tau)^{p'}\, v d\nu\right)^{1/p'}. \end{aligned} \tag{29}$$

(26) and (29) prove the lower bound.

Bound from above. Since $f \in \mathcal{AD}$, by Lemma 1c) we have

$$\begin{aligned} \int_\Omega f g d\nu &= \int_\Omega \frac{fg}{V_\tau} V_\tau d\nu = \int_\Omega \left(P^* \frac{fg}{V_\tau}\right) v d\nu \leq \int_\Omega \left(P^* \frac{g}{V_\tau}\right) f v d\nu \\ &\leq 8\left[\int_\Omega f v d\nu G(\infty)/V(\infty) + \int_\Omega \left[P^*\left(G_\tau V_\tau^{-2} v\right)\right] f v d\nu\right]. \end{aligned}$$

By Hölder's inequality,

$$\int_\Omega f v d\nu G(\infty)/V(\infty) \leq \left(\int_\Omega f^p v d\nu\right)^{1/p} G(\infty)\, V(\infty)^{-1/p}. \tag{30}$$

In Lemma 1b) replace f by $F = G_\tau V_\tau^{-2} v$ and p by p' to get

$$\int_\Omega (P^* F)^{p'}\, v d\nu \leq c \int_\Omega \left(G_\tau V_\tau^{-2} v\right)^{p'} V_\tau^{p'} v^{1-p'} d\nu = c \int_\Omega \left(G_\tau V_\tau^{-1}\right)^{p'} v d\nu.$$

Combining this with Hölder's inequality we get

$$\int_{\Omega} f(y)\,v(y)\,P^{*}F(y)\,d\nu(y) \le \left(\int_{\Omega} f^{p} v\,d\nu\right)^{1/p}\left(\int_{\Omega} (P^{*}F)^{p'} v\,d\nu\right)^{1/p'}$$

$$\le c\left(\int_{\Omega} f^{p} v\,d\nu\right)^{1/p}\left(\int_{\Omega} \left(G_{\tau}V_{\tau}^{-1}\right)^{p'} v\,d\nu\right)^{1/p'}. \qquad (31)$$

(30) and (31) prove what we need in the case $V(\infty) < \infty$.

Part 2. Let $V(\infty) = \infty$. The case $G(\infty) = \infty$ is handled as in Part 1, which means that for the purposes of bound (5) we put $\frac{\infty}{\infty} = 0$. Let $G(\infty) < \infty$. The proof of Part 1 goes through with the following changes. The right side of (26) is zero. For (28), instead of Lemma 1b) we use Lemma 2b). For the upper bound, instead of Lemma 1c) we use Lemma 2c). (30) becomes unnecessary. Further, instead of Lemma 1b) apply Lemma 2b).

Lemma 3. *If $V(\infty) < \infty$, then a)*

$$\int_{\Omega} (Ph)^{p}\,v\,d\nu \le c\int_{\Omega} h^{p}\left(V_{\tau}^{*}\right)^{p} v^{1-p} d\nu. \qquad (32)$$

This is also true with p replaced by p'.

b)

$$I(y) \equiv \left(P\frac{g}{V_{\tau}^{*}}\right)(y) \le 8\left(G(\infty)/V(\infty) + \int_{\Omega(\tau(y))} G_{\tau}^{*}\left(V_{\tau}^{*}\right)^{-2} v\,d\nu\right). \qquad (33)$$

Proof. Here and in the proof of Theorem 3 we use the notation $F^{*} = G_{\tau}^{*}\left(V_{\tau}^{*}\right)^{-2} v$. The partition used here is different from the one applied in Lemma 1. Put $s_0 = 0$ and define $s_0 < s_1 < ... < s_n \to \infty$ by

$$v_k \equiv \int_{\Omega[s_k, s_{k+1}]} v\,d\nu = 2^{-k-1} V(\infty)\,,\ k \ge 0,$$

and $m(s) \ge 0$, for $s \in [0, \infty)$, with $s \in [s_{m(s)}, s_{m(s)+1})$

a) By Theorem A, we need to prove that

$$\begin{aligned} A_1(x) &\equiv \left(\int_{\Omega\setminus\Omega(\tau(x))} v\,d\nu\right)^{1/p}\left(\int_{\Omega(\tau(x))} \left(\left(V_{\tau}^{*}\right)^{p} v^{1-p}\right)^{-p'/p} d\nu\right)^{1/p'} \\ &= V_{\tau}^{*}(x)^{1/p}\left(\int_{\Omega(\tau(x))} \left(V_{\tau}^{*}\right)^{-p'} v\,d\nu\right)^{1/p'} \le c. \end{aligned}$$

Using $\tau(x) < s_{m(\tau(x))+1}$, we have

$$\begin{aligned} J(x) &\equiv \int_{\Omega(\tau(x))} (V_\tau^*)^{-p'} v d\nu \le \sum_{k=0}^{m(\tau(x))} \int_{\Omega[s_k,s_{k+1}]} (V_\tau^*)^{-p'} v d\nu \\ &\le \sum_{k=0}^{m(\tau(x))} V^*(s_{k+1})^{-p'} \int_{\Omega[s_k,s_{k+1}]} v d\nu. \end{aligned}$$

Here

$$V^*(s_{k+1}) = 2^{m(\tau(x))-k-1} V^*\left(s_{m(\tau(x))}\right), \ v_k = 2^{m(\tau(x))-k+1} v_{m(\tau(x))+1}.$$

Hence,

$$\begin{aligned} J(x) &\le c_1 V^*\left(s_{m(\tau(x))}\right)^{-p'} v_{m(\tau(x))+1} \sum_{k=0}^{m(\tau(x))} 2^{(1-p')(m(\tau(x))-k)} \\ &\le c_2 V_\tau^*(x)^{1-p'}. \end{aligned}$$

Thus, $A_1(x) \le c_3 V_\tau^*(x)^{1/p} \left(V_\tau^*(x)^{1-p'}\right)^{1/p'} = c_3$, which proves a).

b) Denote

$$g_n = \int_{\Omega[s_n,s_{n+1}]} g d\nu, \ G^*(s_n) = \sum_{i=n}^{\infty} g_i, \ V^*(s_n) = \sum_{i=n}^{\infty} v_i, \ n \ge 0.$$

The first step is to discretize $I(y)$:

$$I(y) \le \int_{\Omega\left(s_{m(\tau(y))+1}\right)} g/V_\tau^* d\nu = \sum_{i=0}^{m(\tau(y))} \int_{\Omega[s_i,s_{i+1}]} g/V_\tau^* d\nu. \tag{34}$$

For $\tau(x) \in [s_i, s_{i+1})$ we have $m(\tau(x)) = i$ and $V_\tau^*(x) \ge V^*(s_{i+1})$, so

$$I(y) \le \sum_{i=0}^{m(\tau(y))} \frac{g_i}{V^*(s_{i+1})}.$$

Changing the summation order:

$$I(y) \le \sum_{i=0}^{m(\tau(y))} \frac{G^*(s_i) - G^*(s_{i+1})}{V^*(s_{i+1})} \le \frac{G(\infty)}{V^*(s_1)} + \sum_{i=1}^{m(\tau(y))} \frac{G^*(s_i) v_i}{V^*(s_i) V^*(s_{i+1})}$$

$$\leq \frac{G(\infty)}{V^*(s_1)} + \sum_{i=1}^{m(\tau(y))} \frac{G^*(s_i)\, v_i}{V^*(s_{i+1})^2}. \tag{35}$$

Next we bound the sum on the right of (35) by an integral. Note that

$$G^*(s_i)\, v_i = \frac{1}{2} G^*(s_i)\, v_{i-1} = \frac{1}{2} \int_{\Omega[s_{i-1},s_i]} G^*(s_i)\, v d\nu \leq \frac{1}{2} \int_{\Omega[s_{i-1},s_i]} G^*_\tau v d\nu$$

and for $\tau(x) \geq s_{i-1}$

$$V^*(s_{i+1}) = \frac{1}{4} V^*(s_{i-1}) = \frac{1}{4} \int_{\Omega[s_{i-1},\infty]} v d\nu \geq \frac{1}{4} V^*_\tau(x).$$

That is why

$$\begin{aligned} \sum_{i=1}^{m(\tau(y))} \frac{G^*(s_i)\, v_i}{V^*(s_{i+1})^2} &\leq 8 \sum_{i=1}^{m(\tau(y))} \int_{\Omega[s_{i-1},s_i]} G^*_\tau (V^*_\tau)^{-2} v d\nu \\ &= 8 \int_{\Omega[s_0, m(\tau(y))]} F^* d\nu \leq 8 \int_{\Omega(\tau(y))} F^* d\nu. \end{aligned} \tag{36}$$

Since also $G(\infty)/V^*(s_1) = 2G(\infty)/V(\infty)$, (35) and (36) prove b).

Lemma 4. *Let $\int_{\Omega[t,\infty]} v d\nu < \infty$ for any $t > 0$ and $V(\infty) = \infty$.*

a) (32) is true, also with p replaced by p'.

b) (33) holds without the term $G(\infty)/V(\infty)$ on the right.

Proof. a) Define the segments $[t_n, t_{n+1})$ by the condition $v_n = \int_{\Omega[t_n,t_{n+1})} v d\nu = 2^{-n}$, $n \in Z$. Then

$$v_j = 2^{n-j} v_n,\ V^*(t_j) = \int_{\Omega[t_j,\infty]} v d\nu = \sum_{s=j}^{\infty} v_s = 2^{n-j} V^*(t_n).$$

Instead of (25), this time we have

$$\int_{\Omega(\tau(x))} (V^*_\tau)^{-p'} v d\nu \leq \sum_{j=-\infty}^{n(\tau(x))} V^*(t_{j+1})^{-p'} v_j \leq c V^*_\tau(x)^{1-p'}.$$

As in Lemma 3, this inequality implies (32).

b) With the same $\{t_n\}$ as in part a), denote $g_n = \int_{\Omega[t_n,t_{n+1}]} g d\nu$, and then we have

$$G^*(t_n) = \int_{\Omega[t_n,\infty]} g d\nu = \sum_{i=n}^{\infty} g_i,\ V^*(t_n) = \sum_{i=n}^{\infty} v_i.$$

Instead of (34), we have for $t_{n(\tau(y))} \leq \tau(y) < t_{n(\tau(y))+1}$

$$I(y) \leq \int_{\Omega\left(t_{n(\tau(y))+1}\right)} g/V_\tau^* d\nu = \sum_{i=-\infty}^{n(\tau(y))} \int_{\Omega[t_i,t_{i+1}]} g/V_\tau^* d\nu$$

and instead of (35)

$$I(y) \leq \sum_{i=-\infty}^{n(\tau(y))} \frac{G^*(t_i)\, v_i}{V^*(t_{i+1})^2}.$$

Further, using

$$G^*(t_i)\, v_i \leq \frac{1}{2} \int_{\Omega[t_{i-1},t_i]} G_\tau^* v d\nu, \;\; V^*(t_{i+1}) \geq \frac{1}{4} V_\tau^*(x)\,, \text{ for } \tau(x) \geq t_{i-1},$$

we obtain

$$I(y) \leq 8 \sum_{i=-\infty}^{n(\tau(y))} \int_{\Omega[t_{i-1},t_i]} G_\tau^* (V_\tau^*)^{-2}\, v d\nu \leq 8 \int_{\Omega(\tau(y))} G_\tau^* (V_\tau^*)^{-2}\, v d\nu.$$

◀

Proof. [Proof of Theorem 2] **Part 1**. Here we suppose that $V(\infty) < \infty$.

In the proof of the *lower bound*, one part is trivial: since $f \equiv c > 0$ belongs to $\mathcal{AI}$, we have

$$C_{\mathcal{AI}}(g) \geq G(\infty)\, V(\infty)^{-1/p}. \tag{37}$$

Define $f = Ph$ with $h \geq 0$. Then $f \in \mathcal{AI}$, $\int_\Omega fgd\nu = \int_\Omega hP^*gd\nu$ and by Lemma 3a) we have (32). Therefore,

$$\begin{aligned} C_{\mathcal{AI}}(g) &\geq c_1 \sup_{h\geq 0} \frac{\int_\Omega hP^*gd\nu}{\left[\int_\Omega h^p (V_\tau^*)^p v^{1-p} d\nu\right]^{1/p}} = c_1 \sup_{h\geq 0} \frac{\int_\Omega (hV_\tau^*/v)(G_\tau^*/V_\tau^*)\, v d\nu}{\left[\int_\Omega (hV_\tau^*/v)^p\, v d\nu\right]^{1/p}} \\ (\text{put } u &= hV_\tau^*/v \text{ and use duality } (L_p)' = L_{p'}) \\ &= c_1 \sup_{u\geq 0} \frac{\int_\Omega u (G_\tau^*/V_\tau^*)\, v d\nu}{\left(\int_\Omega u^p v d\nu\right)^{1/p}} = c_1 \left(\int_\Omega (G_\tau^*/V_\tau^*)^{p'}\, v d\nu\right)^{1/p'}. \end{aligned} \tag{38}$$

(37)-(38) prove the lower bound.

Bound from above. Using $f \in \mathcal{AI}$ and Lemma 3b), we have

$$\int_\Omega fgd\nu = \int_\Omega \frac{fg}{V_\tau^*} V_\tau^* d\nu = \int_\Omega \left(P\frac{fg}{V_\tau^*}\right) v d\nu \leq \int_\Omega \left(P\frac{g}{V_\tau^*}\right) f v d\nu$$

$$\leq \ 8\left[\int_{\Omega} fvd\nu G(\infty)/V(\infty) + \int_{\Omega} fvPF^{*}d\nu\right], \tag{39}$$

where $F^{*} = G_{\tau}^{*}(V_{\tau}^{*})^{-2}v$. Here by Hölder's inequality we have (30) and

$$\int_{\Omega} fvPF^{*}d\nu \leq \left(\int_{\Omega} f^{p}vd\nu\right)^{1/p}\left(\int_{\Omega}(PF^{*})^{p'}vd\nu\right)^{1/p'}.$$

Now taking in Lemma 3a) $h = F^{*}$ and p' in place of p gives us

$$\int_{\Omega}(PF^{*})^{p'}vd\nu \leq c_1\int_{\Omega}(F^{*})^{p'}(V_{\tau}^{*})^{p'}v^{1-p'}d\nu = c_1\int_{\Omega}(G_{\tau}^{*}/V_{\tau}^{*})^{p'}vd\nu,$$

which allows us to conclude that

$$\int_{\Omega} fvPF^{*}d\nu \leq c_2\left(\int_{\Omega}(G_{\tau}^{*}/V_{\tau}^{*})^{p'}vd\nu\right)^{1/p'}\left(\int_{\Omega} f^{p}vd\nu\right)^{1/p}. \tag{40}$$

(30), (39) and (40) give the desired estimate.

Part 2. Suppose $V(\infty) = \infty$. As in the proof of Theorem 1, we can assume that $G(\infty) < \infty$, which makes (37) obvious. In the proof of Part 1 of this theorem, just replace everywhere Lemma 3 by Lemma 4, and the desired result will follow. ◄

Proof. [Proof of Corollary 1] a) Using duality, we write

$$\begin{aligned}\left(\int_{\Omega}(Kf)^{q}wd\nu\right)^{1/q} &= \sup_{g\geq 0}\frac{\int_{\Omega}(Kf)\,gwd\nu}{\left(\int_{\Omega} g^{q'}wd\nu\right)^{1/q'}} = \sup_{g\geq 0}\frac{\int_{\Omega} fK^{*}(gw)\,d\nu}{\left(\int_{\Omega} g^{q'}wd\nu\right)^{1/q'}}\\ &= \sup_{h\geq 0}\frac{\int_{\Omega} fK^{*}hd\nu}{\left(\int_{\Omega} h^{q'}w^{1-q'}d\nu\right)^{1/q'}}.\end{aligned}$$

Therefore, for the best constant in (7) we have

$$C = \sup_{f\in\mathcal{AD}}\frac{\left(\int_{\Omega}(Kf)^{q}wd\nu\right)^{1/q}}{\left(\int_{\Omega} f^{p}vd\nu\right)^{1/p}} = \sup_{h\geq 0}\frac{1}{\left(\int_{\Omega} h^{q'}w^{1-q'}d\nu\right)^{1/q'}}\sup_{f\in\mathcal{AD}}\frac{\int_{\Omega} fK^{*}hd\nu}{\left(\int_{\Omega} f^{p}vd\nu\right)^{1/p}}.$$

Applying Theorem 1 with $g = K^{*}h$:

$$C \asymp \sup_{h\geq 0}\frac{1}{\left(\int_{\Omega} h^{q'}w^{1-q'}d\nu\right)^{1/q'}}\left\{\int_{\Omega} K^{*}hd\nu\left(\int_{\Omega} vd\nu\right)^{-1/p}\right.$$

$$+\left[\int_{\Omega}\left(\int_{\Omega(\tau(x))}K^{*}hd\nu/V_{\tau}(x)\right)^{p'}v(x)\,d\nu(x)\right]^{1/p'}\Bigg\}.$$

This proves a). For b), we use Theorem 2. ◄

Proof. [Proof of Theorem 3] According to Corollary 1, we have to check one by one the conditions arising from (8) for $K^{*}=T^{*}$. The first condition is a bound on a linear functional of type

$$\int_{\Omega}T^{*}hd\nu\leq C_{1}\left(\int_{\Omega}h^{q'}w^{1-q'}d\nu\right)^{1/q'}\left(\int_{\Omega}vd\nu\right)^{1/p}. \tag{41}$$

The second condition is

$$\left(\int_{\Omega}(PT^{*}h)^{p'}V_{\tau}^{-p'}vd\nu\right)^{1/p'}\leq C_{2}\left(\int_{\Omega}h^{q'}w^{1-q'}d\nu\right)^{1/q'}.$$

Since $PT^{*}h=Ph+\nu(\Omega_{\tau})P^{*}(mh)$ by (9), where all operators are non-negative, this second condition is equivalent to a combination of two:

$$\left(\int_{\Omega}(Ph)^{p'}V_{\tau}^{-p'}vd\nu\right)^{1/p'}\leq C_{3}\left(\int_{\Omega}h^{q'}w^{1-q'}d\nu\right)^{1/q'}, \tag{42}$$

$$\left(\int_{\Omega}[\nu(\Omega_{\tau})P^{*}(mh)]^{p'}V_{\tau}^{-p'}vd\nu\right)^{1/p'}\leq C_{4}\left(\int_{\Omega}h^{q'}w^{1-q'}d\nu\right)^{1/q'}.$$

The last inequality upon replacing $mh=f$ becomes

$$\left(\int_{\Omega}(P^{*}f)^{p'}(\nu(\Omega_{\tau})/V_{\tau})^{p'}vd\nu\right)^{1/p'}\leq C_{4}\left(\int_{\Omega}f^{q'}m^{-q'}w^{1-q'}d\nu\right)^{1/q'}. \tag{43}$$

Regardless of the relationship between p and q, (41) is easily shown to be equivalent to (10). Namely, if (10) is true, then

$$\int_{\Omega}T^{*}hd\nu=\int_{\Omega}[P^{*}(mh)]\cdot 1\cdot d\nu=\int_{\Omega}mh(P1)\,d\nu=\int_{\Omega}hd\nu$$
$$\leq\left(\int_{\Omega}h^{q'}w^{1-q'}d\nu\right)^{1/q'}\left(\int_{\Omega}wd\nu\right)^{1/q}\leq\alpha\left(\int_{\Omega}h^{q'}w^{1-q'}d\nu\right)^{1/q'}\left(\int_{\Omega}vd\nu\right)^{1/p}$$

and (41) holds. On the other hand, by selecting $h=w$ we can verify that (41) implies (10). (42) and (43) are Hardy inequalities for P and P^{*}. Criteria for their validity are supplied by Theorems A and B, but applying those criteria requires a tedious and careful work.

Case $1 < p \le q < \infty$. $p \le q$ implies $\tilde{q} \equiv p' \ge \tilde{p} \equiv q'$, so for (42) the appropriate statement is Theorem Aa), which gives a necessary and sufficient condition for

$$\left(\int_{\Omega} (Pf)^{\tilde{q}}\, \tilde{u} d\mu\right)^{1/\tilde{q}} \le C \left(\int_{\Omega} f^{\tilde{p}} \tilde{v} d\nu\right)^{1/\tilde{p}}. \tag{44}$$

Comparing (42) and (44), we see that

$$\tilde{u} d\mu = V_{\tau}^{-p'} v d\nu,\ \tilde{v} = w^{1-q'}. \tag{45}$$

We plug these in the function from Theorem Aa):

$$\begin{aligned}
\Psi(x) &= \left(\int_{\Omega\setminus\Omega(\tau(x))} \tilde{u} d\mu\right)^{1/\tilde{q}} \left(\int_{\Omega(\tau(x))} \tilde{v}^{-\tilde{p}'/\tilde{p}} d\nu\right)^{1/\tilde{p}'} \\
&= \left(\int_{\Omega\setminus\Omega(\tau(x))} V_{\tau}^{-p'} v d\nu\right)^{1/p'} \left(\int_{\Omega(\tau(x))} w^{(1-q')(-q/q')} d\nu\right)^{1/q} \\
&= \left(\int_{\Omega\setminus\Omega(\tau(x))} V_{\tau}^{-p'} v d\nu\right)^{1/p'} \left(\int_{\Omega(\tau(x))} w d\nu\right)^{1/q}.
\end{aligned}$$

Thus, the validity of (42) is equivalent to the finiteness of

$$\gamma = \sup_{x\in\Omega} \left(\int_{\Omega\setminus\Omega(\tau(x))} V_{\tau}^{-p'} v d\nu\right)^{1/p'} \left(\int_{\Omega(\tau(x))} w d\nu\right)^{1/q}.$$

Since in (43) we have P^* and $\tilde{p} \le \tilde{q}$, it is appropriate to use Theorem Ba), which states that the bound

$$\left(\int_{\Omega} (P^* f)^{\tilde{q}}\, \tilde{u} d\mu\right)^{1/\tilde{q}} \le C^* \left(\int_{\Omega} f^{\tilde{p}} \tilde{v} d\nu\right)^{1/\tilde{p}} \tag{46}$$

is equivalent to $\sup_{x\in\Omega} \Psi^*(x) < \infty$, where

$$\Psi^*(x) = \left(\int_{\Omega(\tau(x))} \tilde{u} d\mu\right)^{1/\tilde{q}} \left(\int_{\Omega\setminus\Omega(\tau(x))} \tilde{v}^{-\tilde{p}'/\tilde{p}} d\nu\right)^{1/\tilde{p}'}.$$

Comparing (43) and (46), we see that

$$\tilde{u} d\mu = \left[\nu(\Omega_{\tau})/V_{\tau}\right]^{p'} v d\nu,\ \tilde{v} = m^{-q'} w^{1-q'}. \tag{47}$$

Thus,

$$\Psi^*(x) = \left(\int_{\Omega(\tau(x))} [\nu(\Omega_\tau)/V]^{p'} v d\nu\right)^{1/p'} \left(\int_{\Omega\setminus\Omega(\tau(x))} m^q w d\nu\right)^{1/q},$$

which gives (11).

The proof in the case under consideration is concluded with the remark that because of (14), $\alpha < \infty$ implies $\gamma < \infty$.

Case $1 < q < p < \infty$. The assumption $q < p$ translates to $\tilde{q} = p' < \tilde{p} = q'$. Hence, for (42) we need to apply Theorem Ab) with $\tilde{r} = r$ and

$$A = \left\{\int_\Omega \left[\left(\int_{\Omega\setminus\Omega(\tau(x))} \tilde{u} d\mu\right)^{1/\tilde{p}} \left(\int_{\Omega(\tau(x))} \tilde{v}^{-\tilde{p}'/\tilde{p}} d\nu\right)^{1/\tilde{p}'}\right]^r \tilde{u}(x)\, d\mu(x)\right\}^{1/r}.$$

Comparison of (42) and (44) gives (45), which leads to (12).

Finally, for (43) we apply Theorem Bb) with weights and measures (47) obtained from comparing (43) and (46). With this choice, the functional from Theorem Bb) becomes

$$\begin{aligned} C^* &\asymp \left\{\int_\Omega \left[\left(\int_{\Omega(\tau(x))} \tilde{u} d\mu\right)^{1/\tilde{p}} \left(\int_{\Omega\setminus\Omega(\tau(x))} \tilde{v}^{-\tilde{p}'/\tilde{p}} d\nu\right)^{1/\tilde{p}'}\right]^r \tilde{u}(x)\, d\mu(x)\right\}^{1/r} \\ &= \left\{\int_\Omega \left[\left(\int_{\Omega(\tau(x))} (m/V_\tau)^{p'} v d\nu\right)^{1/q'} \left(\int_{\Omega\setminus\Omega(\tau(x))} m^q w d\nu\right)^{1/q}\right]^r \right. \\ &\quad \left. \times\, m(x)/V_\tau(x)^{p'} v(x)\, d\nu(x)\right\}^{1/r}, \end{aligned}$$

which is (13).

The proof is complete. ◀

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