Azerbaijan Journal of Mathematics
V. 16, No 1, 2026, January
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2026.1.3

A Geometric Study of Ruled Surfaces Generated by Frenet Vectors via N-Pedal Curves in E^3

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Abstract. This study conducts a geometric investigation of ruled surfaces generated by the tangent, normal, and binormal unit vectors of arc-length parameterized space curves. Utilizing the N-pedal curve construction as a foundational approach, the analysis addresses fundamental geometric properties, including curvature behavior, striction curve geometry, and the distribution parameter. The proposed framework is further applied to computational geometry and geometric modeling, yielding results with relevance to both theoretical research and engineering applications. The findings establish essential geometric principles while offering practical tools for advanced modeling and analysis.

Key Words and Phrases: Frenet frame, space curves, N-pedal curve, ruled surfaces, developable surface, minimal surface.

2010 Mathematics Subject Classifications: 53A04, 53A55, 53A17

1. Introduction

Pedal curves constitute a central topic in differential geometry, defined as the locus of points obtained by projecting a fixed pedal point orthogonally onto the tangents of a generating curve. They display a notable duality with their parent curves and are conceptually related to other dual curve families, such as evolutes, involutes, and Bertrand partner curves [29]. Advances in modern geometry have extended the classical theory of pedal curves through detailed studies of their singularities, curvature behavior, and adaptations within broader geometric contexts. Although their origins trace back to the foundational contributions of Newton and Leibniz, ongoing research continues to underscore their theoretical significance as well as their practical applications in areas including applied mathematics, computer-aided geometric design, and mechanical engineering.

Although the concept of pedal curves originates from classical geometry, recent decades have seen significant modern developments. In 2016, Li and Pei [27] investigated pedal curves of fronts on the sphere. Two years later, in 2018, they further examined pedal curves of frontals in the Euclidean plane [28], establishing additional analytic properties. In 2021, Ceylan and Kara [7] explored pedal and contrapedal curves

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of Bézier curves, providing a link between pedal curve theory and computer-aided geometric design. Most recently, in 2024, Canlı, Şenyurt, and Kaya [5] studied pedal curves obtained from Frenet vectors of space curves and their associated Smarandache curves, while in the same year, Canlı, Şenyurt, Kaya, and Grilli [6] introduced pedal curves generated by alternative frame vectors along with their Smarandache counterparts. Collectively, these contributions demonstrate the continued evolution of pedal curve theory from its classical origins toward broader geometric frameworks and modern applications.

Ruled surfaces form a fundamental category within differential geometry, generated by the continuous motion of a straight line, referred to as the generator, along a guiding curve known as the directrix. They embody a distinctive balance between geometric simplicity and mathematical richness, which accounts for their importance in both theoretical exploration and applied contexts. Owing to their linear construction, ruled surfaces allow for efficient parameterization, exact analytical treatment of geometric problems, and straightforward implementation in engineering and architectural design. While early investigations primarily emphasized developable cases characterized by vanishing Gaussian curvature, more recent studies have extended the scope to include non-developable ruled surfaces, encompassing intricate curvature behaviors and singular structures [10, 33]. Advances in computational methods have further broadened their utility, enabling novel applications in areas such as robotic trajectory planning, computer-aided geometric design, and structural optimization. These developments reaffirm the role of ruled surfaces as central mathematical constructs that not only provide deep theoretical insights but also serve as versatile tools linking abstract geometry with engineering practice. For more details see [2, 17, 18, 19, 20, 21, 22, 23, 25].

Research on ruled surfaces has continued to develop in recent years, with notable contributions spanning both theoretical and applied perspectives. In 2023, Pal and Kumar [31] examined ruled-like surfaces in three-dimensional Euclidean space, laying the groundwork for subsequent investigations. The year 2024 witnessed a surge of interest, with several significant studies: Boyacıoğlu Kalkan and Şenyurt [3] analyzed osculating-type ruled surfaces associated with the Type-2 Bishop frame in E^3; Almoneef and Abdel-Baky [1] introduced a family of timelike constant-axis ruled surfaces within Minkowski 3-space; and Damar [8] proposed an adjoint-curve approach to ruled surfaces. More recent studies from 2025 have advanced the field further: De Carvalho, Domingos, and Santos [9] investigated ruled Ricci surfaces in relation to curves of constant torsion; Pan et al. [32] developed a piecewise ruled approximation method for freeform mesh surfaces in computational geometry; and Elsharkawy, Elsayied, and Refaat [16] introduced quasi ruled surfaces in Euclidean 3-space, extending classical theories to new contexts. Collectively, these works highlight the ongoing evolution of ruled surface theory, bridging classical geometry with modern computational and applied frameworks. There are some applications related to this topic in [4, 11, 12, 13, 14, 15, 24, 26, 30]

The structure of this paper is as follows:

- **Section 2**: Introduces the necessary geometric preliminaries, including the Frenet-frame formalism, the construction of N-pedal curves, and fundamental aspects of ruled surfaces.

- **Section 3**: Develops three principal ruled surfaces derived from the Frenet frame of a unit-speed curve $\alpha(s)$, with the associated N-pedal curve $\alpha_N(s)$ taken as the base curve. The study focuses on the *Tangent Ruled Surface* defined by $\mathbf{T}(s)$, the *Normal Ruled Surface* constructed from $\mathbf{N}(s)$, and the *Binormal Ruled Surface* generated by $\mathbf{B}(s)$. Each surface is examined in detail with respect to its fundamental forms, curvature characteristics, striction curve, and distribution parameter.
- **Section 4**: Explores practical applications of the constructed surfaces in computational design and geometric modeling.
- **Section 5**: Concludes with a summary of the principal contributions and suggests potential future research directions.

2. Preliminaries

This section introduces the fundamental geometric concepts that form the basis of this paper: the Frenet frame for unit-speed curves, pedal curves, and ruled surfaces in $\mathbb{E}^3$ (Euclidean 3-space).

The standard inner product in $\mathbb{E}^3$ is

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3,$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ are arbitrary vectors in $\mathbb{E}^3$ [10].

Let $\alpha(s)$ be a unit-speed curve in $\mathbb{E}^3$ parameterized by arc length s. The set $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is the Frenet frame along the curve $\alpha(s)$, where $\mathbf{T}(s)$, $\mathbf{N}(s)$, and $\mathbf{B}(s)$ are the tangential, normal, and binormal unit vector fields, respectively. These vectors are mutually orthonormal and defined as follows:

$$\mathbf{T}(s) = \alpha'(s), \quad \mathbf{N}(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \quad \mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s), \tag{1}$$

where differentiation with respect to s is indicated by a prime [10, 33].

For a unit-speed curve, the Frenet–Serret equations are

$$\begin{aligned} \mathbf{T}'(s) &= \kappa(s)\mathbf{N}(s), \\ \mathbf{N}'(s) &= -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s), \\ \mathbf{B}'(s) &= -\tau(s)\mathbf{N}(s). \end{aligned} \tag{2}$$

The curvature and torsion of the curve $\alpha(s)$ are

$$\kappa(s) = \|\mathbf{T}'(s)\|, \qquad \tau(s) = -\langle \mathbf{B}'(s), \mathbf{N}(s) \rangle.$$

The N-pedal curve α_N of a regular unit-speed curve $\alpha(s)$ with respect to a point P in $\mathbb{E}^3$ is

$$\alpha_N = \alpha(s) + \langle P - \alpha(s), \mathbf{N}(s) \rangle \mathbf{N}(s),$$

where P is the pedal point and $\langle P-\alpha(s),\mathbf{N}(s)\rangle$ is the signed distance from P to the normal line of $\alpha(s)$ at $\alpha(s)$ [7, 29].

If P is the origin, then

$$\chi(s)=-\langle\alpha(s),\mathbf{N}(s)\rangle,$$

and

$$\alpha_N=\alpha(s)+\chi(s)\mathbf{N}(s). \tag{3}$$

This formulation provides a simpler representation, where $\chi(s)$ can be interpreted as the signed distance from the curve point $\alpha(s)$ to the foot of the perpendicular dropped from the origin onto the normal line at that point, see Figure(1).
The unit tangent vector $\mathbf{T_2}$ of the N-pedal curve associated with $\alpha(s)$ can be expressed in terms of the Frenet frame of $\alpha(s)$ as [5, 6]

$$\mathbf{T_2}=\omega_2\big(1-\kappa\chi\big)\mathbf{T}+\omega_2\chi'\mathbf{N}+\omega_2\tau\chi\mathbf{B},\qquad \omega_2=\frac{1}{\sqrt{\big(1-\kappa\chi\big)^2+(\chi')^2+(\tau\chi)^2}}. \tag{4}$$

Example 1. *[5] With respect to the origin $O(0,0)$, the N-pedal curve of an ellipse is $\alpha(t)=(2\cos t,\sin t)$, see Figure (2).*
The N-pedal curve is given by

$$\alpha_N(t)=\Big(\frac{6\cos t\sin^2 t}{1+3\sin^2 t},\ -\frac{3\sin t\cos^2 t}{1+3\sin^2 t}\Big).$$

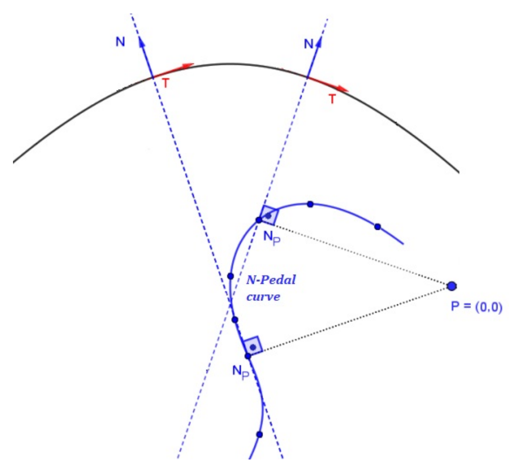


Figure 1: The regular curve (black) and its N-pedal curve (blue).

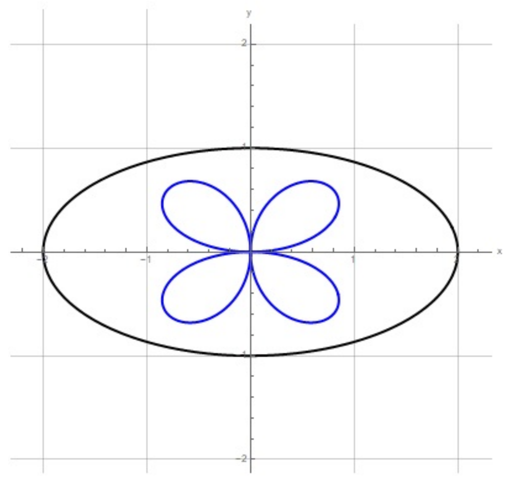


Figure 2: The N-pedal curve (blue) of the ellipse (black) with respect to the origin.

As $\mathbf{X}$ moves along $\alpha(s)$, its associated N-pedal curve α_N generates a ruled surface with regular parametrization

$$R(s,v) = \alpha_N + v\,\mathbf{X}(s), \tag{5}$$

where α_N is the base curve and $\mathbf{X}(s)$ the ruling of the surface R [3, 31]. The striction curve and distribution parameter of R are

$$\beta_X(s) = \alpha_N - \frac{\langle \mathbf{T_2}, \mathbf{X}' \rangle}{\|\mathbf{X}'\|^2}\mathbf{X}(s), \tag{6}$$

$$\lambda_X(s) = \frac{\det(\mathbf{T_2}, \mathbf{X}, \mathbf{X}')}{\|\mathbf{X}'\|^2}. \tag{7}$$

The standard unit normal field $\mathbf{n}$ on a surface R is

$$\mathbf{n} = \frac{R_s \times R_v}{\|R_s \times R_v\|}, \tag{8}$$

where R_s and R_v are the partial derivatives of R with respect to s and v.

The geometry of the ruled surface $R(s,v)$ is described by its fundamental forms [16, 33]. The first fundamental form (FFF) is

$$I = E\,ds^2 + 2F\,ds\,dv + G\,dv^2, \tag{9}$$

with $E = \langle R_s, R_s\rangle$, $F = \langle R_s, R_v\rangle$, and $G = \langle R_v, R_v\rangle$. The second fundamental form (SFF) is

$$II = L\,ds^2 + 2M\,ds\,dv + N\,dv^2, \tag{10}$$

with $L = \langle R_{ss}, \mathbf{n}\rangle$, $M = \langle R_{sv}, \mathbf{n}\rangle$, and $N = \langle R_{vv}, \mathbf{n}\rangle$. The third fundamental form (TFF) is

$$III = e\,ds^2 + 2f\,ds\,dv + g\,dv^2, \tag{11}$$

with $e = \langle \mathbf{n}_s, \mathbf{n}_s\rangle$, $f = \langle \mathbf{n}_s, \mathbf{n}_v\rangle$, and $g = \langle \mathbf{n}_v, \mathbf{n}_v\rangle$.

The Gaussian curvature K and mean curvature H of the ruled surface are

$$K = \frac{LN - M^2}{EG - F^2}, \qquad H = \frac{EN - 2FM + GL}{2(EG - F^2)}. \tag{12}$$

3. Ruled Surfaces Generated by N-Pedal Curves

This section introduces three types of ruled surfaces generated by the N-pedal curve, where $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are the Frenet frame for the unit-speed curve $\alpha(s)$, and discusses their fundamental properties.

Definition 1. *Let α_N be the N-pedal curve of the unit-speed curve $\alpha(s)$ with Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$. Then the parametric representations of the ruled surfaces R^T, R^N, and R^B using* (5) *are*

$$R^T(s,v) = \alpha_N + v\mathbf{T}(s), \qquad R^N(s,v) = \alpha_N + v\mathbf{N}(s), \qquad R^B(s,v) = \alpha_N + v\mathbf{B}(s). \tag{13}$$

These are the Tangent, Normal, *and* Binormal *ruled surfaces, respectively.*

3.1. R^T Tangent Ruled Surface

Definition 2. *The Tangent Ruled Surface R^T is*

$$R^T(s,v) = \alpha(s) + v\mathbf{T}(s) + \chi(s)\mathbf{N}(s), \tag{14}$$

derived from (13) *and* (3).

Theorem 1. *The FFF of the surface R^T is*

$$I = \big(1-\chi\kappa\big)^2 + \big(\chi' + v\kappa\big)^2 + \big(\chi\tau\big)^2 \, ds^2 + 2\big(1-\chi\kappa\big) ds\, dv + dv^2. \tag{15}$$

Proof. From (14) and (2),

$$R^T_s = \big(1-\chi\kappa\big)\mathbf{T} + \big(\chi' + v\kappa\big)\mathbf{N} + \chi\tau\mathbf{B}, \qquad R^T_v = \mathbf{T}(s). \tag{16}$$

From (9), the coefficients are

$$E = \big(1-\chi\kappa\big)^2 + \big(\chi' + v\kappa\big)^2 + \big(\chi\tau\big)^2, \qquad F = 1-\chi\kappa, \qquad G = 1. \tag{17}$$

◄

Theorem 2. *The SFF of R^T is*

$$II = \frac{\chi\tau(\chi'' + \kappa + v\kappa' - \chi(\kappa^2+\tau^2)) - (\chi' + v\kappa)((2\chi' + v\kappa)\tau + \chi\tau')}{\sqrt{\chi^2\tau^2 + (\chi' + v\kappa)^2}} ds^2 + \frac{2\chi\kappa\tau}{\sqrt{(\chi\tau)^2 + (\chi' + v\kappa)^2}} ds\, dv. \tag{18}$$

Theorem 3. *The Gaussian curvature K and mean curvature H of R^T are*

$$K = \frac{-(\chi\kappa\tau)^2}{\big((\chi\tau)^2 + (\chi' + v\kappa)^2\big)^2},$$
$$H = \frac{\tau\left(\chi^2(\kappa^2-\tau^2) + \chi(\chi'' + v\kappa' - \kappa) - 2(\chi')^2 - 3\chi' v\kappa - v^2\kappa^2\right) - \chi\tau'(\chi' + v\kappa)}{2\big((\chi\tau)^2 + (\chi' + v\kappa)^2\big)^{3/2}}. \tag{19}$$

Corollary 1. *(a) The surface R^T is developable iff $\kappa = 0$ and $\tau \neq 0$, with*

$$H = \frac{\chi\tau(\chi'' - \chi\tau^2) - \chi'(2\chi'\tau + \chi\tau')}{2\left(\chi^2\tau^2 + (\chi')^2\right)^{3/2}}.$$

(b) The surface R^T is both developable and minimal iff the base curve $\alpha(s)$ is planar $(\tau = 0)$.

Theorem 4. *The striction curve $\beta_T(s)$ for R^T is*

$$\beta_T(s) = \alpha(s) - \frac{\omega_2\chi'}{\kappa}\mathbf{T} + \chi\,\mathbf{N}.$$

Theorem 5. *The distribution parameter λ_T for R^T is*

$$\lambda_T = \frac{\omega_2\tau\chi}{\kappa}.$$

3.2. R^N Normal Ruled Surface

Definition 3. *The Normal Ruled Surface R^N is*

$$R^N(s,v) = \alpha(s) + (\chi(s) + v)\,\mathbf{N}(s), \tag{20}$$

derived from (13) *and* (3)*.*

Theorem 6. *The FFF of R^N is*

$$I = \Big[\big(1-(\chi+v)\kappa\big)^2 + (\chi')^2 + \big((\chi+v)\tau\big)^2\Big]ds^2 + 2\chi'\,ds\,dv + dv^2. \tag{21}$$

Theorem 7. *The SFF of R^N is*

$$II = L\,ds^2 + 2M\,ds\,dv + N\,dv^2, \tag{22}$$

with

$$\begin{aligned} L &= \frac{\big(-2\chi'\kappa - (\chi+v)\kappa'\big)\big(-(\chi+v)\tau\big) + \big(2\chi'\tau + (\chi+v)\tau'\big)\big(1-(\chi+v)\kappa\big)}{\sqrt{(\chi+v)^2\tau^2 + \big(1-(\chi+v)\kappa\big)^2}}, \\ M &= \frac{\tau}{\sqrt{\big((\chi+v)\tau\big)^2 + \big(1-(\chi+v)\kappa\big)^2}}, \qquad N = 0. \end{aligned} \tag{23}$$

Theorem 8. *The Gaussian curvature K and mean curvature H of R^N are*

$$\begin{aligned} K &= -\frac{\tau^2}{\Big(\big((\chi+v)\tau\big)^2 + \big(1-(\chi+v)\kappa\big)^2\Big)^2}, \\ H &= \frac{(\chi+v)\left[(\chi+v)\kappa'\tau + \tau'\left(1-(\chi+v)\kappa\right)\right]}{2\Big((1-(\chi+v)\kappa)^2 + ((\chi+v)\tau)^2\Big)^{3/2}}. \end{aligned}$$

Corollary 2. *(a) R^N is minimal iff $\kappa = 0$ and $\tau \neq 0$, with*

$$K = -\frac{\tau^2}{\big(1+\tau^2(\chi+v)^2\big)^2}.$$

(b) R^N is both developable and minimal iff the base curve $\alpha(s)$ is planar ($\tau = 0$).

Theorem 9. *The striction curve $\beta_N(s)$ for R^N is*

$$\beta_N(s) = \alpha(s) + \left(\chi + \frac{\omega_2\kappa}{\kappa^2+\tau^2} - \omega_2\chi\right)\mathbf{N}.$$

3.3. R^B Binormal Ruled Surface

Definition 4. *The Binormal Ruled Surface R^B is*

$$R^B(s,v) = \alpha(s) + \chi(s)\,\mathbf{N}(s) + v\,\mathbf{B}(s). \tag{24}$$

Theorem 10. *The FFF of R^B is*

$$I = \Big[\big(1-\chi\kappa\big)^2 + \big(\chi' - v\tau\big)^2 + \big(\chi\tau\big)^2\Big]\,ds^2 + 2\chi\tau\,ds\,dv + dv^2. \tag{25}$$

Theorem 11. *The SFF of R^B is*

$$II = L\,ds^2 + 2M\,ds\,dv + N\,dv^2, \tag{26}$$

with

$$\begin{aligned} L &= \frac{\big(-2\chi'\kappa - \chi\kappa' + v\tau\kappa\big)(\chi' - v\tau) - \big((1-\chi\kappa)\kappa + \chi'' - v\tau' - \chi\tau^2\big)(1-\chi\kappa)}{\sqrt{(\chi'-v\tau)^2 + (1-\chi\kappa)^2}},\\ M &= \frac{\tau(1-\chi\kappa)}{\sqrt{(\chi'-v\tau)^2 + (1-\chi\kappa)^2}}, \qquad N = 0. \end{aligned} \tag{27}$$

Theorem 12. *The Gaussian curvature K and mean curvature H of R^B are*

$$K = -\frac{\tau^2(1-\chi\kappa)^2}{\big((\chi'-v\tau)^2 + (1-\chi\kappa)^2\big)^2},$$

$$\begin{aligned} H =& \frac{\big(-2\chi'\kappa + v\tau\kappa - \chi\kappa'\big)(\chi'-v\tau) - \big((1-\chi\kappa)\kappa + \chi'' - v\tau' - \chi\tau^2\big)(1-\chi\kappa)}{\big((\chi'-v\tau)^2 + (1-\chi\kappa)^2\big)^{3/2}}\\ &- \frac{2\tau^2\chi(1-\chi\kappa)}{\big((\chi'-v\tau)^2 + (1-\chi\kappa)^2\big)^{3/2}}. \end{aligned}$$

Corollary 3. *If the base curve $\alpha(s)$ is planar ($\tau = 0$), then R^B is developable ($K = 0$) with*

$$H = \frac{\chi'\big(-2\chi'\kappa - \chi\kappa'\big) - \big((1-\chi\kappa)\kappa + \chi''\big)(1-\chi\kappa)}{\big((\chi')^2 + (1-\chi\kappa)^2\big)^{3/2}}.$$

4. Examples

Example 2. *Let $\alpha(s)$ be a general helix:*

$$\alpha(s) = \left(4\cos\frac{s}{5},\ 4\sin\frac{s}{5},\ \frac{3s}{5}\right).$$

By (1)*, the Frenet frame is*

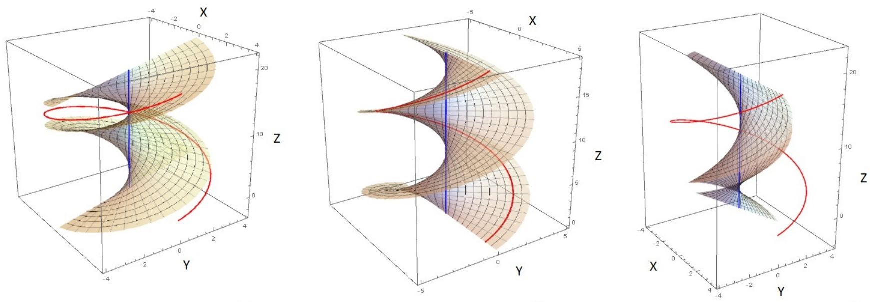
$$\mathbf{T}(s) = \left(-\frac{4}{5}\sin\frac{s}{5}, \frac{4}{5}\cos\frac{s}{5}, \frac{3}{5}\right),$$
$$\mathbf{N}(s) = \left(-\cos\frac{s}{5}, -\sin\frac{s}{5}, 0\right),$$
$$\mathbf{B}(s) = \left(\frac{3}{5}\sin\frac{s}{5}, -\frac{3}{5}\cos\frac{s}{5}, \frac{4}{5}\right).$$

From (3)*,*

$$\alpha_N(s) = \left(0, 0, \tfrac{3s}{5}\right),$$

where $\chi(s) = -\langle \alpha(s), \mathbf{N}(s)\rangle = 4$*. Consequently,*

$$R^T(s,v) = \alpha_N(s) + v\,\mathbf{T}(s) = \left(-\frac{4v}{5}\sin\frac{s}{5},\ \frac{4v}{5}\cos\frac{s}{5},\ \frac{3}{5}(s+v)\right),$$
$$R^N(s,v) = \alpha_N(s) + v\,\mathbf{N}(s) = \left(-v\cos\tfrac{s}{5},\ -v\sin\tfrac{s}{5},\ \tfrac{3s}{5}\right),$$
$$R^B(s,v) = \alpha_N(s) + v\,\mathbf{B}(s) = \left(\frac{3v}{5}\sin\frac{s}{5},\ -\frac{3v}{5}\cos\frac{s}{5},\ \frac{3s}{5} + \frac{4v}{5}\right).$$



Tangent Ruled Surface (R^T) *Normal Ruled Surface* (R^N) *Binormal Ruled Surface* (R^B)

Figure 3: The helix (red) and T-pedal curve (blue); viewpoint $(3, -2, 1.5)$.

Example 3. *Let* $\alpha(s)$ *be a regular curve:*

$$\alpha(s) = \left(\frac{3}{2}\cos\frac{s}{2} + \frac{1}{6}\cos\frac{3s}{2}, \frac{3}{2}\sin\frac{s}{2} + \frac{1}{6}\sin\frac{3s}{2}, \sqrt{3}\cos\frac{s}{2}\right).$$

By (1)*, the Frenet frame is*

$$\mathbf{T}(s) = \left(-\sin^3\frac{s}{2}, \cos^3\frac{s}{2}, -\frac{\sqrt{3}}{2}\sin\frac{s}{2}\right),$$
$$\mathbf{N}(s) = \left(-\cos\frac{s}{2}, -\sin\frac{s}{2}, 0\right),$$
$$\mathbf{B}(s) = \left(\frac{\sqrt{3}}{2}\sin\frac{s}{2}, -\frac{\sqrt{3}}{2}\cos\frac{s}{2}, \frac{1}{2}\right).$$

Then, by (3),

$$\alpha_N(s) = \left(3\cos\frac{s}{2} - \frac{10}{3}\cos^3\frac{s}{2},\ -2\sin\frac{s}{2} + \frac{10}{3}\sin^3\frac{s}{2},\ \frac{\sqrt{3}}{3}\cos\frac{s}{2}\right),$$

where $\chi(s) = -\langle \alpha(s), \mathbf{N}(s)\rangle = \frac{4\sqrt{3}}{3}\cos\frac{s}{2}$. *Thus,*

$$\begin{aligned}
R^T(s,v) = &\left(3\cos\frac{s}{2} - \frac{10}{3}\cos^3\frac{s}{2} + v\left(-\frac{3}{2}\sin\frac{s}{2} + \sin^3\frac{s}{2}\right),\right.\\
&\left. -2\sin\frac{s}{2} + \frac{10}{3}\sin^3\frac{s}{2} + v\cos^3\frac{s}{2},\ \frac{\sqrt{3}}{3}\cos\frac{s}{2} - \frac{\sqrt{3}}{2}v\sin\frac{s}{2}\right),\\
R^N(s,v) = &\left(3\cos\frac{s}{2} - \frac{10}{3}\cos^3\frac{s}{2} - \frac{\sqrt{3}}{2}v\cos s,\right.\\
&\left. -2\sin\frac{s}{2} + \frac{10}{3}\sin^3\frac{s}{2} - \frac{\sqrt{3}}{2}v\sin s,\ \frac{\sqrt{3}}{3}\cos\frac{s}{2} - \frac{v}{2}\right),\\
R^B(s,v) = &\left(3\cos\frac{s}{2} - \frac{10}{3}\cos^3\frac{s}{2} - \frac{v}{2}\cos\frac{s}{2}\left(1 + 2\sin^2\frac{s}{2}\right),\right.\\
&\left. -2\sin\frac{s}{2} + \frac{10}{3}\sin^3\frac{s}{2} - v\sin^3\frac{s}{2},\ \frac{\sqrt{3}}{3}\cos\frac{s}{2} + \frac{\sqrt{3}}{2}v\cos\frac{s}{2}\right).
\end{aligned}$$

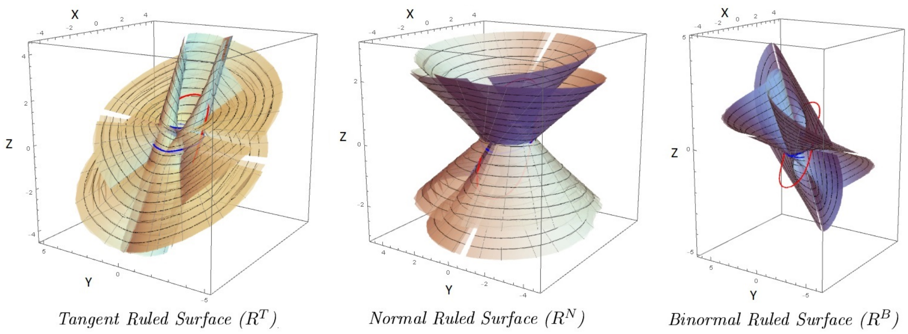


Figure 4: Regular curve (red) and T-pedal curve (blue); viewpoint $(3, -2, -1)$.

5. Conclusion

The present study develops a differential-geometric framework for ruled surfaces generated by the tangent, normal, and binormal vector fields of arc-length parameterized space curves, with the N-pedal curve construction as a central tool. Fundamental properties such as curvature behavior, striction curve geometry, and the distribution parameter are explicitly characterized, strengthening the theoretical foundation of classical surface geometry and extending its utility to computational geometry and parametric surface modeling.

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Received 27 July 2025
Accepted 12 October 2025