Azerbaijan Journal of Mathematics
V. 16, No 1, 2026, January
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2026.1.172

Induced Mappings and Sequential Properties of the Space of G-Permutation Degree

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Abstract. In this paper, we study preservation of some mappings under the functor of G-permutation degree SP^n_G. We prove that if the mapping $f : X \to Y$ is almost-open (resp., pseudo-open, monotone), then the mapping $\mathsf{SP}^n_G f : \mathsf{SP}^n_G X \to \mathsf{SP}^n_G Y$ is also almost-open (resp., pseudo-open, monotone). In addition, we prove that the space X^n is sequential (resp., Fréchet-Urysohn) if and only if the space $\mathrm{SP}^n_G X$ is sequential (resp. Fréchet-Urysohn). Also, we prove that if the space X^n is strongly Fréchet-Urysohn, then the space $\mathsf{SP}^n_G X$ is also a strongly Fréchet-Urysohn space.

Key Words and Phrases: almost open mapping, pseudo-open mapping, monotone mapping, functor, sequential space, (strongly) Fréchet-Urysohn space.

2010 Mathematics Subject Classifications: 18F60, 54A20, 54C10, 54D55

1. Introduction and preliminary notes

Throughout this paper, all spaces are assumed to be T_1-spaces. Topological terminology and notation follow [6, 8, 9].

Let S_n be the symmetric group of all permutations of the set $\{1, 2, \dots, n\}$ and G be a subgroup of S_n. For a topological space X, a relation ρ_G on the space X^n is defined as follows: for $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $\mathbf{y} = (y_1, y_2, \dots, y_n)$ in X^n

$$\mathbf{x}\rho_G\mathbf{y} \iff \exists\ \sigma \in G \ \text{ with } y_i = x_{\sigma(i)},\ i \le n. \tag{1}$$

This relation is an equivalence relation (called the G-symmetric equivalence relation). The G-symmetric equivalence class of elements $\mathbf{x} = (x_1, x_2, \dots, x_n) \in X^n$ is denoted by $[\mathbf{x}]_G = [(x_1, x_2, \dots, x_n)]_G$. The quotient set X/ρ_G equipped with the quotient topology is denoted by $\mathsf{SP}^n_G X$ and called the *space of n-G-permutation degree* or simply the *space of G-permutation degree.*

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The quotient mapping of X^n onto $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ is denoted by π_n^s and defined as

$$\pi_n^s((x_1, x_2, \dots, x_n)) = [(x_1, x_2, \dots, x_n)]_G \tag{2}$$

for every $(x_1, x_2, \dots, x_n) \in X^n$.

Let $f : X \to Y$ be a continuous mapping. Define the mapping $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f$: $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} Y$ by

$$\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f\,([(x_1, x_2, \dots, x_n)]_G) = [(f(x_1), f(x_2), \dots, f(x_n))]_G \tag{3}$$

for every $[(x_1, x_2, \dots, x_n)]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$.

It is known that the operation $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ so constructed is a normal functor in the category **Comp** of compact Hausdorff spaces and their continuous mappings [8]. This functor is called the *functor of G-permutation degree.*

Note that points of the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$ are finite subsets (equivalence classes) of the space X^n. Thus, this space is similar to the exponential space $\mathsf{exp}_{\mathsf{n}} X$ of all non-empty n-element subsets of X equipped with the well known Vietoris topology [6, 9]. In fact, the spaces $\mathsf{exp}_2 X$ and $\mathsf{SP}^2_{\mathsf{S}_\mathsf{n}} X$ are homeomorphic, while it is not the case for $n \geq 3$ [8]. The operation $\mathsf{exp}_{\mathsf{n}}$ is also a covariant functor in the category **Comp**. Let us observe that the functor $\exp_{\mathrm{n}}$ is a factor functor of the functor $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ [7, 8].

Recently, the spaces of $G-$permutation degree have been investigated from different points of view in a number of papers: metric and uniform structures [3], e-density and τ-base [4, 5], tightness-type properties [14] (the functor $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ preserves minitightness, functional tightness, $T-$tightness, weak tightness), network-type properties [15] ($\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ preserves *cs*-network, *cs**-network, *cn*-network and *ck*-network), generalized metric properties [16].

In this paper, we study preservation of almost open, pseudo-open and monotone mappings under the functor of G-permutation degree $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ (Section 2). In Section 3, we prove that the above functor preserves sequential spaces, Fréchet-Urysohn spaces, and strongly Fréchet-Urysohn spaces.

2. Induced mappings on $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$

In this section, we investigate preservation of three kinds of mappings (almost open, pseudo-open and monotone) under influence of the functor $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}|$.

A mapping $f : X \to Y$ is called *almost open* if for every $y \in Y$ there exists $x \in f^{\leftarrow}(y)$ such that $f(U)$ is a (not necessarily open) neighborhood of y for every neighborhood U of x. [Recall that this class of mappings plays an important role in the theory of topological vector spaces. For some information about these mappings see [12, 17].]

Theorem 1. *If a mapping $f : X \to Y$ is almost open, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f : \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} Y$.*

Proof. Let $f : X \to Y$ be an almost open mapping. Suppose that $[\mathbf{y}]_G = [(y_1, y_2, \ldots, y_n)]_G$ is an arbitrary element of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} Y$. Since the functor $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ preserves preimages, we have

$$(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f)^{\leftarrow} \left([(y_1, y_2, \ldots, y_n)]_G\right) = \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} \left([(f^{\leftarrow}(y_1), f^{\leftarrow}(y_2), \ldots, f^{\leftarrow}(y_n))]_G\right). \tag{4}$$

Since $f : X \to Y$ is an almost open mapping, for every $i \le n$ there is $x_i \in f^{\leftarrow}(y_i)$ such that for any neighborhood U_i of x_i, $f(U_i)$ is a neighborhood of y_i. By (4) we have $x_i \in f^{\leftarrow}(y_i)$ for all $i \le n$.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Then $[\mathbf{x}]_G \in (\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f)^{\leftarrow} ([(y_1, y_2, \ldots, y_n)]_G)$. Let $[U_1, U_2, \ldots, U_n]_G$ be a neighborhood of $[\mathbf{x}]_G$. By the above, $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f)([U_1, U_2, \ldots, U_n]_G)$ is a neighborhood of $\mathbf{y}$ which means that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f$ is an almost open mapping. ◀

In [18], it was shown that for every almost open mapping $f : X \to Y$, the mapping $\mathsf{exp}_{\mathsf{n}} f$ is almost open. By this fact and Theorem 1, we get the following corollary.

Corollary 1. *Let $f : X \to Y$ be an almost open mapping. Then the following hold:*

(1) $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f : \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} Y$ *is an almost open mapping;*

(2) $\mathsf{exp}_{\mathsf{n}} f : \mathsf{exp}_{\mathsf{n}} X \to \mathsf{exp}_{\mathsf{n}} Y$ *is an almost open mapping.*

A mapping $f : X \to Y$ is called *pseudo-open* [2] if for every $y \in Y$ and every neighborhood U of $x \in f^{\leftarrow}(y)$, $f(U)$ is a (not necessarily open) neighborhood of y.

Clearly, pseudo-open mappings are stronger form of almost open mappings.

Theorem 2. *If a mapping $f : X \to Y$ is pseudo-open, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f : \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} Y$.*

Proof. Let $f : X \to Y$ be a pseudo-open mapping. Suppose that $[\mathbf{y}]_G = [(y_1, y_2, \ldots, y_n)]_G$ is an element of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} Y$ and

$$\mathbf{x} = [x_1, x_2, \ldots, x_n] \in (\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f)^{\leftarrow} \left([(y_1, y_2, \ldots, y_n)]_G\right) =$$

$$= \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} \left([(f^{\leftarrow}(y_1), f^{\leftarrow}(y_2), \ldots, f^{\leftarrow}(y_n))]_G\right).$$

Take any neighborhood $[U_1, U_2, \ldots, U_n]_G$ of $\mathbf{x}$. Then $x_i \in U_i$ for every $i \le n$. Since $f : X \to Y$ is a pseudo-open mapping, we have $f(x_i) \in \mathrm{Int}(f(U_i)) = V_i$ for every $i = 1, 2, \ldots, n$. It follows that $[(y_1, y_2, \ldots, y_n)]_G \in [(V_1, V_2, \ldots, V_n)]_G$.

On the other hand, we have

$$\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f\left([(U_1, U_2, \ldots, U_n)]_G\right) = [(f(U_1), f(U_2), \ldots, f(U_n))]_G$$

and

$$[(V_1, V_2, \ldots, V_n)]_G \subset [(f(U_1), f(U_2), \ldots, f(U_n))]_G.$$

It means that the set $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f\left([(U_1, U_2, \ldots, U_n)]_G\right)$ is a neighborhood of the point $[(y_1, y_2, \ldots, y_n)]_G$. Theorem 2 is proved. ◀

In [18] it was shown that for every pseudo-open mapping $f : X \to Y$, the mapping $\exp_{\mathsf{n}} f$ is pseudo-open. By this fact and Theorem 2, we get the following corollary.

Corollary 2. *Let $f : X \to Y$ be a pseudo-open mapping. Then:*
(1) $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f : \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} Y$ *is a pseudo-open mapping;*
(2) $\mathsf{exp}_{\mathsf{n}} f : \mathsf{exp}_{\mathsf{n}} \to \mathsf{exp}_{\mathsf{n}} Y$ *is a pseudo-open mapping.*

We say that a mapping $f : X \to Y$ is *monotone* [6, 13] if $f^{\leftarrow}(y)$ is a connected subset of X for every $y \in Y$.

Theorem 3. *If the mapping $f : X \to Y$ is monotone, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f : \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} Y$.*

Proof. Suppose that the mapping $f : X \to Y$ is monotone. Let $[\mathbf{y}]_G = [(y_1, y_2, \ldots, y_m)]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} Y$. We have

$$(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f)^{\leftarrow}\left([(y_1, y_2, \ldots, y_n)]_G\right) = \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\left([(f^{\leftarrow}(y_1), f^{\leftarrow}(y_2), \ldots, f^{\leftarrow}(y_n))]_G\right)$$

and every $f^{\leftarrow}(y_i)$, $i \le n$, is a connected subset of X. Connectedness is preserved under the Cartesian product, so the set $f^{\leftarrow}(y_1) \times f^{\leftarrow}(y_2) \times \ldots \times f^{\leftarrow}(y_n)$ is connected in X^n. Since the mapping $\pi^s_n : X^n \to \mathsf{SP}^{\mathsf{n}} X$ is continuous, we conclude that $\pi^s_n\left(f^{\leftarrow}(y_1) \times f^{\leftarrow}(y_2) \times \ldots \times f^{\leftarrow}(y_n)\right) = (\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f)^{\leftarrow}([\mathbf{y}]_G)$ is connected, i.e. $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f$ is a monotone mapping. ◀

3. Sequential properties of the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$

In this section we assume that all topological spaces are Hausdorff.

Let X be a topological space, P be a subset of X and $\{x^m\}_{m \in \mathbb{N}}$ be a sequence in X. We say that $\{x^m\}_{m \in \mathbb{N}}$ is *eventually* in P if there exists $m_0 \in \mathbb{N}$ such that $\{x^m : m \ge m_0\} \subset P$.

$P \subset X$ is called a *sequential neighborhood* of $x \in X$ if every sequence converging to x is eventually in P [10]. A set $P \subset X$ is said to be *sequentially open* if P is a sequential neighborhood of every point in P. A set $P \subset X$ is said to be *sequentially closed* if $X \setminus P$ is sequentially open in X.

A space X is said to be a *sequential space* [10] if every sequentially open subset is open in X, or, equivalently, a subset F of X is closed if (hence and only if) it contains the limit points of all sequences in F. Clearly, every first countable space is sequential, every metric space is sequential and every discrete space is sequential.

Example 1. Let X be any set, τ_d be the discrete topology and τ be a non-sequential topology in X. Consider the identity mapping $id_X : (X, \tau_d) \to (X, \tau)$. This mapping is continuous. It follows that a continuous image of a sequential space is not always a sequential space.

For every topological space (X, τ), a new topology σ_τ can be introduced as follows:

$$U \in \sigma_\tau \text{ if and only if } U \text{ is a sequentially open subset of } (X, \tau).$$

The space (X, σ_τ) is called a *sequential coreflection* of the space (X, τ) and is denoted by σX [11]. Clearly, the space σX is a sequential space. Moreover, X and σX have the same convergent sequences.

Theorem 4. *For a space X, X^n is a sequential space if and only if $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$ is a sequential space.*

Proof. ($\Rightarrow$) Suppose that X^n is a sequential space. If $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} U \subset \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$ is sequentially open and $\{\mathbf{x^m}\} \to \mathbf{x^0} \in (\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} U)$, then $\{\pi^s_{n,G}(\mathbf{x^m})\} \to \pi^s_{n,G}(\mathbf{x^0}) \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} U$ and $\{\pi^s_{n,G}(\mathbf{x^m})\}_{m\in\mathbb{N}}$ is eventually in $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} U$. Since X^n is sequential, $\{\mathbf{x^m}\}_{m\in\mathbb{N}}$ is eventually in the set $(\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} U)$, and it follows that $(\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} U)$ is open in X^n. As $\pi^s_{n,G} : X^n \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$ is a quotient mapping, the set $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} U$ is open in $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$. It means $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$ is a sequential space.

($\Leftarrow$) Let σX^n be the sequential coreflection of X^n and $id_{X^n} : \sigma X^n \to X^n$ be the identity mapping. It is easy to check that the mapping id_{X^n} is continuous. Suppose that $f = \pi^s_{n,G} \circ id_{X^n} : \sigma X^n \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$. Since the mapping $\pi^s_{n,G}$ is an open and closed surjection (and a perfect mapping for the Hausdorff space X^n), the mapping f is closed. Consequently, f is a perfect mapping. Then id_{X^n} is a perfect mapping. It follows that X^n is a sequential space. Theorem 4 is proved. ◄

It is well known that the product of two sequential spaces need not be a sequential space.

However, in [10] the following was given.

Lemma 1. *If a product space is sequential, so is every of its factors.*

In [1], the following theorem was proved.

Theorem 5. ([1]) *If X and Y are sequential spaces and Y is locally compact, then $X \times Y$ is sequential.*

Corollary 3. *Let X be a locally compact space. If X is a sequential space, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$.*

Since a compact Hausdorff space is locally compact, we have the following corollary.

Corollary 4. *Let X be a compact Hausdorff space. If X is a sequential space, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$.*

A topological space X is called a *Fréchet-Uryhohn space* [10, 13] if the closure of any subset A of X is the set of limits of sequences in A.

It is easy to check that every first countable space (and so every metric space) is a Fréchet-Urysohn space. Moreover, every Fréchet-Urysohn space is sequential.

There exists a space X that is a sequential space, but not Fréchet-Urysohn.

Example 2. Let R be the set $\mathbb{R}$ of real numbers with the topology T which is generated by the usual metric topology and all sets of the form $\{0\} \cup U$, where U a usual open neighborhood of the sequence $\{\frac{1}{n}\}$. It can be easily checked that this space is sequential. Let $A = \mathbb{R} \setminus \left(\{\frac{1}{n} : n = 1, 2, \ldots\} \cup \{0\}\right)$. For every positive integer n there exists a sequence $\{x^n_j\}_{j\in\mathbb{N}}$ in A converging to $\frac{1}{n}$. Clearly, $0 \in \overline{A}$. However, every sequence of $\mathbb{R}$ converging to 0 is eventually constant or subsequence of $\{\frac{1}{n}\}$. Hence, $(\mathbb{R}, T)$ is not a Fréchet-Urysohn space.

Proposition 1. ([11]) *A sequential space is Fréchet-Urysohn if and only if it is hereditarily sequential.*

In general, Fréchet-Urysohn property is not preserved under quotient mappings. Moreover, the product of two Fréche-Urysohn spaces will not be always Fréchet-Urysohn. Even the square of a compact Fréchet-Urysohn space need not be Fréchet-Urysohn

Theorem 6. *X^n is a Fréchet-Utysohn space if and only if $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is a Fréchet-Urysohn space.*

Proof. ($\Rightarrow$) Suppose that X^n is a Fréchet-Urysohn space. Let $[\mathbf{x}]_G \in \overline{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A} \subset \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. If $(\pi^s_{n,G})^{\leftarrow}([\mathbf{x}]_G) \cap \overline{(\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A)} = \emptyset$, put $U = X^n \setminus \overline{(\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A)}$. Then $[\mathbf{x}]_G \in \mathrm{Int}\pi^s_{n,G}(U) \subset \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X \setminus \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A$ and we have a contradiction. So there is $\mathbf{x_0} \in (\pi^s_{n,G})^{\leftarrow}([\mathbf{x}]_G) \cap \overline{(\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^\mathsf{n}_\mathsf{G} A)}$. Consider a sequence $\{\mathbf{x}^m\}_{m\in\mathbb{N}} \subset (\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^\mathsf{n}_\mathsf{G} A)$ converges to $\mathbf{x_0}$. It follows that $\{\pi^s_{n,G}(\mathbf{x}^m)\}_{m\in\mathbb{N}} \subset \mathsf{SP}^\mathsf{n}_\mathsf{G} A$ and $\{\pi^s_{n,G}(\mathbf{x}^m)\}_{m\in\mathbb{N}}$ converging to $[\mathbf{x}]_G$. Therefore, $\mathsf{SP}^\mathsf{n}_\mathsf{G} X$ is a Fréchet-Urysohn space.

($\Leftarrow$) Assume $\mathsf{SP}^\mathsf{n}_\mathsf{G} X$ is a Fréchet-Urysohn space. Let A be a subset of X^n and $\mathbf{x} \in \overline{A}$. Suppose that $(\pi^s_{n,G})^{\leftarrow}([\mathbf{x}]_G) = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\} \subset X^n$ and $\mathbf{x} = \mathbf{x}_1$. Since X^n is Hausdorff, there exists an open neighborhood $U_\mathbf{x}$ of $\mathbf{x}$ such that $\overline{U_\mathbf{x}} \cap \{\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_k\} = \emptyset$. Hence, $\mathbf{x} \in U_\mathbf{x} \cap \overline{A} \subset \overline{U_\mathbf{x} \cap A}$. Since the mapping $\pi^s_{n,G}$ is closed, it implies that $[\mathbf{x}]_G \in \pi^s_{n,G}(\overline{U_\mathbf{x} \cap A}) = \overline{\pi^s_{n,G}(U_\mathbf{x} \cap A)}$. As $\mathsf{SP}^\mathsf{n}_\mathsf{G} X$ is a Fréchet-Urysohn space, there exists a sequence $\{\mathbf{a}^m\}_{m\in\mathbb{N}} \subset U_\mathbf{x} \cap A$ such that the sequence $\{\pi^s_{n,G}(\mathbf{a}^m)\}_{m\in\mathbb{N}}$ converges to the point $[\mathbf{x}]_G$. It follows that there exists a subsequence $\{\mathbf{a}^{m_i}\}_{m_i\in\mathbb{N}}$ of the sequence $\{\mathbf{a}^m\}_{m\in\mathbb{N}}$, which converges to the point $\mathbf{a} \in X^n$. It is seen that $\pi^s_{n,G}(\mathbf{a}) = [\mathbf{x}]_G$. Then $\mathbf{a} \in \overline{U_\mathbf{x}} \cap (\pi^s_{n,G})^{\leftarrow}([\mathbf{x}]_G) = \mathbf{x}$. Consequently, the subsequence $\{\mathbf{a}^{m_i}\}_{m_i\in\mathbb{N}} \subset A$ converges to $\mathbf{x}$. It means X^n is a Fréchet-Urysohn space. Theorem 6 is proved. ◄

We can define a mapping $f : \sigma X^n \to \sigma \mathsf{SP}^\mathsf{n}_\mathsf{G} X$ by $f(\mathbf{x}) = \pi^s_{n,G}(\mathbf{x}) = [\mathbf{x}]_G$ for every point $\mathbf{x} \in X^n$. We can also give the following proposition (related to this mapping and Theorem 6)

Proposition 2. *σX^n is a Fréchet-Urysohn space if and only if $\sigma \mathsf{SP}^\mathsf{n}_\mathsf{G} X$ is a Fréchet-Urysohn space.*

A space X is *strongly Fréchet-Urysohn* [13] if for any decreasing family $\{A_m\}_{m\in\mathbb{N}}$ of subsets of X and any point $x \in \bigcap_{m\in\mathbb{N}} \overline{A_m}$ we may pick points $x^m \in A_m$ in such a way that the sequence $\{x^m\}_{m\in\mathbb{N}}$ converges to x. It is known that a product of two strongly Fréchet-Urysohn spaces need not be Fréchet-Urysohn, even not sequential.

Theorem 7. *If $\mathsf{SP}^\mathsf{n}_\mathsf{G} X$ is a strongly Fréchet-Urysohn space, then so is X^n.*

Proof. Suppose that $\mathsf{SP}^\mathsf{n}_\mathsf{G} X$ is a strongly Fréchet-Urysohn space. Let $\{A_m\}_{m\in\mathbb{N}} \subset X^n$ be a decreasing sequence of subsets of X^n and $\mathbf{x} \in \bigcap_{m\in\mathbb{N}} \overline{A_m}$. Put $(\pi^s_{n,G})^{\leftarrow}\left(\pi^s_{n,G}(\mathbf{x})\right) = \{\mathbf{x}_1, \mathbf{x}_2 \dots, \mathbf{x}_k\}$ with $\mathbf{x}_1 = \mathbf{x}$, where $k \le n$. X^n is a Hausdorff space and so there exists an open neighborhood $U_\mathbf{x}$ of $\mathbf{x}$ such that $\overline{U_\mathbf{x}} \cap \{\mathbf{x}_2, \mathbf{x}_3 \dots, \mathbf{x}_k\} = \emptyset$. It follows that $\mathbf{x} \in U_\mathbf{x} \cap \overline{A_m} \subset \overline{U_\mathbf{x} \cap A_m}$ for every $m \in \mathbb{N}$. Then $\pi^s_{n,G}(\mathbf{x}) \in \pi^s_{n,G}\left(\overline{U_\mathbf{x} \cap A_m}\right) = \overline{\pi^s_{n,G}\left(U_\mathbf{x} \cap A_m\right)}$. Since $\mathrm{SP}^\mathsf{n}_\mathsf{G} X$ is a strongly Fréchet-Urysohn space, for every $m \in \mathbb{N}$ there exists $\mathbf{y}_m \in U_\mathbf{x} \cap A_m$ such that the sequence $\{\pi^s_{n,G}(\mathbf{y}_m)\}_{m\in\mathbb{N}}$ converges to $\pi^s_{n,G}(\mathbf{x})$. As $\{A_m\}_{m\in\mathbb{N}}$ is a decreasing sequence of subsets, we can consider the sequence $\{\mathbf{y}_m\}_{m\in\mathbb{N}}$ which is convergent in X^n and then the set $\overline{U_\mathbf{x}} \cap (\pi^s_{n,G})^{\leftarrow}\left(\pi^s_{n,G}(\mathbf{x})\right) = \{\mathbf{x}\}$ includes its limit. It means that $\{\mathbf{y}_m\}_{m\in\mathbb{N}}$ converges to the point $\mathbf{x}$. Therefore, X^n is a strongly Fréchet-Urysohn space. Theorem 7 is proved. ◄

References

[1] Antosik, P., Boehme, T.K., Mohanadi, F. (1985) *Concerning sequential spaces*, Qatar University Science Bulletin, **5**, 41–48.

[2] Arhangel'kii, A.V. (1963) *Some types of factor mappings, and the relations between classes of topological spaces*, Soviet Mathematics Doklady, **4**, 1726–1729.

[3] Beshimov, R.B., Georgiou, D.N., Sereti, F., Zhuraev, R.M. (2024) *Metric, stratifiable and uniform spaces of G-permutation degree*, Mathematica Slovaca, **74**, 235–248.

[4] Beshimov, R.B., Georgiou, D.N., Juraev, R.M., Manasipova, R.Z., Sereti, F. (2025) *Some topological properties of e-space and description of τ-closed sets*, Filomat, **39**, 2625–2637.

[5] Beshimov, R.B., Juraev, R.M., Manasipova, R.Z. (2025) *On τ-base and e-density of topological spaces*, Filomat, **39**, 3353–3358.

[6] Engelking, R. (1989) *General topology*, Sigma Series in Pure Mathematics, Vol. 6, Heldermann Verlag, Berlin.

[7] Fedorchuk, V.V. (1981) *Covariant functors in the category of compacta, absolute retracts and Q-manifolds*, Uspekhi Matematicheskikh Nauk, **36**, 177–195 (Russian Mathematical Surveys, **36**, 211–233).

[8] Fedorchuk, V.V., Filippov, V.V. (1989) *Topology of hyperspaces and its applications*, Znanie (Mathematics, Cybernetics 4), Moscow, 48 pp. (*in Russian*).

[9] Fedorchuk, V.V., Filippov, V.V. (2006) *General topology: basic constructions*, Fizmatlit, Moscow (*in Russian*).

[10] Franklin, S.P. (1965) *Spaces in which sequences suffice*, Fundamenta Mathematicae, **57**, 107–115.

[11] Franklin, S.P. (1967) *Spaces in which sequences suffice II*, Fundamenta Mathematicae, **61**, 51–56.

[12] Ge, Y. (2007) *Weak forms of open mappings and strong forms of sequence-covering mappings*, Matematicki Vesnik, **59**, 1–8.

[13] Hart, K.P., Nagata, J., Vaughan, J.E. (eds.) (2004) *Encyclopedia of general topology*, Elsevier Science Ltd., Amsterdam.

[14] Kočinac, Lj.D.R., Mukhamadiev, F.G., Sadullaev, A.K. (2022) *Tightness-type properties of the space of permutation degree*, Mathematics, **10**, 3341.

[15] Kočinac, Lj.D.R., Mukhamadiev, F.G., Sadullaev, A.K., Meyliev, Sh.U. (2023) *Some network-type properties of the space of G-permutation degree*, Applied General Topology, **24**, 229–237.

[16] Kočinac, Lj.D.R., Mukhamadiev, F.G., Sadullaev, A.K. (2024) *Some classes of topological spaces and the space of G-permutation degree*, Georgian Mathematical Journal, **31**, 285–291.

[17] Lin, S. (2002) *Point-countable covers and sequence-covering mappings*, Chinese Science Press, Beijing (*in Chinese*).

[18] Toshbuvayev, B.M. (2024) *Acting of covariant functors on mappings*, International Journal of Applied Mathematics, **37**, 47–51.

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Received 06 April 2025
Accepted 15 November 2025