Azerbaijan Journal of Mathematics
V. 16, No 1, 2026, January
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2026.1.154

On the Asymptotics of the Solution to a Boundary Value Problem for a Quasilinear Hyperbolic Equation in an Infinite Strip

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Abstract. In an infinite strip, a boundary value problem for a perturbed quasilinear second order equation of hyperbolic type degenerating into a parabolic equation is considered. Asymptotic expansion of generalized solution is constructed to within any positive degree of small parameter, and the residual term is estimated.

Key Words and Phrases: asymptotics, boundary layer type function, residual term.
2010 Mathematics Subject Classifications: 35L81, 35L72

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1. Introduction

When studying numerous real phenomena with irregular transitions from one physical characteristics to an other, it is necessary to study boundary value problems for differential equations containing a small parameter for higher derivatives (see e.g., [4, 29]).

Mathematical models of real physical, chemical, biological and other processes, are usually idealized, since when composing these models one has to neglect these or other small quantities. Naturally, these arises a question as to how much the discarding of these small quantities distort the true picture of the phenomenon. Thus, there arises a mathematical problem of dependence of solution of differential equations on small parameters. Various methods for solving this problem can be combined under the general name "small parameter methods". Lyapunov and Pioncare (see [15]) mentioned first among the authors that laid the foundations of small parameter methods.

Originally, singularly perturbed boundary value problems were studied from various points of view by Tikhonov, [28], Pontryagin [16], Vishik and Lusternik

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[31, 32], Lomov [8], Il'in [5], etc. There are various methods for construction of asymptotic expansions of the solutions to boundary value problems for singularly perturbed linear differential equations. Among them, the method developed by Vishik and Lusternik (see [31],[32]) has an undoubted advantage.

There are a lot of monographs dedicated to the asymptotic methods for solving singularly perturbed differential equations, for example, Moiseev [9], Cole [2], Mishenko and Rozov [10], Naife [12]. We can also mention the books by Javadov, Sabzaliev, Javadova [7], and Sabzaliev and Sabzalieva [17].

It is well known that linear models do not always adequately describe real processes. Therefore, the study of boundary value problems for singularly perturbed nonlinear differential equations has a special practical and theoretical importance. The Vishik-Lusternik method for constructing the asymptotics in a small parameter of solutions to boundary value problems for linear differential equations is transferred to some classes of nonlinear differential equations. However the study of nonlinear singularly perturbed boundary value problems by this method is accompanied by bulky calculations. The study of each linear equation requires special approach from the researcher. When studying such nonlinear equations one has to overcome some complex problems that are easily solved for linear equations. In [33], Vishik and Lusternik offered a method for constructing the solution of the following boundary value problem for nonlinear differential equation:

$$\varepsilon y^{''} + \varphi(x, y)y' + \psi(x, y) = 0,\ y(0) = A,\ y(1) = B.$$

The asymptotics of solution to this problem involving the powers of the parameter A was studied by Vazov.

Lets give a brief overview of results obtained in the theory of nonlinear singularly perturbed differential equations.

In [6], Yu Chen constructed asymptotics in a small parameter of the solution of a mixed problem for the following quasilinear hyperbolic equation:

$$\varepsilon\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right) - \varphi(t, x, u)\frac{\partial u}{\partial t} + \psi(t, x, u) = 0.$$

In [2], Trenogin established the asymptotics in a small parameter of the solution to the boundary value problem

$$\frac{\partial u}{\partial t} - \varepsilon b(x, t)\frac{\partial^2 u}{\partial x^2} + c(x, t, u) = 0, \quad u(x, 0) = \varphi(x),\ u(0, t) = u(l, t) = 0.$$

In [36], Lunin constructed the total asymptotics of the solution to the Dirichlet problem for the following elliptic equation:

$$-\varepsilon^4 \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right)^3 - \varepsilon^2 \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x^2} + F(x, u) = 0.$$

The paper [11] considers the problem $u_t = \varepsilon\Delta + |u|^{p-1} u$, $u(x,t) = 0$ in $\partial\Omega \times (0, +\infty)$, $u(x, 0) = \varphi(x)$ in Ω , with $p > 1$, $\varepsilon > 0$ in a bounded domain Ω of R^N, where $\varphi(x)$ is a continuous function.

In [35], Khapaev studied the following singularly perturbed problem

$$\varepsilon^2(u_t - \Delta u) = f(u, x, t),$$

where $\dim x = 2,\ x \in D,\ 0 < t < +\infty,\ u|_{\partial D} = 0,\ u(x, 0) = 0.$

For the elliptic equation

$$\varepsilon^2\Delta u = F(u, x, y, \varepsilon),$$

Denisov [3] has considered the Dirichlet problem.

The Cauchy problem for the quasilinear parabolic equation

$$\frac{\partial u}{\partial t} + \frac{\partial\varphi(u)}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad x \in R,$$

was considered by Zakharov in [37].

We should also note the author's papers [18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. Today, constructing the asymptotics of the solution of singularly perturbed boundary value problem is very important, and the research in this direction is successfully continued. Let's mention some of them.

In [38], asymptotic behavior of solutions to the Cauchy problem for a quasilinear parabolic equation with a small parameter at a higher derivative has been studied near singular points.

In [34], a singularly perturbed periodic problem for a Burgers type reaction-diffusion-advection parabolic equation with a modular advection and linear amplification has been considered. The conditions of the existence, uniqueness and Lyapunov asymptotic stability of the periodic solution with an internal transition layer have been obtained and asymptotic approximation of the solution has been constructed.

In [39], formal asymptotic expansion of the solution to the Cauchy problem for a singularly perturbed differential-operator transfer equation with small diffusion and nonlinearity is constructed in a critical state. It is shown that the principal term of the asymptotic of solution is determined as the solution of the Cauchy problem for a Burgers type parabolic equation. The estimates for residual terms are given.

In [14], singularly perturbed special type transfer equations. Formal asymptotic expansion of the solution to the Cauchy problem is constructed for a singularly perturbed differential-operator transfer equation with small nonlinearity and diffusion in the case of many spatial variables. The principal term of the asymptotics is described by the solution of the quasilinear parabolic equation. The residual term is estimated.

The above review shows the following characteristics of considered singularly perturbed nonlinear differential equations. First, a great majority of these equations degenerate for $\varepsilon > 0$ into functional or ordinary differential equations.

Secondly, in all these equations the derivatives of the sought function appear in the equation linearly, while only the sought function itself appears in the equation nonlinearly. Thirdly, a great majority of the considered boundary value problems refer to the equations of elliptic and parabolic type. Finally, almost all boundary value problems are considered only in bounded domains.

Much less works have been dedicated to the study of singularly perturbed hyperbolic equations than the works related to elliptic and parabolic equations.

In [1, 2, 13], boundary value problems for nonlinear singularly perturbed differential equations are considered only in bounded domains.

In [26], complete asymptotics of the solution to the mixed problem in a rectangle is constructed for a singularly perturbed quasilinear hyperbolic equation. Mixed problem in a semi-infinite strip for a quasilinear hyperbolic equation has been studied in [27].

In the present paper, we consider a boundary value problem in an infinite strip for a quasilinear hyperbolic equation degenerating into a parabolic equation.

2. Problem statement and iteration processes

In the present paper, we consider the following boundary value problem in the infinite strip $\Pi = \{(t,x)|0 \leq t \leq T, -\infty < x < \ < +\infty\}$:

$$L_\varepsilon U \equiv \varepsilon \frac{\partial^2 u}{\partial t^2} + \varepsilon^{p-1} \left(\frac{\partial u}{\partial t} \right)^p + \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + au - f(t,x) = 0, \tag{1}$$

$$u|_{t=0} = 0, \quad \left. \frac{\partial u}{\partial t} \right|_{t=0}, \quad (-\infty < x < +\infty), \tag{2}$$

$$\lim_{|x| \to +\infty} u = 0, \ (0 \leq t \leq T), \tag{3}$$

where $\varepsilon > 0$ is a small parameter, $p = 2k + 1, k$ is an arbitrary positive integer, $a > 0$ is a constant, and $f(t,x)$ is a given function.

Our aim is to construct an asymptotic expansion of generalized solution to problem (1)-(3). It should be noted that it is not possible to construct an asymptotic solution for the considered problem in the traditional way. In this connection, the first iteration process and the iteration process for constructing layer type functions are embedded inside each other. Furthermore, the "notion of satisfying a boundary condition approximately to within any positive power of a small parameter", is introduced. In what follows, this notion is used to estimate the residual term. It should be noted that such an approach was used in [22].

In the first iterative process, approximate solution of equation (1) is sought in the form

$$W = W_0(t,x) + \varepsilon W_1(t,x) + ... + \varepsilon^n W_n(t,x), \tag{4}$$

and the functions $W_i(t,x); i = 0,1,...,n$ are chosen so that

$$L_\varepsilon W = O(\varepsilon^{n+1}). \tag{5}$$

To define W_i, we obtain the following recurrently connected equations

$$L_0 W_i \equiv \frac{\partial W_i}{\partial t} - \frac{\partial^2 W_i}{\partial x^2} + aW_i = f_i(t,x); \quad i = 0,1,...,n. \tag{6}$$

Here $f_0(t,x) = f(t,x); f_j(t,x)$ are the known functions depending on the first and second derivatives of $W_0, W_1, ..., W_{j-1}, \quad j = 1,2,...,n$. Equations (6) will be solved under the following boundary conditions:

$$\lim_{|x|\to+\infty} W_i = 0; \quad i = 0,1,...,n. \tag{7}$$

For equations (6) we will use the first initial condition from (2). Then the second initial condition from (2) will be lost. To compensate the lost initial condition a boundary layer type function should be constructed near $t = 0$.

To construct a boundary layer type function near the boundary $t = 0$, the second decomposition of the operator L_ε should be written near this boundary. We make change of variables $t = \varepsilon\tau, \quad x = x$. Let $r_i(\tau,x)$ be some smooth functions. Then for the function $r = \sum_{j=0}^{n+1} \varepsilon^j r_j(\tau,x)$ the expansion of $L_\varepsilon\ (r)$ in powers of ε in the coordinates (τ,x) will be of the form

$$L_{\varepsilon,1} r \equiv \varepsilon^{-1} \left\{ \frac{\partial^2 r_0}{\partial \tau^2} + \left(\frac{\partial r_0}{\partial \tau}\right)^{2k+1} + \frac{\partial r_0}{\partial \tau} + \sum_{j=0}^{n+1} \left[\frac{\partial r_0}{\partial \tau} + \right.\right.$$

$$\left.\left. +(2k+1)\left(\frac{\partial r_0}{\partial \tau}\right)^{2k} \frac{\partial r_j}{\partial \tau} + \frac{\partial r_j}{\partial \tau} + Q_j(r_0, r_1, ..., r_{j-1}) \right] \right. + O(\varepsilon^{n+1}). \tag{8}$$

Here $Q_j(r_0, r_1, ..., r_{j-1})$ denote the known functions depending on $r_0, r_1, ..., r_{j-1}$ and their derivatives. Explicit formulas for Q_j are rather bulky, therefore we give formulas only for Q_1 and Q_2:

$$Q_1 = -\frac{\partial r_0}{\partial x^2} + ar_0 - f(0,x),$$

$$Q_2 = -\frac{\partial^2 r_1}{\partial x^2} + ar_1 - \frac{\partial f(0,x)}{\partial t} + \frac{(2k+1)(2k)}{2!}\left(\frac{\partial r_0}{\partial \tau}\right)^{2k+1}\left(\frac{\partial r_1}{\partial \tau}\right)^2.$$

Assume that the functions $W_i(t,x)$ have been already found. Then a new expansion of the function $W = \sum\limits_{i=0}^{n} \varepsilon^i W_i(\varepsilon\tau, x)$ in powers of ε in the coordinates (τ, x) will be of the form

$$W = \sum_{i=0}^{n+1} \varepsilon^i \omega_i(\tau, x) + O(\varepsilon^{n+2}). \tag{9}$$

Here $\omega_0 = W_0(0,x)$ is independent of τ, while the other $\omega_i(\tau,x)$'s are defined by the formula $\omega_i(\tau,x) = \sum\limits_{s+r=j} \frac{1}{s!}\frac{\partial^s W_r(0,x)}{\partial t^s}\tau^s; \quad j = 1,2,..,n+1.$

A boundary layer type function near boundary $t = 0$ should be sought in the form

$$V = \varepsilon\left[V_0(\tau,x) + \varepsilon V_1(\tau,x) + ... + \varepsilon^n V_n(\tau,x)\right], \tag{10}$$

as an approximate solution to the equation

$$L_{\varepsilon,1}(W+V) - L_{\varepsilon,1}W = O(\varepsilon^{n+1}). \tag{11}$$

From (9) and (10) it follows that

$$W + V = \sum_{j=0}^{n+1} \varepsilon^j g_j(\tau,x) + O(\varepsilon^{n+2}), \tag{12}$$

where $g_0 = \omega_0 = W_0(0,x)$ is independent of τ, and the functions $g_j(\tau,x)$ are defined by the formula $g_j(\tau,x) = \omega_j(\tau,x) + V_{j-1}(\tau,x); \quad j = 1,2,...,n+1.$

Having substituted the expressions (12),(9) for the functions $W, W+V$ in (11), and taking into account (8), we obtain the following recurrently connected equations for defining the functions $V_0, V_1, ..., V_n$:

$$\frac{\partial^2 V_0}{\partial \tau^2} + \frac{\partial V_0}{\partial \tau} = 0, \tag{13}$$

$$\frac{\partial^2 V_j}{\partial \tau^2} + \frac{\partial V_j}{\partial \tau} = h_j; \quad j = 1,2,...,n. \tag{14}$$

Here h_j denotes the function $h_j = \frac{\partial^2 V_{j-1}}{\partial x^2} - aV_{j-1}$ for $j = 1,2,...,2k$, and the function $h_j = \frac{\partial^2 V_{j-1}}{\partial x^2} - aV_{j-1} + g_{2k+s}$ for $j = 2k+1, 2k+2, ..., n$, with the functions g_{2k+s}, $s=1,2,..n-2k$ depending on $V_0, V_1, ..., V_{s-1}; \omega_0, \omega_1, ..., \omega_s$ and their derivatives. For example, $g_{2k+1}$and g_{2k+2} have the form

$$g_{2k+1}=\left(\frac{\partial \omega_1}{\partial \tau}+\frac{\partial V_0}{\partial \tau}\right)^{2k+1}-\left(\frac{\partial \omega_1}{\partial \tau}\right)^{2k+1},$$

$$g_{2k+2}=\left(\frac{\partial \omega_1}{\partial \tau}+\frac{\partial V_0}{\partial \tau}\right)^{2k}\cdot\left(\frac{\partial \omega_2}{\partial \tau}+\frac{\partial V_1}{\partial \tau}\right)-\left(\frac{\partial \omega_1}{\partial \tau}\right)^{2k}\frac{\partial \omega_2}{\partial \tau}.$$

When constructing asymptotics of the problem considered in the traditional way, we first should use the initial condition $W_i|_{t=0}=0$ boundary conditions (7) and conduct the first iteration process for all equations (6), and then construct a boundary layer type function V near $t=0$ to satisfy the second initial condition from (2), i.e. $\frac{\partial}{\partial t}(W+V)\big|_{t=0}=0$, that was lost. Then the constructed sum $W+V$ may not satisfy the initial condition $(W+V)|_{t=0}=0$, which is satisfied by the function W. To overcome this difficulty, we proceed as follows.

We find initial conditions for equations (6) from the equality

$$(W+V)|_{t=0}=0. \tag{15}$$

Having substituted the expressions (4),(10) for the functions W, V to the equality (15) and comparing the coefficients at same powers of ε whose degrees are less than $n+1$, we have

$$W_0|_{t=0}=0, \tag{16}$$

$$W_i|_{t=0}=-\left.V_{i-1}\right|_{t=0}; \quad i=1,2,...,n. \tag{17}$$

Note that if the functions $W_i; i=0,1,...,n$ satisfy conditions (16), (17), then the sum $W+V$ satisfies the condition

$$(W+V)|_{t=0}=\varepsilon^{n+1}\varphi(x), \tag{18}$$

where $\varphi(x)=V_n(0,x)$. In this case, we say that the function $W+V$ satisfies the boundary condition (15) approximately with the ε^{n+1}-th accuracy.

Boundary conditions for equations (13),(14) are found from

$$\frac{\partial}{\partial t}(W+V)\bigg|_{t=0}=0 \tag{19}$$

and have the from

$$\frac{\partial V_i}{\partial \tau}\bigg|_{\tau=0}=-\left.\frac{\partial W_i}{\partial t}\right|_{t=0}; \quad i=0,1,...,n. \tag{20}$$

Since the boundary conditions for equations obtained in iteration processes are already known, we can construct the functions $W_i, V_i; i = 0, 1, ..., n$.

The function W_0 is the solution of equation (6) (for $i = 0$) satisfying condition (16) and boundary condition (7) (for $i = 0$). Let us note that the convergence of all subsequent functions $V_0, W_1, V_1, ..., W_n, V_n$ to zero as $|x| \to +\infty$ will be provided due to the decrease of the function $W_0(t, x)$ with respect to x. And the decrease of the function $W_0(t, x)$ with respect to x will be achieved by means of the conditions imposed on the function $f(t, x)$.

We can write explicit formula defining the bounded solution of equation (6) that satisfies the initial condition (16). In order to ensure the role that function $W_0(t, x)$ should play, the following statement should be proven without using this formula.

Lemma 1. *Let $f(t, x)$ be a function given in Π with continuous derivatives with respect to t up to the $(n + 2)$-th order infinitely differentiable with respect to x and satisfying the condition*

$$\sup_x \left(1 + |x|^l\right) \left|\frac{\partial^k f(t, x)}{\partial t^{k_i} \partial x^{k_2}}\right| \leq C^{(1)}_{l_{k_1 k_2}} < +\infty, \tag{21}$$

where l is a non-negative number, $k = k_1 + k_2$, $k_1 \leq n + 2, k_2$ is an arbitrary non-negative integer, and $C^{(1)}_{l_{k_1 k_2}}$ is a positive number. Then the function $W_0(t, x)$, being the solution of problem (6),(7) (for $i = 0$), (16), has in Π continuous derivatives with respect to t up to the $(n + 3)$-th order is infinitely differentiable with respect to x, and satisfies the condition

$$\sup_x \left(1 + |x|^l\right) \left|\frac{\partial^k W_0(t, x)}{\partial t^{k_i} \partial x^{k_2}}\right| \leq C^{(2)}_{l_{k_1 k_2}} < +\infty, \tag{22}$$

where $k_1 \leq n + 3$, $C^{(2)}_{l_{k_1 k_2}} = const > 0$.

Proof. Applying the Fourier transform with respect to x, we reduce the problem (6),(7) (for $i = 0$) to

$$\frac{d\widetilde{W_0}}{dt} + \left(a + \lambda^2\right) \widetilde{W_0} = \widetilde{f}(t, \lambda), \quad \widetilde{W_0}\Big|_{t=0} = 0. \tag{23}$$

Here

$$\widetilde{W_0}(t, \lambda) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} W_0(t, x) \exp(-i\lambda x) dx, \;\; \widetilde{f}(t, \lambda) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} f(t, x) \exp(-i\lambda x) dx.$$

The solution of problem (23) is of the form

$$\widetilde{W}_0(t,\lambda) = \int_0^t \widetilde{f}(t,\lambda) \exp\left[-(a+\lambda^2)(t-\tau)\right] d\tau. \tag{24}$$

$W_0(t,x)$ is found as an inverse Fourier transform of the function $\widetilde{W}_0(t,\lambda)$ by the following formula:

$$W_0(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \widetilde{W}(t,\lambda) \exp(i\lambda x) d\lambda. \tag{25}$$

Condition (21) implies that the function $\widetilde{f}(t,\lambda)$ and all its derivatives with respect to t up to the $(n+2)$-th order and those with respect to the variable λ belong to the Schwartz space (denoted by S_λ). Obviously, to prove Lemma 1 it is enough to show that the function $\widetilde{W}_0(t,\lambda)$ and all its derivatives with respect to t up to the $(n+3)$-th order belong to S_λ. The validity of the following formula can be easily proved:

$$\frac{\partial^k \widetilde{W}_0(t,\lambda)}{\partial \lambda^k} = \int_0^t \left[\sum_{j=0}^{k} a_j(t-\tau,\lambda) \frac{\partial^{k-j} \widetilde{f}(t,\lambda)}{\partial \lambda^{k-j}}\right] \exp\left[-(a+\lambda^2)(t-\tau)\right] d\tau, \tag{26}$$

where $a_j(t-\tau,\lambda)$ is a polynomial of degree j with respect to λ and $t-\tau$. Since $\widetilde{f}(t,\lambda)$ belongs to the space S_λ and the functions $a_j(t-\tau,\lambda)$ have a polynomial growth with respect to λ, it follows from (26) that $W_0(t,\lambda) \in S_\lambda$. Now we prove that $\frac{\partial^k \widetilde{W}_0(t,\lambda)}{\partial \lambda^k} \in S_\lambda; k_1 = 1,2,...,n+3$. It can be easily proved that the derivatives $\widetilde{W}_0(t,\lambda)$ with respect to t of any order are expressed by the formula

$$\frac{\partial^{k_1} \widetilde{W}_0(t,\lambda)}{\partial \lambda^{k_1}} = \left[-(a+\lambda^2)\right]^{k_1} \widetilde{W}_0(t,\lambda) + \sum_{j=0}^{k_1-1} \left[-(a+\lambda^2)\right]^{k_1-j-1} \frac{\partial^j \widetilde{f}(t,\lambda)}{\partial t^j}. \tag{27}$$

The functions $\varphi_m(\lambda) = \left[-(a+\lambda^2)\right]^m$ have a polynomial growth with respect to λ. Belonging of the function $\widetilde{W}_0(t,\lambda)$ to the space S_λ has been proven above. Thus, each term on the right-hand side of (27) is a product of two functions, one of which has a polynomial growth and the another one belongs to the space S_λ. Therefore, the relation $\frac{\partial^{k_1} \widetilde{W}_0(t,\lambda)}{\partial t^{k_1}} \in S_\lambda$, is valid. Hence it follows that $\frac{\partial^{k_1} \widetilde{W}_0(t,x)}{\partial t^{k_1}} \in S_x; k_1 = 0,1,...,n+3$.

Lemma 1 is proved. ◄

Knowing the function W_0, we can define the function V_0 as a boundary layer solution (13),(20) (for $i = 0$). Obviously, V_0 is defined by the formula

$$V_0(\tau, x) = \frac{\partial W_0(0, x)}{\partial t} e^{-\tau} \tag{28}$$

The function $V_0(\tau, x)$ defined by (28) is infinitely differentiable with respect to both variables. Furthermore, the function $V_0(\tau, x)$ belongs to the space S_x for each $\tau \in [0, +\infty)$. Hence, in particular, it follows that $\lim\limits_{|x|\to+\infty} V_0(\tau, x) = 0$. Knowing the functions W_0 and V_0, we can define the function $W_1(t, x)$. From (6),(7),(17) for $i = 1$ it follows that $W_1(t, x)$ is the solution of the following boundary value problem:

$$\frac{\partial W_1}{\partial t} - \frac{\partial^2 W_1}{\partial x^2} + aW_1 = f_1(t, x);\ \ W_1|_{t=0} = \varphi_1(x);\ \lim_{|x|\to+\infty} W_1 = 0, \tag{29}$$

where $f_1 = -\frac{\partial^2 W_0}{\partial x^2}, \varphi_1(x) = -V_0(0, x)$. We can look for the function W_1 in the form $W_1 = W_1^{(1)} + W_1^{(2)}$, where the functions $W_1^{(1)}$ and $W_1^{(2)}$ are the solutions of the following boundary value problems:

$$\frac{\partial W_1^{(1)}}{\partial t} - \frac{\partial^2 W_1^{(1)}}{\partial x^2} + aW_1^{(2)} = 0;\ \ W_1^{(1)}\Big|_{t=0} = \varphi_1(x);\ \lim_{|x|\to+\infty} W_1^{(1)} = 0, \tag{30}$$

$$\frac{\partial W_1^{(2)}}{\partial t} - \frac{\partial^2 W_1^{(2)}}{\partial x^2} + aW_1^{(2)} = f_1(t, x);\ \ W_1^{(2)}\Big|_{t=0} = 0;\ \lim_{|x|\to+\infty} W_1^{(2)} = 0. \tag{31}$$

Applying the Fourier transform, we reduce the problem (30) to

$$\frac{\partial \widetilde{W}_1^{(1)}}{\partial t} + (a + \lambda^2)\widetilde{W}_1^{(1)} = 0;\ \ \widetilde{W}_1^{(1)}\Big|_{t=0} = \widetilde{\varphi}_1(\lambda),$$

whose solution is of the form $\widetilde{W}_1^{(1)}(t, \lambda) = \widetilde{\varphi}_1(\lambda) \exp\left[-(a + \lambda^2)t\right]$.

Here $\widetilde{W}_1(t, \lambda), \widetilde{\varphi}_1(\lambda)$ denote Fourier transforms of the functions $W_1(t, x), \varphi_1(x)$, respectively. Since $\varphi_1(x) = -V_0(0, x)$ and the function $V_0(\tau, x)$ belongs to the space S_x, we have $\varphi_1(x) \in S_x$, consequently $\widetilde{\varphi}_1(\lambda) \in S_\lambda$. The function $\widetilde{W}_1^{(1)}(t, \lambda)$ is infinitely differentiable with respect to both arguments t and λ. Derivatives of any order with respect to t of the function $\widetilde{W}_1(t, \lambda)$ are defined by the equality $\frac{\partial^{k_1} \widetilde{W}_1}{\partial t^{k_1}} = [-(a + \lambda^2)]^{k_1} \varphi_1(\lambda) exp[-(a + \lambda^2)t]$. Derivatives of arbitrary order with respect to t of the function $\widetilde{W}_1^{(1)}(t, \lambda)$ consist of a finite sum, each term of which is a product of three functions: the first of them is $\exp\left[-(a + \lambda^2)t\right]$, the second one is the function $\widetilde{\varphi}_1(\lambda)$ or its derivatives, and the third one is a polynomial with respect to λ and t. Therefore, for each t from $[0,T]$ and for arbitrary non-negative integer k_1 the relation $\frac{\partial^{k_1}\widetilde{W}_1(t,\lambda)}{\partial t^{k_1}} \in S_\lambda$ is valid, hence we have $\frac{\partial^{k_1}W_1(t,x)}{\partial t^{k_1}} \in S_x$.

As for the problem (31), we note that the right-hand side $f_1(t,x)$ of the equation for $W_1^{(2)}$ satisfies the condition (21) in Lemma 1 for $k_1 \le n$. Therefore, by the same lemma, the function $\widetilde{W}_1^{(2)}(t,x)$ being the solution of problem (31), together with its derivatives with respect to t up to the $(n+1)$-th order belongs to the space S_x. The function $W_1(t,x)$ being the sum of $W_1^{(1)}(t,x)$ and $W_1^{(2)}(t,x)$ possesses the same prosperty. From (14) and (20) for $i=1$ it follows that the function $V_1(\tau,x)$ is a boundary layer type solution of the following problem:

$$\frac{\partial^2 V_1}{\partial \tau^2} + \frac{\partial V_1}{\partial \tau} = \left[\frac{\partial^3 W_0(0,x)}{\partial t \partial x^2} - a^2 \frac{\partial W_0(0,x)}{\partial t}\right]\exp(-\tau),\ \left.\frac{\partial V_1}{\partial \tau}\right|_{\tau=0} = -\frac{\partial W_1(0,x)}{\partial t}.$$

Obviously, the boundary layer type solution of the last problem is defined by the formula

$$V_1(\tau,x) = \left\{\frac{\partial W_1(0,x)}{\partial t} + \left[a\frac{\partial W_0(0,x)}{\partial t} - \frac{\partial^3 W_0(0,x)}{\partial t \partial x^2}\right]\tau\right\}\exp(-\tau).$$

It follows from the last formula that the function $V_1(\tau,x)$ belongs to the space S_x for each $\tau \in [0,+\infty)$. Therefore, the condition $\lim\limits_{|x|\to+\infty} V_1(\tau,x) = 0$ is fulfilled. Continuing this process, one by one we construct the functions $W_2, V_2, ..., W_n, V_n$. This time the function $W = \sum\limits_{i=0}^{n} \varepsilon^i W_i$ will satisfy the boundary condition

$$\lim_{|x|\to+\infty} W = 0, \quad (0 \le t \le T). \tag{32}$$

We can prove that all the functions $V_1(\tau,x)$ are defined by the formula

$$V_1(\tau,x) = \left[b_{i0}(x) + b_{i1}(x)\tau + ... + b_{ii}(x)\tau^i\right]\exp(-\tau), \tag{33}$$

and the coefficients $b_{is}(x)$ are expressed homogeneously by the function

$$\frac{\partial^{1+2r} W_s(0,x)}{\partial t \partial x^{2r}};\ r+s \le i, \quad i = 0,1,...,n. \tag{34}$$

Since all the functions of the form (34) belong to the space S_x, we have $b_{is}(x) \in S_x$. Therefore, from (33) it follows that $\lim\limits_{|x|\to+\infty} V_i = 0, .i = 0,1,...,n$. Hence it follows that the function $V = \sum\limits_{i=0}^{n} \varepsilon^i V_i$ satisfies the boundary condition

$$\lim_{|x|\to+\infty} V = 0, \quad (0 \le t \le T). \tag{35}$$

From (32) and (35) it follows that the sum $W+V$, in addition to initial conditions (18), and (19), satisfies the boundary condition

$$\lim_{|x|\to+\infty}(W+V)=0, \quad (0\le t\le T) \tag{36}$$

as well. We denote $\widetilde{u}=W+V=\sum\limits_{i=0}^{n}\varepsilon^i W_i+\sum\limits_{i=0}^{n}\varepsilon^{1+i}V_i$ and call the difference $u-\widetilde{u}=z$ a residual term, where u is the solution of problem (1)-(3). Then we obtain the following asymptotic expansion in a small parameter of the generalized solution to problem (1)-(3):

$$u=\sum_{i=0}^{n}\varepsilon^i W_i+\sum_{i=0}^{n}\varepsilon^{1+i}V_i+z. \tag{37}$$

Now we need to estimate the residual term.

3. Estimating the residual term and formulation of the main result

The following lemma is valid.

Lemma 2. *The following estimate is valid for the residual term z in (37)*

$$\varepsilon\int\limits_{-\infty}^{+\infty}\left(\left.\frac{\partial z}{\partial t}\right|_{t=T}\right)^2dx+\varepsilon^{2k}\iint\limits_{\Pi}\left(\frac{\partial z}{\partial t}\right)^{2k+2}dtdx+\iint\limits_{\Pi}\left(\frac{\partial z}{\partial t}\right)^2dtdx+$$

$$+\int\limits_{-\infty}^{+\infty}\left(\left.\frac{\partial z}{\partial x}\right|_{t=T}\right)^2dx+\int\limits_{-\infty}^{+\infty}\left(z|_{t=T}\right)^2dx<C\varepsilon^{2(n+1)}, \tag{38}$$

where $C>0$ is a constant independent of ε.

Proof. Adding (5) and (11) together, we see that $\tilde{u}$ satisfies the equation $L_\varepsilon\widetilde{u}=O(\varepsilon^{n+1})$. Subtracting the last equation from (1), we obtain

$$\varepsilon\frac{\partial^2 z}{\partial t^2}+\varepsilon^{2k}\left[\left(\frac{\partial u}{\partial t}\right)^{2k+1}-\left(\frac{\partial\widetilde{u}}{\partial t}\right)^{2k+1}\right]+\frac{\partial z}{\partial t}-\frac{\partial z}{\partial x^2}+az=\varepsilon^{n+1}F_0(\varepsilon,t,x), \tag{39}$$

where $|F_0(\varepsilon,t,x)|\le C_1$ for any $\varepsilon\in[0,\varepsilon_0)$. Moreover, $C_1>0$ is independent of ε.

From (2),(3),(18),(19),(36) it follows that z satisfies the following boundary conditions:

$$z|_{t=0}=\varepsilon^{n+1}\varphi(x), \quad \left.\frac{\partial z}{\partial t}\right|_{t=0}=0, \quad \lim_{|x|\to\infty}z=0. \tag{40}$$

Recall that $\varphi(x) = -V_n(0,x)$. We introduce an auxiliary function

$$z_1 = \varepsilon^{n+1}\left[t^2 e^{-x^2} - V_n(0,x)\right] \tag{41}$$

and represent the residual term z as

$$z = z_1 + z_2. \tag{42}$$

Then from (39)-(42) it follows that z_2 is the solution of the following problem

$$\varepsilon\frac{\partial^2 z_2}{\partial t^2} + \varepsilon^{2k}\left\{\left[\frac{\partial(\widetilde{u}+z_1+z_2)}{\partial t}\right]^{2k+1} - \left(\frac{\partial\widetilde{u}}{\partial t}\right)^{2k+1}\right\}+$$

$$+\frac{\partial z_2}{\partial t} - \frac{\partial^2 z_2}{\partial x^2} + az_2 = \varepsilon^{n+1}F_1(\varepsilon,t,x) \tag{43}$$

$$z_2|_{t=0} = 0, \quad \left.\frac{\partial z_2}{\partial t}\right|_{t=0} = 0, \quad \lim_{|x|\to\infty} z_2 = 0. \tag{44}$$

Here $F_1(\varepsilon,t,x)$ denotes a function bounded in Π for any $\varepsilon \in [0,\varepsilon_0)$.

We write equation (43) in the form

$$\varepsilon\frac{\partial^2 z}{\partial t^2} + \varepsilon^{2k}\left\{\left[\frac{\partial(\widetilde{u}+z_1+z_2)}{\partial t}\right]^{2k+1} - \left[\frac{\partial(\widetilde{u}+z_1)}{\partial t}\right]^{2k+1}\right\}+$$

$$+\frac{\partial z_2}{\partial t} - \frac{\partial^2 z_2}{\partial x^2} + az_2 = \varepsilon^{n+1}F_1(\varepsilon,t,x) + F_2(\varepsilon,t,x), \tag{45}$$

where

$$F_2(\varepsilon,t,x) = -\varepsilon^{2k}\left\{\left[\frac{\partial(\widetilde{u}+z_1)}{\partial t}\right]^{2k+1} - \left(\frac{\partial\widetilde{u}}{\partial t}\right)^{2k+1}\right\}.$$

Using the expansion a $a^{2k+1} - b^{2k+1} = (a-b)(a^{2k} + a^{2k-1}b + ... + ab^{2k-1} + b^{2k})$ and explicit expression (41) for the function z_1, we can represent the function $F_2(\varepsilon,t,x)$ in the form

$$F_2(\varepsilon,t,x) = \varepsilon^{n+1}F_3(\varepsilon,t,x), \tag{46}$$

where

$$F_3(\varepsilon,t,x) = -\varepsilon^{2k}(2t)e^{-x^2}\left\{\left[\frac{\partial(\widetilde{u}+z_1)}{\partial t}\right]^{2k} + \left[\frac{\partial(\widetilde{u}+z_1)}{\partial t}\right]^{2k-1}\frac{\partial\widetilde{u}}{\partial t}+\right.$$

$$\left.+... + \frac{\partial(\widetilde{u}+z_1)}{\partial t}\left(\frac{\partial\widetilde{u}}{\partial t}\right)^{2k-1} + \left(\frac{\partial\widetilde{u}}{\partial t}\right)^{2k}\right\}.$$

Substituting the expression (46) for $F_2(\varepsilon, t, x)$ to the right-hand side of (45), we have

$$\varepsilon\frac{\partial^2 z}{\partial t^2} + \varepsilon^{2k}\left\{\left[\frac{\partial(\widetilde{u} + z_1 + z_2)}{\partial t}\right]^{2k+1} - \left[\frac{\partial(\widetilde{u} + z_1)}{\partial t}\right]^{2k+1}\right\} +$$

$$+\frac{\partial z_2}{\partial x^2} - \frac{\partial^2 z_2}{\partial x^2} + az_2 = \varepsilon^{n+1}\Phi(\varepsilon, t, x), \tag{47}$$

where $\Phi(\varepsilon, t, x) = F_2(\varepsilon, t, x) + F_3(\varepsilon, t, x)$ is a function bounded in Π for any $\varepsilon \in [0, \varepsilon_0)$.

Multiplying both sides of equation (47) scalarly by $\frac{\partial z_2}{\partial t}$ and integrating by parts, after some transformations we get the validity of the estimate (38) for z_2. The validity of (38) for z follows from its validity for z_2 and (41),(42).

Lemma 2 is proved. ◄

Summarizing the results obtained, we get the following statement.

Theorem 1. *Assume that the function $f(t, x)$ given in Π has continuous derivatives with respect to t up to the $(n + 2)$-th order is infinitely differentiable with respect to x and satisfies the condition (21). Then for the solution of problem (1)-(3) the asymptotic representation (37) is valid, where the functions W_i, are defined by the first iteration process, V_j are boundary layer type functions near $t = 0$, z is a residual term and the estimate (38) is valid for it.*

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Received 05 September 2025
Accepted 13 November 2025