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On the Inverse Scattering Problem for the Schrödinger Equation with Increasing Potential

A.Kh. Khanmamedov*, A.F. Mamedova

Abstract. The one-dimensional Schrödinger equation on the entire axis is considered, the potential of which at the left end grows as a quadratic function, and at the right end as a linear function. The inverse scattering problem is studied by the method of transformation operators. The main integral equations of the Marchenko type are obtained, allowing the inverse problem to be solved.

Key Words and Phrases: Schrödinger equation, Airy function, Macdonald function, growing potential, inverse scattering problem, main integral equations.

2010 Mathematics Subject Classifications: 34A55, 34L40

1. Introduction

The inverse scattering problems for the one-dimensional Schrödinger equation with decreasing potentials and potentials in the form of step functions was thoroughly investigated by numerous authors (see [2, 10, 12] and the references therein). The inverse scattering problem for the Schrödinger operator with infinitely increasing potential is also of significant interest. An important step in this direction was made in [9]. More precisely, the inverse scattering problem was solved for the one-dimensional Schrödinger equation whose potential increases on one side. In this case, according to the results obtained in [6], unlike the Schrödinger equations with decreasing potentials and with potentials in the form of step functions, it is impossible to establish conditions for the scattering data guaranteeing that the potential belongs to the indicated class with conditions imposed on both sides. In turn, this fact is connected with the absence of the main Marchenko-type equation on the left side. In this direction, we note the

*Corresponding author.

works [3, 4, 5, 6, 7, 8, 11], which indicate a class of potentials growing at one end or at both ends, for which the inverse scattering problem is solved effectively.

In this work, the inverse scattering problem for the one-dimensional Schrödinger equation is considered, the potential of which at $x \to \pm\infty$ behaves like $-(-x)^{\frac{3\mp1}{2}}$. Consider the equation

$$-y'' + q(x)y = \lambda y, \qquad -\infty < x < \infty, \tag{1}$$

where the real potential $q(x)$ satisfies the condition

$$\int_{-\infty}^{0} \left(1 + |x|^5\right) e^{2x^2} \left|q(x) + x^2\right| dx + \int_{0}^{\infty} \left(1 + |x|^4\right) e^{2x^{\frac{3}{2}}} \left|q(x) - x\right| dx < \infty. \tag{2}$$

In this paper, we study the inverse scattering problem for the equation (1). The main Marchenko-type integral equations are deduced on both sides. An algorithm for solving the inverse scattering problem is constructed.

2. The scattering problem

Let us consider equation (1). Note that when $x \to \pm\infty$, this equation becomes

$$-y'' \pm x^{\frac{3\mp1}{2}} y = \lambda y \ ,$$

the solutions of which are the functions $e_+^0(x,\lambda) = \sqrt{\pi}\mathrm{A}i(x-\lambda)$ and $e_-^0(x,\lambda) = D_{\frac{i\lambda}{2}-\frac{1}{2}}\left(-\sqrt{2}e^{-\frac{i\pi}{4}}x\right)$, where $\mathrm{A}i(x)$ is the Airy function and $U(a,x) = D_{-a-\frac{1}{2}}(x)$ is the MacDonald function. It is known [1, 13] that the function $e_+^0(x,\lambda)$ for $x \to +\infty$ decreases like $e^{-2x^{\frac{3}{2}}}$ and the function $e_-^0(x,\lambda)$ for $x \to +\infty$ has order $O\left(|x|^{-\frac{1}{2}-Im\lambda}\right)$.

Let us now consider solutions of equation (1) with asymptotics

$$e_\pm(x,\lambda) = e_\pm^0(x,\lambda)\left[1 + o(1)\right], \ x \to \pm\infty.$$

As follows from [4, 6], under condition (2), such solutions exist and the following representations through transformation operators are valid:

$$e_\pm(x,\lambda) = e_\pm^0(x,\lambda) \pm \int_x^{\pm\infty} K^\pm(x,t)\ e_\pm^0(t,\lambda)\,dt. \tag{3}$$

In this case, the kernels $K^\pm(x,t)$ are real-valued continuous functions and satisfy the following relations:

$$\left|K^\pm(x,t)\right| \le \frac{1}{2}\sigma^\pm\left(\frac{x+t}{2}\right) e^{\sigma_1^\pm(x)} \tag{4}$$

$$K^{\pm}(x,x) = \pm\frac{1}{2}\int_x^{\pm\infty}\left[q(t) \mp t^{\frac{3\mp1}{2}}\right]dt, \tag{5}$$

where $\sigma^{+}(x) = \pm\int_x^{\pm\infty}\left|q(t) \mp t^{\frac{3\mp1}{2}}\right|dt$, $\sigma_1^{\pm}(x) = \pm\int_x^{\infty}\sigma^{\pm}(t)\,dt$.

Let us study the relationship between the solutions $e_{\pm}(x,\lambda)$. Due to the reality of the potential $q(x)$, we conclude that for real values of λ, the solution to equation (1) is also $\overline{e_{-}(x,\lambda)}$. Since the Wronskian of two solutions of equation (1) does not depend on x, it coincides with its value at $x \to -\infty$. Then, using (3)-(4) and taking into account the known [4] formulas

$$U\left(-\frac{i\lambda}{2},0\right) = \frac{2^{\frac{i\lambda-1}{4}}\sqrt{\pi}}{\Gamma\left(\frac{3}{4}-\frac{i\lambda}{4}\right)},\ U_x'\left(-\frac{i\lambda}{2},0\right) = -\frac{2^{\frac{i\lambda+1}{4}}\sqrt{\pi}}{\Gamma\left(\frac{1}{4}-\frac{i\lambda}{4}\right)}, \tag{6}$$

we find that the solutions $e_{-}(x,\lambda)$ and $\overline{e_{-}(x,\lambda)}$ are linearly independent and their Wronskian is equal to $i\sqrt{2}e^{\frac{\pi\lambda}{4}}$:

$$W\left\{e_{-}(x,\lambda), \overline{e_{-}(x,\lambda)}\right\} = i\sqrt{2}e^{\frac{\pi\lambda}{4}}. \tag{7}$$

Since the solution $e_{+}(x,\lambda)$ takes real values for all real values of λ, then the expansion takes place:

$$e_{+}(x,\lambda) = a(\lambda)\overline{e_{-}(x,\lambda)} + \overline{a(\lambda)}e_{-}(x,\lambda), \tag{8}$$

where the coefficient $a(\lambda)$ is determined by the formula

$$a(\lambda) = -\frac{W\left[e_{+}(x,\lambda), e_{-}(x,\lambda)\right]}{i\sqrt{2}e^{\frac{\pi\lambda}{4}}}. \tag{9}$$

From relations (3), (4), (6), (9), as well as from the known properties of the Airy function [1] of the first kind and the parabolic cylinder function [1], it follows that $a(\lambda)$ has the asymptotics

$$a(\lambda) = \frac{i}{\sqrt{2}}e^{-\frac{\pi\lambda}{4}}e_{-}^{0}(0,\lambda)\frac{de_{+}^{0}(0,\lambda)}{dx}\left[1+O\left(|\lambda|^{-\frac{1}{2}}\right)\right],\ \lambda\to\infty. \tag{10}$$

Let us examine other properties of the function $a(\lambda)$. By virtue of (3), (4), (9), for real values of λ, the function $a(\lambda)$ is continuous. Moreover, it admits an analytic continuation to the upper half-plane. It should be noted that a function $a(\lambda)$ that is continuous in the closed upper half-plane has no zeros there. Indeed, for real values of λ the function $a(\lambda)$ does not vanish, since otherwise, according to (8), for some v the identity $e_{+}(x,\lambda \equiv 0)$ would hold, which is impossible. If, for $\lambda = \lambda_0$, $Im\lambda_0 > 0$, $a(\lambda)$ is equal to zero, then the solutions $e_+(x, \lambda_0)$ and $e_-(x, \lambda_0)$ are linearly dependent and, hence, $e_+(x, \lambda_0) = ce_-(x, \lambda_0) = \psi(x)$, where c is a constant. Since $e_+(x, \lambda_0)$ exponentially decreases as $x \to +\infty$, and $e_-(x, \lambda_0)$ exponentially decreases as $x \to -\infty$, $\psi(x)$ is an eigenfunction of the equation (1) with eigenvalue λ_0. But for real $q(x)$, equation (1) is self-adjoint, so λ_0 cannot be a complex number.

Now let $Im\lambda > 0$. Following Titchmarsh [13], we further deduce that the functions $\psi_1(x, \lambda)$ and $\psi_2(x, \lambda)$, defined up to a factor in the general theory [13], coincide with $e_+(x, \lambda_0)$ and $e_-(x, \lambda_0)$, respectively. The functions $e_+(x, \lambda_0)$ and $e_-(x, \lambda_0)$ hence serve as eigenfunctions of the continuous spectrum of the equation (1). Repeating the corresponding reasoning from the work [6], we obtain the following formulas for the expansion in eigenfunctions of the continuous spectrum for problem (1):

$$\frac{1}{2\sqrt{2}\pi}\int_{-\infty}^{\infty} e^{-\frac{\pi\lambda}{4}} |t(\lambda)|^2 e_+(x, \lambda) e_+(y, \lambda) d\lambda = \delta(x - y), \tag{11}$$

$$\frac{1}{\sqrt{2}\pi} Re \int_{-\infty}^{\infty} \left[\overline{e_-(x, \lambda)} + r(\lambda) e_-(x, \lambda)\right] e_-(y, \lambda) e^{-\frac{\pi\lambda}{4}} d\lambda = \delta(x - y), \tag{12}$$

where $t(\lambda) = a^{-1}(\lambda)$, $r(\lambda) = \frac{\overline{a(\lambda)}}{a(\lambda)}$. The functions $t(\lambda) = a^{-1}(\lambda)$ and $r(\lambda) = \frac{\overline{a(\lambda)}}{a(\lambda)}$ are called the transmission and reflection coefficients, respectively. The inverse scattering problem for the Schrödinger equation (1) is formulated as the problem of reconstruction of the potential $q(x)$ according to the transmission coefficient $t(\lambda)$.

3. Derivation of the main equations

In the solution of the inverse problem, an important role is played by the so-called main Marchenko-type integral equations, the derivations of which we will now deal with.

Let

$$F^+(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left[2^{-\frac{3}{2}} e^{-\frac{\pi\lambda}{4}} |t(\lambda)|^2 - 1\right] e_+^0(x, \lambda) e_+^0(y, \lambda) d\lambda. \tag{13}$$

Theorem 1. *For each fixed x, the function $K^+(x, y)$ in (3) satisfies the integral equation*

$$F^+(x, y) + K^+(x, y) + \int_x^{\infty} K^+(x, t) F^+(t, y) dt = 0, y > x. \tag{14}$$

Proof. To deduce the main integral equations (14), we use the formula of expansion (11) in the eigenfunctions of the scattering problem. By using the well-known properties of the operators of transformation [10, 12] and relations (3) and (4), we get

$$e_+^0(y,\lambda) = e_+(y,\lambda) + \int_y^\infty K(y,t)e_+(t,\lambda)dt,$$

where the kernel $K(y,t)$ satisfies the inequality similar to (4). By using the decomposition formula (11), we obtain

$$\begin{aligned}&\tfrac{1}{2\sqrt{2}\pi}\int_{-\infty}^{\infty}|t(\lambda)|^2 e_+(x,\lambda)\,e_+^0(y,\lambda)\,e^{-\frac{\pi\lambda}{4}}d\lambda = \delta(x-y)+\\ &+\int_y^\infty K(y,t)\left(\tfrac{1}{2\sqrt{2}\pi}\int_{-\infty}^{\infty}|t(\lambda)|^2 e_+(x,\lambda)\,e_+(y,\lambda)\,d\lambda\right)\delta(x-t)\,dt=\\ &=\delta(x-y)+\int_y^\infty K(y,t)\,\delta(x-t)\,dt=\delta(x-y)+K(y,x)=\delta(x-y).\end{aligned} \tag{15}$$

On the other hand, from (3) and from the well-known [6] expansion formula

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e_+^0(x,\lambda)\,e_+^0(y,\lambda)\,d\lambda = \delta(x-y)$$

we have

$$\begin{aligned}&\tfrac{1}{\pi}\int_{-\infty}^{\infty} e_+(x,\lambda)\,e_+^0(y,\lambda)\,d\lambda = \tfrac{1}{\pi}\int_{-\infty}^{\infty} e_+^0(x,\lambda)\,e_+^0(y,\lambda)\,d\lambda+\\ &+\int_x^\infty K^+(x,t)\left(\tfrac{1}{\pi}\int_{-\infty}^{\infty} e_+^0(t,\lambda)\,e_+^0(y,\lambda)\,d\lambda\right)dt=\\ &=\tfrac{1}{\pi}\int_{-\infty}^{\infty} e_+^0(x,\lambda)\,e_+^0(y,\lambda)\,d\lambda+\int_x^\infty K^+(x,t)\,\delta(t-y)\,dt=\\ &=\delta(x-y)+K^+(x,y).\end{aligned}$$

Comparing this relation with (15), we obtain

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left[2^{-\frac{3}{2}}e^{-\frac{\pi\lambda}{4}}|t(\lambda)|^2-1\right]e_+(x,\lambda)\,e_+^0(y,\lambda)\,d\lambda = -K^+(x,y).$$

Further, from (3), (13) it follows that

$$\begin{aligned}&\tfrac{1}{\pi}\int_{-\infty}^{\infty}\left[2^{-\frac{3}{2}}e^{-\frac{\pi\lambda}{4}}|t(\lambda)|^2-1\right]e_+(x,\lambda)\,e_+^0(y,\lambda)\,d\lambda=\\ &=\tfrac{1}{\pi}\int_{-\infty}^{\infty}\left[2^{-\frac{3}{2}}e^{-\frac{\pi\lambda}{4}}|t(\lambda)|^2-1\right]e_+^0(x,\lambda)\,e_+^0(y,\lambda)\,d\lambda+\\ &+\int_x^\infty K^+(x,t)\left(\tfrac{1}{\pi}\int_{-\infty}^{\infty}\left[2^{-\frac{3}{2}}e^{-\frac{\pi\lambda}{4}}|t(\lambda)|^2-1\right]e_+^0(t,\lambda)\,e_+^0(y,\lambda)\,d\lambda\right)dt=\\ &=F^+(x,y)+\int_x^\infty K^+(x,t)\,F^+(t,y)\,dt.\end{aligned}$$

From the last two relations we obtain the main equation (14). The theorem is proved. ◄

Let us now consider the function $r_0(\lambda)$, defined by the formula

$$r_0(\lambda) = \frac{e^{\frac{i\pi}{4}}\Gamma\left(\frac{1}{2} - \frac{i\lambda}{2}\right)}{\sqrt{2\pi} e^{\frac{\pi\lambda}{4}}}.$$

As follows from [4], the expansion formula

$$\frac{1}{\sqrt{2\pi}} Re \int_{-\infty}^{\infty} e_-^0(x,\lambda)\left[\overline{e_-^0(y,\lambda)} + r_0(\lambda) e_-^0(y,\lambda)\right] e^{-\frac{\pi\lambda}{4}} d\lambda = \delta(x-y). \qquad (16)$$

is valid. Using the expansion formulas (12), (16), we prove the following theorem.

Theorem 2. *For each fixed x, the function $K^-(x,y)$ in (4) satisfies the integral equation*

$$F^-(x,y) + K^-(x,y) + \int_{-\infty}^{x} K^-(x,t) F^-(t,y)\, dt = 0, y < x, \qquad (17)$$

where the kernel $F^-(x,y)$ is defined by the formula

$$F^-(x,y) = \frac{1}{\sqrt{2\pi}} Re \int_{-\infty}^{\infty} [r(\lambda) - r_0(\lambda)]\, e_-^0(x,\lambda)\, e_-^0(y,\lambda)\, e^{-\frac{\pi\lambda}{4}} d\lambda. \qquad (18)$$

Proof. By using the well-known properties of the operators of transformation [10, 12] and relations (3) and (4), we get

$$e_-^0(y,\lambda) = e_-(y,\lambda) + \int_{-\infty}^{y} K_0(y,t) e_-(t,\lambda) dt,$$

where the kernel $K_0(y,t)$ satisfies the inequality similar to (4). For $y < x$, the last formula with regard to (12) then implies

$$\begin{aligned} &\tfrac{1}{\sqrt{\pi}} Re \int_{-\infty}^{\infty} \left[\overline{e_-(x,\lambda)} + r(\lambda) e_-(x,\lambda)\right] e_-^0(y,\lambda)\, e^{-\frac{\pi\lambda}{4}} d\lambda = \delta(x-y) + \\ &+ \int_{-\infty}^{y} K_0(y,t) \left(\tfrac{1}{\sqrt{\pi}} Re \int_{-\infty}^{\infty} \left[\overline{e_-(x,\lambda)} + r(\lambda) e_-(x,\lambda)\right] e_-(t,\lambda)\, e^{-\frac{\pi\lambda}{4}} d\lambda\right) dt = \\ &= \delta(x-y) + \int_{-\infty}^{y} K_0(y,t)\, \delta(x-t)\, dt = \delta(x-y) + K_0(y,x) = \delta(x-y). \end{aligned}$$

On the other hand, using (3), (16), (18), we obtain

$$\begin{aligned} &\tfrac{1}{\sqrt{2\pi}} Re \int_{-\infty}^{\infty} \left[\overline{e_-(x,\lambda)} + r(\lambda) e_-(x,\lambda)\right] e_-^0(y,\lambda)\, e^{-\frac{\pi\lambda}{4}} d\lambda = \\ &= \tfrac{1}{\sqrt{2\pi}} Re \int_{-\infty}^{\infty} \left[\overline{e_-^0(x,\lambda)} + r_0(\lambda) e_-^0(x,\lambda)\right] e_-^0(y,\lambda)\, e^{-\frac{\pi\lambda}{4}} d\lambda + \\ &+ \int_{-\infty}^{x} K^-(x,t) \left(\tfrac{1}{\sqrt{2\pi}} Re \int_{-\infty}^{\infty} \left[\overline{e_-^0(t,\lambda)} + r_0(\lambda) e_-^0(t,\lambda)\right] e_-^0(y,\lambda)\, e^{-\frac{\pi\lambda}{4}} d\lambda\right) dt + \\ &+ \tfrac{1}{\sqrt{2\pi}} Re \int_{-\infty}^{\infty} [r(\lambda) - r_0(\lambda)]\, e_-^0(x,\lambda)\, e_-^0(y,\lambda)\, e^{-\frac{\pi\lambda}{4}} d\lambda + \\ &+ \int_{-\infty}^{x} K^-(x,t) \left(\tfrac{1}{\sqrt{2\pi}} Re \int_{-\infty}^{\infty} [r(\lambda) - r_0(\lambda)]\, e_-^0(t,\lambda)\, e_-^0(y,\lambda)\, e^{-\frac{\pi\lambda}{4}} d\lambda\right) dt = \\ &= \delta(x-y) + K^-(x,y) + F^-(x,y) + \int_{-\infty}^{x} K^-(x,t)\, F^-(t,y)\, dt. \end{aligned}$$

Comparing the last equality with the penultimate one, we obtain the main equation (17). The theorem is proved. ◄

Remark 1. *The study of equations* (14), (17) *together with estimates* (4) *leads to the discovery of the basic properties of the functions* $F^{\pm}(x,y)$. *By imposing these properties as conditions on the functions* $F^{\pm}(x,y)$, *as in [6, 7], one can prove the unique solvability of equations* (14), (17) *in the spaces* $L_p(x,+\infty)$, $L_p(-\infty,x)$, $p=1,2$, *respectively.*

Remark 2. *The basic integral equations allow us to solve the inverse problem. Indeed, let the transmission coefficient* $t(\lambda)$ *be given. Using formulas* (13), (18), *we determine the functions* $F^{\pm}(x,y)$. *Using the basic equations* (14), (17), *we find the functions* $K^{\pm}(x,y)$. *Then we restore the potential* $q(x)$ *using any of the formulas* (5).

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Agil Kh. Khanmamedov
Baku State University, Baku, Azerbaijan
Institute of Mathematics and Mechanics, Baku, Azerbaijan
Azerbaijan University, Baku, Azerbaijan
Baku Engineering University, Khirdalan, Azerbaijan
Western Caspian University, Baku, Azerbaijan
Azerbaijan Technical University, Baku, Azerbaijan
E-mail: agil_khanmamedov@yahoo.com

Afag F. Mamedova
Azerbaijan State University of Economics (UNEC), Baku, Azerbaijan
E-mail: Mammadova Afaq@unec.edu.az

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