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On the Properties of Hybrid Δ_h Legendre-Laguerre Polynomials and Their Applications

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Abstract. This study delves into the formulation and analysis of a novel class of polynomials, the Δ_h Legendre-Laguerre polynomials, denoted as ${}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u, v, w)$. Through a rigorous investigation, explicit expressions and fundamental properties of these polynomials are derived. Additionally, the study establishes significant interrelations between the Δ_h Legendre-Laguerre polynomials and other well-known polynomial families, thereby enhancing their relevance across diverse mathematical and scientific fields. Furthermore, the monomiality principle and symmetric identities associated with these hybrid special polynomials are systematically explored.

Key Words and Phrases: Δ_h sequences, monomiality principle, explicit forms, determinant form.

2010 Mathematics Subject Classifications: 33E20, 33B10, 33E30, 11T23

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1. Introduction and preliminaries

Special polynomial sequences play a crucial role in modeling and describing the behavior of complex systems across various disciplines, including quantum mechanics and statistical mechanics. These polynomials have also been employed in analyzing intricate systems in fields such as statistics and mathematical physics. Within mathematics, polynomial sequences are essential in areas like algebraic combinatorics, entropy studies, and combinatorial analysis. Notable examples include the Laguerre, Chebyshev, and Jacobi polynomials, which serve as solutions to specific ordinary differential equations in approximation theory and physics. Among these, Laguerre polynomials form a distinguished class of orthogonal polynomials with significant applications in both mathematics and physics. Named after the French mathematician Edmond Laguerre, who introduced them

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in the 19th century, these polynomials arise as solutions to the Laguerre differential equation, a second-order linear differential equation. Laguerre polynomials exhibit several fundamental properties:

They constitute an orthogonal set on $[0, +\infty)$ with respect to the weight function e^{-u}, meaning their weighted inner product equals zero unless they have the same degree.

A recurrence relation allows the computation of higher-degree Laguerre polynomials using lower-degree ones, facilitating efficient polynomial generation and numerical computations.

Their generating function enables the expansion of certain functions into a series of Laguerre polynomials, which aids in solving differential equations and obtaining closed-form solutions. Laguerre polynomials have extensive applications, particularly in physics, where they appear in solutions to the Schrödinger equation for the hydrogen atom and other quantum systems exhibiting spherical symmetry. Additionally, they arise in mathematical and engineering problems involving diffusion equations, wave propagation, and heat conduction.

Polynomial sequences serve as fundamental tools in modeling and analyzing complex system behavior across diverse scientific disciplines, including quantum mechanics and statistical mechanics. These specialized polynomials play a crucial role in studying intricate systems within fields such as mathematical physics, statistical analysis, and combinatorial mathematics. In various branches of mathematics, including algebraic combinatorics, entropy calculations, and combinatorial structures, polynomial sequences are indispensable. Several well-known families, such as Laguerre, Chebyshev, and Jacobi polynomials, emerge as solutions to specific ordinary differential equations, particularly in approximation theory and physics. Among these, Laguerre polynomials hold special significance due to their wide-ranging applications in mathematical and physical sciences.

Recent research in mathematical physics has extensively focused on two-variable special polynomials, which exhibit unique mathematical properties and have diverse applications across various scientific fields. These polynomials are widely explored in algebraic geometry, where they play a crucial role in understanding complex mathematical structures. Notable examples include bivariate Chebyshev, Hermite, and Laguerre polynomials, each possessing distinct characteristics that make them valuable in different domains such as signal processing, numerical analysis, and approximation theory. Bivariate Laguerre polynomials extend the classical Laguerre polynomials to two variables, satisfying a corresponding bivariate Laguerre differential equation. These polynomials have significant applications in quantum mechanics, potential theory, and random matrix theory, especially in modeling systems with two degrees of freedom. A defining feature of these polynomials is their orthogonality with respect to a weight function, which makes them essential in mathematical physics, probability theory, and approximation methods. Their ability to represent and analyze multivariate functions makes them indispensable tools in solving problems across mathematical and scientific disciplines.

Mathematical physics has seen growing interest in two-variable special polynomials, given their unique properties and broad applications [3, 4, 5, 6, 7, 8, 9, 10, 11]. These polynomials, including bivariate Chebyshev, Hermite, and Laguerre polynomials, play a crucial role in signal processing, numerical analysis, and approximation theory.

The introduction of two-variable Legendre $\mathbb{W}_\phi(u,v)$ and Laguerre polynomials $\mathbb{S}_\phi(u,v)$ [12] is particularly intriguing, both for their intrinsic mathematical properties and for their applications in physics. Their development further enriches the framework of multivariate function analysis, offering new perspectives on fundamental mathematical structures.

The generating equation that specifies the 2-variable Laguerre polynomials (2VLP) $\mathbb{W}_\phi(u,v)$:

$$e^{v\xi}J_0(\xi\sqrt{-u}) = \sum_{\phi=0}^{\infty}\mathbb{W}_\phi(u,v)\frac{\xi^\phi}{\phi!}, \tag{1}$$

where $J_0(u\xi)$ is the 0^{th} order ordinary Bessel function of first kind [13] defined by

$$J_\phi(2\sqrt{u}) = \sum_{\nu=0}^{\infty}\frac{(-1)^\nu\ (\sqrt{u})^{\phi+\nu}}{\nu!\ (\phi+\nu)!}. \tag{2}$$

We also note that

$$\exp(-\alpha D_u^{-1}) = J_0(2\sqrt{\alpha u}), \qquad D_u^{-\phi}\{1\} := \frac{u^\phi}{\phi!} \tag{3}$$

is the inverse derivative operator.

Or, alternatively, by

$$e^{v\xi}C_0(-u\xi) = \sum_{\phi=0}^{\infty}\mathbb{W}_\phi(u,v)\frac{\xi^\phi}{\phi!}, \tag{4}$$

where $C_0(u\xi)$ is the 0^{th} order Tricomi function of first kind [13] with

$$C_0(-u\xi) = e^{D_u^{-1}\xi}. \tag{5}$$

Thus, in view of equation (3) or (5), the generating expression for Laguerre polynomials can be written as:

$$e^{v\xi}\ e^{D_u^{-1}\xi} = \sum_{\phi=0}^{\infty}\mathbb{W}_\phi(u,v)\frac{\xi^\phi}{\phi!}. \tag{6}$$

Further, the 2-variable Legendre polynomials (2VLeP) $\mathbb{S}_\phi(u,v)$ are specified by means of the following generating equation:

$$e^{v\xi}J_0(\xi\sqrt{-u})=\sum_{\phi=0}^{\infty}\mathbb{S}_\phi(u,v)\frac{\xi^\phi}{\phi!}, \tag{7}$$

where $J_0(u\xi)$ is the 0^{th} order ordinary Bessel function of first kind [13] given in (5).

Or, alternatively by

$$e^{v\xi}C_0(-u\xi^2)=\sum_{\phi=0}^{\infty}\mathbb{S}_\phi(u,v)\frac{\xi^\phi}{\phi!}, \tag{8}$$

where $C_0(u\xi)$ is the 0^{th} order Tricomi function of first kind [13] given in (5).

Thus, in view of equation (3) or (5), the generating expression for Legendre polynomials can be written as:

$$e^{v\xi}\ e^{D_u^{-1}\xi^2}=\sum_{\phi=0}^{\infty}\mathbb{S}_\phi(u,v)\frac{\xi^\phi}{\phi!}. \tag{9}$$

The monomiality principle is a fundamental concept in polynomial theory, providing a structured framework for expressing and manipulating polynomials. This principle states that any polynomial can be uniquely represented as a linear combination of monomials, where each monomial consists of a single variable raised to a non-negative integer power. This representation simplifies polynomial structures, facilitating their analysis in various mathematical contexts. By decomposing complex polynomial expressions into their monomial components, key properties such as degree, leading coefficient, and roots can be easily determined, offering deeper insights into polynomial behavior and enabling the development of advanced mathematical techniques and algorithms. Beyond its theoretical significance, the monomiality principle is widely applied in computational mathematics, numerical analysis, and engineering. Algorithms for polynomial interpolation, approximation, and numerical integration often leverage the monomial basis to enhance efficiency and accuracy. Similarly, in fields such as signal processing, control theory, and image analysis, polynomials play a crucial role in modeling complex systems and phenomena, with the monomial representation serving as an intuitive and computationally convenient framework. Moreover, in physics, polynomials are extensively used to describe fundamental laws and processes, highlighting the broad applicability of the monomiality principle in both theoretical research and practical problem-solving across scientific and engineering disciplines. The exploration of the monomiality principle, its operational rules, and its applications in hybrid special polynomials has been a focal point in recent studies. The introduction of the poweroid concept further extends the scope of this principle, offering new insights into polynomial structures and their applications in various mathematical and physical domains. Steffenson originally put out the idea of monomiality in 1941 [20]; Dattoli subsequently expanded on this idea [21, 22]. The $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ operators are multiplicative and derivative operators that are crucial in this context for a polynomial set $g_k(u_1)_{k\in\mathbb{N}}$. The following expressions are satisfied by these operators:

$$g_{k+1}(u_1) = \hat{\mathcal{J}}\{g_k(u_1)\} \tag{10}$$

and

$$k\ g_{k-1}(u_1) = \hat{\mathcal{K}}\{g_k(u_1)\}. \tag{11}$$

Hence, applying multiplicative and derivative operations to the polynomial set $\{g_k(u_1)\}_{m\in\mathbb{N}}$ yields a quasi-monomial domain. The following formula plays a crucial role in this quasi-monomial structure:

$$[\hat{\mathcal{K}}, \hat{\mathcal{J}}] = \hat{\mathcal{K}}\hat{\mathcal{J}} - \hat{\mathcal{J}}\hat{\mathcal{K}} = \hat{1}. \tag{12}$$

It exhibits a Weyl group structure.

If the set $\{g_k(u_1)\}_{k\in\mathbb{N}}$ is quasi-monomial, the operators $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ help determine its significance. Consequently, the following axioms hold:

(i) $g_k(u_1)$ gives differential equation

$$\hat{\mathcal{J}}\hat{\mathcal{K}}\{g_k(u_1)\} = k\ g_k(u_1), \tag{13}$$

provided $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ exhibits differential traits.

(ii) The expression

$$g_k(u_1) = \hat{\mathcal{J}}^k\ \{1\}, \tag{14}$$

gives the explicit form, with $g_0(u_1) = 1$.

(iii) Further, the expression

$$e^{w\hat{\mathcal{J}}}\{1\} = \sum_{k=0}^{\infty} g_k(u_1)\frac{w^k}{k!}, \quad |w| < \infty, \tag{15}$$

demonstrates generating expression behavior and is obtained by applying identity (14).

These methodologies, grounded in mathematical physics, quantum mechanics, and classical optics, continue to hold significant relevance in contemporary research. They serve as powerful analytical tools for examining intricate phenomena within these domains, with their validation underscoring their essential contribution to the advancement of scientific knowledge.Very recently, a large interest is shown by mathematicians to introduce Δ_h forms of special polynomials. Some extensions of the special polynomials were studied in [13]. After that, by using the classical finite difference operator Δ_h, a new form of the special polynomials, known as the Δ_h special polynomials of different polynomials is introduced in [1, 2, 17, 18, 19, 23]. These Δ_h special polynomials have been studied because of their remarkable applications not only in different branches of mathematics but also in physics and statistics.

These Δ_h-Appell polynomials are represented as:

$$\mathbb{A}_{\phi}^{[h]}(u) := \mathbb{A}_{\phi}(u), \quad \phi \in \mathbb{N}_0, \tag{16}$$

and defined by

$$\mathbb{A}_{\phi}^{[h]}(u) = \phi h \mathcal{A}_{\phi-1}(u), \quad \phi \in \mathbb{N}_0, \tag{17}$$

where Δ_h is the finite difference operator:

$${}_{\Delta_h}\mathbb{H}^{[h]}(u) = \mathbb{H}(u+h) - \mathbb{H}(u). \tag{18}$$

The Δ_h-Appell polynomials $\mathbb{A}_{\phi}(u)$ are specified by the following generating function [16]:

$$\gamma(\xi)(1+h\xi)^{\frac{u}{h}} = \sum_{\phi=0}^{\infty} \mathbb{A}_{\phi}^{[h]}(u)\frac{\xi^{\phi}}{\phi!}, \tag{19}$$

where

$$\gamma(\xi) = \sum_{\phi=0}^{\infty} \gamma_{\phi,h}\frac{\xi^{\phi}}{\phi!}, \qquad \gamma_{0,h} \neq 0. \tag{20}$$

Motivated by Costabile [16], here we introduced the three variable Δ_h Legendre-Laguerre polynomials:

$$(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} = \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}_{\omega}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!} \tag{21}$$

through the generating function concept.

A generalized falling factorial sum $\sigma_k(\phi; h)$ is defined by means of the following generating function [24]:

$$\frac{(1+h\xi)^{\frac{\phi+1}{h}} - 1}{(1+h\xi)^{\frac{1}{h}} - 1} = \sum_{k=0}^{\infty} \sigma_k(\phi; h)\frac{\xi^k}{k!}. \tag{22}$$

The article is designed as follows: In Section 2, we discuss how the Δ_h Legendre-Laguerre polynomials ${}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)$ are generated and explore recurrence relations that govern their behavior. Section 3 presents formulas for summing or evaluating these Δ_h Legendre-Laguerre polynomials ${}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)$ over certain ranges or with specific constraints. These formulas can be useful for calculating the values of the polynomials efficiently. Section 4 discusses the monomiality principle, which relates to how Δ_h Legendre-Laguerre polynomials ${}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)$ behave under certain operations. The determinant form for these polynomials is also established. In Section 5, symmetric identities for these polynomials are derived, providing key insights into their structural properties. The conclusion section summarizes the main findings of the study while also discussing their implications, applications, and potential directions for future research related to Laguerre-Appell polynomials. Each of these sections offers a deeper exploration of the mathematical properties and characteristics of these polynomials, contributing to a broader understanding of their theoretical and practical significance.

2. Δ_h Legendre-Laguerre polynomials

This section explores a novel class of three-variable Δ_h Legendre-Laguerre polynomials, establishing their key properties and expanding the scope of polynomial theory. It not only enriches existing knowledge but also opens new directions for research and applications.

A central contribution is the construction of the generating function for Δ_h Legendre-Laguerre polynomials, ${}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)$, a vital tool in combinatorics, analysis, and mathematical physics. Generating functions reveal structural insights, facilitating the study of orthogonality, recurrence relations, and special function identities.

By linking these polynomials to their generating function, this research deepens the understanding of polynomial families and their applications. The results presented here highlight distinctive properties, fostering further mathematical and scientific exploration. To establish the generating function, we first prove the following result:

Theorem 1. *For the three variable Δ_h Legendre-Laguerre polynomials ${}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)$, the following generating relation holds true:*

$$(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} = \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}, \tag{23}$$

or equivalently

$$(1+h\xi)^{\frac{v}{h}} C_0\left(\frac{u}{h}\log(1+h\xi)\right) C_0\left(\frac{-w}{h}\log(1+h\xi^2)\right) = \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v)\frac{\xi^\phi}{\phi!}. \quad (24)$$

Proof. Consider the expression $(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}$, where D_u^{-1} and D_w^{-1} denote the inverse difference operators with respect to u and w, respectively.

Thus, expanding each factor about $u=v=w=0$ by means of the Newton series for finite differences, we obtain a formal power series in the variable ξ. After forming the product of these two expansions and collecting like powers of ξ, the resulting series can be written in the form $\sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)\frac{\xi^\phi}{\phi!}$.

By definition, the coefficient of $\frac{\xi^\phi}{\phi!}$ in this expansion is precisely the polynomial ${}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)$ given in (23). Hence, the above identity provides the generating function for the Δ_h Legendre–Laguerre polynomials in three variables, completing the proof. ◄

Theorem 2. *For the three variable Δ_h Legendre-Laguerre polynomials ${}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)$, the following relations hold true:*

$$\frac{{}_v\Delta_h}{h}{}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w) = \phi\ {}_{\mathbb{S}}\mathbb{L}_{\phi-1}^{[h]}(u,v,w)$$
$$\frac{{}_u\Delta_h}{h}{}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w) = \phi(\phi-1)\ {}_{\mathbb{S}}\mathbb{L}_{\phi-2}^{[h]}(u,v,w), \quad D_u^{-1}\to u,\ D_w^{-1}\to w. \quad (25)$$

Proof. By differentiating (23) w.r.t. v and taking into consideration (9), we have

$$\begin{aligned}{}_v\Delta_h(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} &= (1+h\xi)^{\frac{v+h}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} \\ &\quad -(1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}(1+wh\xi^2)^{\frac{D_u^{-1}}{h}} \\ &= (1+h\xi-1)(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} \\ &= h\xi\ (1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}. \quad (26)\end{aligned}$$

On inserting the right-hand side of the equation (23) in (26), we obtain

$${}_v\Delta_h\sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)\frac{\xi^\phi}{\phi!} = h\sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi+1}}{\phi!}. \quad (27)$$

By substituting $\phi \to \phi-1$ in the right-hand side of (26) and equating coefficients of like powers of ξ, we obtain assertion (24).

Next, we derive the explicit form of the three-variable Δ_h Legendre-Laguerre polynomials ${}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)$ by establishing the following result:

Theorem 3. *For the three variable Δ_h Legendre-Laguerre polynomials ${}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)$, the following relations hold true:*

$$ {}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w) = \sum_{d=0}^{[\frac{v}{h}]} \binom{\phi}{d} \binom{\frac{v}{h}}{d} h^d \; {}_{\mathbb{S}}\mathbb{L}_{\phi-d}^{[h]}(u,0,w). \tag{28} $$

Proof. Expand the generating equation (23) as follows:

$$ (1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}\{1\} = \sum_{d=0}^{[\frac{v}{h}]} \binom{\frac{v}{h}}{d} \frac{(h\xi)^d}{d!} \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,0,w)\frac{\xi^\phi}{\phi!}. \tag{29} $$

Then, it can be rewritten as follows:

$$ \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)\frac{\xi^\phi}{\phi!} = \sum_{\phi=0}^{\infty}\sum_{d=0}^{[\frac{v}{h}]} \binom{\frac{v}{h}}{d} h^d \; {}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,0,w)\frac{\xi^{\phi+d}}{\phi! \; d!}. \tag{30} $$

Substituting $n \to \phi - d$ in the right-hand side of the previous expression yields

$$ \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)\frac{\xi^\phi}{\phi!} = \sum_{\phi=0}^{\infty}\sum_{d=0}^{[\frac{v}{h}]} \binom{\frac{v}{h}}{d} h^d \; {}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,0,w)\frac{\xi^{\phi}}{(\phi-d)! \; d!}. \tag{31} $$

Multiplying and dividing by $\phi!$ the right-hand side of (31) and equating the coefficients of like exponents of ξ, we derive (28). ◀

3. Summation formulae

This section establishes summation formulae, also known as sigma notation, which play a crucial role in mathematical analysis. These formulae offer systematic methods for computing sums involving special polynomials of two variables, revealing intricate relationships, hidden symmetries, and structural patterns. Their applications span combinatorics, probability theory, and mathematical physics, aiding in efficient computations and deeper theoretical insights. Essentially, summation formulae serve as fundamental tools for advancing mathematical theory and its applications.

We now demonstrate these formulae by proving the following results:

Theorem 4. *For $\phi \geq 0$, we have*

$$\mathbb{sL}_{\phi}^{[h]}(u,v,w) = \sum_{\gamma=0}^{\phi} \binom{\phi}{\gamma} \left(-\frac{v}{h}\right)_{\gamma} (-h)^{\gamma} \mathbb{sL}_{\phi-\gamma}^{[h]}(u,0,w). \tag{32}$$

Proof. From (23), we have

$$\sum_{\phi=0}^{\infty} \mathbb{sL}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!} = (1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}$$

$$= \sum_{\phi=0}^{\infty} \mathbb{sL}_{\phi}^{[h]}(u,0,w)\frac{\xi^{\phi}}{\phi!}\sum_{\gamma=0}^{\infty}\left(-\frac{v}{h}\right)_{\gamma}(-h)^{\gamma}\frac{\xi^{\gamma}}{\gamma!}$$

$$= \sum_{\phi=0}^{\infty}\left(\sum_{\gamma=0}^{\phi}\binom{\phi}{\gamma}\left(-\frac{v}{h}\right)_{\gamma}(-h)^{\gamma}\mathbb{sL}_{\phi}^{[h]}(u,0,w)\right)\frac{\xi^{\phi}}{\phi!}. \tag{33}$$

Comparing the coefficients of ϕ, we get (32). ◄

Theorem 5. *For $\phi \geq 0$, we have*

$$\mathbb{sL}_{\phi}^{[h]}(u,v+1,w) = \sum_{\gamma=0}^{\phi} \binom{\phi}{\gamma} \left(-\frac{1}{h}\right)_{\gamma} (-h)^{\gamma} \mathbb{sL}_{\phi-\gamma}^{[h]}(u,v,w). \tag{34}$$

Proof. From (23), we have

$$\sum_{\phi=0}^{\infty} \mathbb{sL}_{\phi}^{[h]}(u,v+1,w)\frac{\xi^{\phi}}{\phi!} - \sum_{\phi=0}^{\infty} \mathbb{sL}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!} = (1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}$$

$$\times\left((1+h\xi)^{\frac{1}{h}}-1\right) = \sum_{\phi=0}^{\infty}\mathbb{sL}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}\left(\sum_{\gamma=0}^{\infty}\left(-\frac{1}{h}\right)_{\gamma}(-h)^{\gamma}\frac{\xi^{\gamma}}{\gamma!}-1\right)$$

$$= \sum_{\phi=0}^{\infty}\left(\sum_{\gamma=0}^{\phi}\binom{\phi}{\gamma}\left(-\frac{1}{h}\right)_{\gamma}(-h)^{\gamma}\mathbb{sL}_{\phi-\gamma}^{[h]}(u,v,w)\right)\frac{\xi^{\phi}}{\phi!} - \sum_{\phi=0}^{\infty}\mathbb{sL}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}. \tag{35}$$

Comparing the coefficients of ϕ, we obtain (34). ◄

In view of expression

$$\frac{[\log(1+\xi)]^k}{k!} = \sum_{i=k}^{\infty} S_1(i,k)\frac{\xi^i}{i!}, \quad |\, \xi \,| < 1, \tag{36}$$

it follows that

$$(v)_i = \sum_{k=0}^{i} (-1)^{i-k} S_1(i,k) v^k. \tag{37}$$

We now explore the link between Stirling numbers of the first kind and three-variable Δ_h Legendre-Laguerre polynomials.

Theorem 6. *For $\phi \geq 0$, we have*

$$_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w) = \sum_{\gamma=0}^{\phi} \binom{\phi}{\gamma} {}_{\mathbb{S}}\mathbb{L}_{\phi-\gamma}^{[h]}(u,0,w) \sum_{j=0}^{\gamma} v^j S_1(\gamma,j) h^{\gamma-j}. \tag{38}$$

Proof. From (23), we have

$$\begin{aligned}
\sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}_{n}^{[h]}(u,v,w)\frac{\xi^\phi}{\phi!} &= e^{\frac{v}{h}\log(1+h\xi)}(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} \\
&= (1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} \sum_{j=0}^{\infty} \left(\frac{v}{h}\right)^j \frac{[\log(1+h\xi)]^j}{j!} \\
&= \sum_{\phi=0}^{\infty} \mathbb{SL}_{\phi}^{[h]}(u,0,w)\frac{\xi^\phi}{\phi!} \sum_{\gamma=0}^{\infty}\sum_{j=0}^{\gamma} \left(\frac{v}{h}\right)^j S_1(\gamma,j) h^\gamma \frac{\xi^\gamma}{\gamma!} \\
&= \sum_{\phi=0}^{\infty} \left(\sum_{\gamma=0}^{\phi} \binom{\phi}{\gamma} {}_{\mathbb{S}}\mathbb{L}_{\phi-\gamma}^{[h]}(u,0,w) \sum_{j=0}^{\gamma} \left(\frac{v}{h}\right)^j S_1(\gamma,j) h^\gamma \right) \frac{\xi^\phi}{\phi!}. \qquad (39)
\end{aligned}$$

Comparing the coefficients of ϕ, we obtain (38). ◄

Theorem 7. *For $\phi \geq 0$, we have*

$$_{\mathbb{S}}\mathbb{L}_{n}^{[h]}(u,0,w) = \sum_{\gamma=0}^{\phi} \binom{\phi}{\gamma} {}_{\mathbb{S}}\mathbb{L}_{\phi-\gamma}^{[h]}(u,v,w) \sum_{j=0}^{\gamma} \left(-\frac{v}{h}\right)^j S_1(\gamma,j) h^{\gamma}. \tag{40}$$

Proof. From (23), we have

$$\begin{aligned}
(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} &= e^{-\frac{v}{h}\log(1+h\xi)} \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)\frac{\xi^\phi}{\phi!} \\
&= \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}_{n}^{[h]}(u,v,w)\frac{\xi^\phi}{\phi!} \sum_{j=0}^{\infty} \left(-\frac{v}{h}\right)^j \frac{[\log(1+h\xi)]^j}{j!}
\end{aligned}$$

$$= \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}_{n}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}\sum_{\gamma=0}^{\infty}\sum_{j=0}^{\gamma}\left(-\frac{v}{h}\right)^{j} S_1(\gamma,j)h^{\gamma}\frac{\xi^{\gamma}}{\gamma!}$$

$$= \sum_{\phi=0}^{\infty}\left(\sum_{\gamma=0}^{\phi}\binom{\phi}{\gamma}{}_{\mathbb{S}}\mathbb{L}_{\phi-\gamma}^{[h]}(u,v,w)\sum_{j=0}^{\gamma}\left(-\frac{v}{h}\right)^{j} S_1(\gamma,j)h^{\gamma}\right)\frac{\xi^{\phi}}{\phi!}. \quad (41)$$

Comparing the coefficients of ϕ, we obtain (40). ◄

Theorem 8. *For $\phi \geq 0$, we have*

$${}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w) = \sum_{\gamma=0}^{\phi}\sum_{l=0}^{\gamma}\binom{\phi}{\gamma}(-h)^{\gamma}{}_{\mathbb{S}}\mathbb{L}_{\phi-\gamma}^{[h]}(u,0,w)(1)^{\gamma-l}S_1(\gamma,l)\left(-\frac{v}{h}\right)^{l}. \quad (42)$$

Proof. From (23), we have

$$\sum_{\phi=0}^{\infty}{}_{\mathbb{S}}\mathbb{L}_{n}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!} = (1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}$$

$$= \sum_{\phi=0}^{\infty}{}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,0,w)\frac{\xi^{\phi}}{\phi!}\sum_{\gamma=0}^{\infty}\left(-\frac{v}{h}\right)_{\gamma}(-h)^{\gamma}\frac{\xi^{\gamma}}{\gamma!}$$

$$= \sum_{\phi=0}^{\infty}\left(\sum_{\gamma=0}^{\phi}\binom{\phi}{\gamma}\left(-\frac{v}{h}\right)_{\gamma}(-h)^{\gamma}{}_{\mathbb{S}}\mathbb{L}_{\phi-\gamma}^{[h]}(u,0,w)\right)\frac{\xi^{\phi}}{\phi!}. \quad (43)$$

Comparing the coefficients of ϕ, we get

$${}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w) = \sum_{\gamma=0}^{\phi}\binom{\phi}{\gamma}\left(-\frac{v}{h}\right)_{\gamma}(-h)^{\gamma}{}_{\mathbb{S}}\mathbb{L}_{\phi-\gamma}^{[h]}(u,0,w). \quad (44)$$

Using the above equality (37), we get'

$${}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w) = \sum_{\gamma=0}^{\phi}\sum_{l=0}^{\gamma}\binom{\phi}{\gamma}(-h)^{\gamma}{}_{\mathbb{S}}\mathbb{L}_{\phi-\gamma}^{[h]}(u,0,w)(1)^{\gamma-l}S_1(\gamma,l)\left(-\frac{v}{h}\right)^{l}. \quad (45)$$

This completes the proof of the theorem. ◄

Theorem 9. *For $\phi \geq 0$, we have*

$${}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,s,w) = \sum_{l=0}^{\phi}\sum_{j=0}^{l}\binom{\phi}{l}h^{l}{}_{\mathbb{S}}\mathbb{L}_{\phi-l}^{[h]}(u,v,w)\left(\frac{s-v}{h}\right)^{j}S_1(l,j). \quad (46)$$

Proof. From (23), we have

$$(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} = e^{-\frac{v}{h}\log(1+h\xi)}\sum_{\phi=0}^{\infty}{}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}. \qquad (47)$$

Replacing v by s and comparing the resulting equations, we get

$$e^{\frac{s}{h}\log(1+h\xi)}(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+ht^2)^{\frac{D_w^{-1}}{h}} = e^{\frac{x-v}{h}\log(1+h\xi)}\sum_{\phi=0}^{\infty}{}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}.$$

By using equations (23) and (36) in the the above equation, we get

$$\sum_{\phi=0}^{\infty}{}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,s,w)\frac{\xi^{\phi}}{\phi!} = \sum_{\phi=0}^{\infty}{}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}\sum_{j=0}^{\infty}\left(\frac{s-v}{h}\right)^{j}\frac{[\log(1+h\xi)]^{j}}{j!}$$

$$\sum_{\phi=0}^{\infty}{}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,s,w)\frac{\xi^{\phi}}{\phi!} = \sum_{\phi=0}^{\infty}\sum_{l=0}^{\phi}\sum_{j=0}^{l}\binom{\phi}{l}h^{l}{}_{\mathbb{S}}\mathbb{L}_{\phi-l}^{[h]}(u,v,w)\left(\frac{s-v}{h}\right)^{j}S_1(l,j)\frac{\xi^{\phi}}{\phi!}.$$

Finally, comparing the coefficients of equal powers of ϕ, we come to assertion (46) of Theorem 9. ◄

4. Monomiality Principle

Here, we validate the monomiality property for three-variable Δ_h Legendre-Laguerre polynomials, denoted as ${}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)$. These polynomials form a vital mathematical framework for studying various phenomena. Establishing their monomiality helps uncover fundamental properties governing their behavior and applications.

This section presents our validation results, reinforcing the significance of Δ_h Legendre-Laguerre polynomials. Through rigorous analysis, we confirm the robustness of their monomiality, strengthening their reliability for theoretical and practical applications.

Theorem 10. *The three-variable Δ_h Legendre–Laguerre polynomials ${}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)$ admit explicit multiplicative (raising) and derivative (lowering) operator representations, given as follows:*

$$\hat{M_{\mathbb{S}\mathbb{L}}} = \left(\frac{v-D_u^{-1}}{1+{}_v\Delta_h} + \frac{2\ D_w^{-1}{}_v\Delta_h}{h+{}_v\Delta_h{}^2}\right) \qquad (48)$$

and

$$\hat{D_{\mathbb{S}\mathbb{L}}} = \frac{{}_v\Delta_h}{h}. \qquad (49)$$

Proof. Considering the expression (18) and taking derivatives w.r.t. v of the expression (23), we have

$$
\begin{aligned}
{}_v\Delta_h\Big\{(1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}\Big\} &= (1+h\xi)^{\frac{v+h}{h}}(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} \\
&\quad - (1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} \\
&= (1+h\xi-1)(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} \\
&= h\xi\ (1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}. \quad (50)
\end{aligned}
$$

Thus, we have

$$
\frac{{}_v\Delta_h}{h}\Big[(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}\Big] = \xi\Big[(1+ht)^{\frac{v}{-}D_u^{-1}h}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}\Big], \quad (51)
$$

which gives the identity

$$
\frac{{}_v\Delta_h}{h}\Big[{}_{\mathbb{S}}\mathbb{L}_\phi^{[h]}(u,v,w)\Big] = \xi\Big[{}_{\mathbb{S}}\mathbb{L}_\phi^{[h]}(u,v,w)\Big]. \quad (52)
$$

Now, differentiating expression (23) w.r.t. ξ, we have

$$
\frac{\partial}{\partial\xi}\Big\{(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}\Big\} = \frac{\partial}{\partial\xi}\left\{\sum_{\phi=0}^{\infty}{}_{\mathbb{S}}\mathbb{L}_\phi^{[h]}(u,v,w)\frac{\xi^\phi}{\phi!}\right\}, \quad (53)
$$

$$
\left(\frac{v}{1+h\xi}-\frac{D_u^{-1}}{1+h\xi}+2\frac{D_w^{-1}\xi}{1+h\xi^2}\right)\left\{\sum_{\phi=0}^{\infty}{}_{\mathbb{S}}\mathbb{L}_\phi^{[h]}(u,v,w)\frac{\xi^\phi}{\phi!}\right\} = \sum_{\phi=0}^{\infty}\phi\ {}_{\mathbb{S}}\mathbb{L}_\phi^{[h]}(u,v,w)\frac{\xi^{\phi-1}}{\phi!}. \quad (54)
$$

Using the identity expression (52) and replacing $\phi \to \phi+1$ in the r.h.s. of previous expression (54), we get the validity of (48).

Further, in view of identity expression (52), we have

$$
\frac{{}_v\Delta_h}{h}\Big[{}_{\mathbb{S}}\mathbb{L}_\phi^{[h]}(u,v,w)\Big] = \Big[\phi\ {}_{\mathbb{S}}\mathbb{L}_{\phi-1}^{[h]}(u,v,w)\Big], \quad (55)
$$

which gives expression for the derivative operator (49). ◄

Next, we deduce the differential equation for the three variable Δ_h Legendre-Laguerre polynomials ${}_{\mathbb{S}}\mathbb{L}_n^{[h]}(u,v,w)$ by proving the following result:

Theorem 11. *The three variable Δ_h Legendre-Laguerre polynomials ${}_{\mathbb{S}}\mathbb{L}_\phi^{[h]}(u,v,w)$ satisfy the differential equation*

$$
\left(\frac{v-D_u^{-1}}{1+{}_v\Delta_h}+\frac{2\ D_w^{-1}{}_v\Delta_h}{h+{}_v\Delta_h{}^2}-\frac{\phi h}{{}_v\Delta_h}\right){}_{\mathbb{S}}\mathbb{L}_\phi^{[h]}(u,v,w) = 0. \quad (56)
$$

Proof. Inserting (48) and (49) in the expression (13), we get the validity of (56). ◄

5. Symmetric identities

In this Section, we investigate symmetric identities inherent to the three-variable Δ_h special polynomials. These identities unveil intriguing relationships between the variables and coefficients within the polynomials, shedding light on their underlying symmetrical properties. By exploring how the polynomials behave under transformations that interchange the variables or coefficients, we uncover profound connections that extend beyond their initial definitions. These symmetric identities not only deepen our understanding of the polynomials themselves but also offer valuable insights into broader mathematical structures and phenomena. Through systematic examination and rigorous derivation, we establish a comprehensive framework for understanding and exploiting the symmetrical properties of these two-variable special polynomials, paving the way for further advancements in both theoretical analyses and practical applications.

Theorem 12. *For $a \neq b$, $a, b > 0$ and $u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{C}$, we have*

$$\sum_{\gamma=0}^{\phi} \binom{\phi}{\gamma} a^{\phi-\gamma} b^{\gamma} {}_{\mathbb{S}}\mathbb{L}_{\phi-\gamma}^{[h]}(au_1, av_1, aw_1) {}_{\mathbb{S}}\mathbb{L}_{\gamma}^{[h]}(bu_2, bv_2, bw_2)$$

$$= \sum_{\gamma=0}^{\phi} \binom{\phi}{\gamma} a^{\gamma} b^{\phi-\gamma} {}_{\mathbb{S}}\mathbb{L}_{\phi-\gamma}^{[h]}(au_2, av_2, aw_2) {}_{\mathbb{S}}\mathbb{L}_{\gamma}^{[h]}(bu_1, bv_1, bw_1). \tag{57}$$

Proof. Let

$$A(\xi) = (1+h\xi)^{\frac{ab(v_1+v_2)}{h}} C_0\left(\frac{abu_1}{h}\log(1+h\xi)\right) C_0\left(\frac{abu_2}{h}\log(1+h\xi)\right)$$

$$\times C_0\left(\frac{-abw_1}{h}\log(1+h\xi^2)\right) C_0\left(\frac{-abw_2}{h}\log(1+h\xi^2)\right) \tag{58}$$

$$= \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(bu_1, bv_1, bw_1) \frac{(a\xi)^{\phi}}{\phi!} \sum_{\gamma=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}_{\gamma}^{[h]}(au_2, av_2, aw_2) \frac{(b\xi)^{\gamma}}{\gamma!}$$

$$= \sum_{\phi=0}^{\infty} \left(\sum_{\gamma=0}^{\phi} \binom{\phi}{\gamma} a^{\phi-\gamma} b^{\gamma} {}_{\mathbb{S}}\mathbb{L}_{\phi-\gamma}^{[h]}(au_1, av_1, aw_1) {}_{\mathbb{S}}\mathbb{L}_{\gamma}^{[h]}(bu_2, bv_2, bw_2) \right) \frac{\xi^{\phi}}{\phi!}. \tag{59}$$

Similarly, we have

$$A(\xi)=\sum_{\phi=0}^{\infty}\left(=\sum_{\gamma=0}^{\phi}\binom{\phi}{\gamma}a^{\gamma}b^{\phi-\gamma}{}_{\mathbb{S}}\mathbb{L}_{\phi-\gamma}^{[h]}(au_2,av_2,aw_2){}_{\mathbb{S}}\mathbb{L}_{\gamma}^{[h]}(bu_1,bv_1,bw_1)\right)\frac{\xi^{\phi}}{\phi!}. \tag{60}$$

Comparing the coefficients of ξ on both sides of last equations, we get (57). ◄

Theorem 13. *For $a\neq b$, $a,b>0$ and $u,v,w\in\mathbb{C}$, we have*

$$\sum_{k=0}^{\phi}\sum_{\gamma=0}^{k}\binom{\phi}{k}\binom{k}{\gamma}a^{\phi-\gamma}b^{\gamma+1}\beta_{\phi-k}(h){}_{\mathbb{S}}\mathbb{L}_{k-m}^{[h]}(bu,bv,bw)\sigma_{\gamma}(a-1;h)$$

$$=\sum_{k=0}^{\phi}\sum_{\gamma=0}^{k}\binom{\phi}{k}\binom{k}{\gamma}b^{\phi-\gamma}a^{\gamma+1}\beta_{\phi-k}(h){}_{\mathbb{S}}\mathbb{L}_{k-\gamma}^{[h]}(au,av,aw)\sigma_{\gamma}(b-1;h). \tag{61}$$

Proof. Consider

$$B(\xi)=\frac{ab\xi(1+h\xi)^{\frac{abv}{h}}C_0\left(\frac{abu}{h}\log(1+h\xi)\right)C_0\left(\frac{-abw}{h}\log(1+h\xi^2)\right)\left((1+h\xi)^{\frac{ab}{h}}-1\right)}{((1+h\xi)^{\frac{a}{h}}-1)((1+h\xi)^{\frac{b}{h}}-1)}$$

$$=\frac{ab\xi}{((1+h\xi)^{\frac{a}{h}}-1)}(1+h\xi)^{\frac{abv}{h}}C_0\left(\frac{abu}{h}\log(1+h\xi)\right)C_0\left(\frac{-abw}{h}\log(1+h\xi^2)\right)$$

$$\times\frac{((1+h\xi)^{\frac{ab}{h}}-1)}{((1+h\xi)^{\frac{b}{h}}-1)}=b\sum_{\phi=0}^{\infty}\beta_{\phi}(h)\frac{(a\xi)^{\phi}}{\phi!}\sum_{k=0}^{\infty}{}_{\mathbb{S}}\mathbb{L}_{k}^{[h]}(bu,bv,bw)\frac{(a\xi)^{k}}{k!}\sum_{\gamma=0}^{\infty}\sigma_{\gamma}(a-1;h)\frac{(b\xi)^{\gamma}}{\gamma!}$$

$$=b\sum_{\phi=0}^{\infty}\beta_{\phi}(h)\frac{(a\xi)^{\phi}}{\phi!}\sum_{k=0}^{\infty}\sum_{\gamma=0}^{k}\binom{k}{\gamma}a^{k-\gamma}b^{\gamma}\mathbb{S}_{k-\gamma}^{[h]}(bu,bv,bw)\sigma_{\gamma}(a-1;h)\frac{\xi^{k}}{k!}$$

$$=\sum_{\phi=0}^{\infty}\left(\sum_{k=0}^{\phi}\sum_{\gamma=0}^{k}\binom{\phi}{k}\binom{k}{\gamma}a^{\phi-\gamma}b^{\gamma+1}\beta_{\phi-k}(h){}_{\mathbb{S}}\mathbb{L}_{k-m}^{[h]}(bu,bv,bw)\sigma_{\gamma}(a-1;h)\right)\frac{\xi^{\phi}}{\phi!}. \tag{62}$$

Similarly, we have

$$B(\xi)=\sum_{\phi=0}^{\infty}\left(\sum_{k=0}^{\phi}\sum_{\gamma=0}^{k}\binom{\phi}{k}\binom{k}{\gamma}b^{\phi-\gamma}a^{\gamma+1}\beta_{\phi-k}(h){}_{\mathbb{S}}\mathbb{L}_{k-\gamma}^{[h]}(au,av,aw)\sigma_{\gamma}(b-1;h)\right)\frac{\xi^{\phi}}{\phi!}. \tag{63}$$

Comparing the coefficients of ξ on both sides of last equations, we get (61). ◄

6. Conclusion

The introduction of Δ_h Legendre-Laguerre polynomials marks a significant step forward in polynomial theory, particularly in quantum mechanics and entropy modeling. By leveraging the monomiality principle and operational techniques, these polynomials reveal novel mathematical structures and insights.

This research provides explicit formulas and fundamental properties, linking Δ_h Legendre-Laguerre polynomials with established polynomial families, thus enriching mathematical theory. Future studies could further investigate their structural and algebraic aspects, extending their applications in quantum mechanics, statistical mechanics, and computational science.

The interdisciplinary potential of these polynomials highlights opportunities for collaboration across scientific domains, paving the way for deeper exploration and real-world applications.

References

[1] Alam, N., Wani, S.A., Khan, W.A., Zaidi, H.N. (2024) *Investigating the properties and dynamic applications of Δ_h Legendre–Appell polynomials*, Mathematics, **12**, Article 1973.

[2] Alam, N., Wani, S.A., Khan, W.A., Gassem, F., Altaleb, A. (2024) *Exploring properties and applications of Laguerre special polynomials involving the Δ_h form*, Symmetry, **16**, Article 1154.

[3] Wani, S.A., Khan, S. (2019) *Properties and applications of the Gould–Hopper–Frobenius–Euler polynomials*, Tbilisi Mathematical Journal, **12(1)**, 93–104.

[4] Khan, S., Riyasat, M., Wani, S.A., (2017) *On some classes of differential equations and associated integral equations for the Laguerre–Appell polynomials*, Advances in Pure and Applied Mathematics, **9(3)**.

[5] Ramírez, W., Cesarano, C. (2022) *Some new classes of degenerated generalized Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials*, Carpathian Mathematical Publications, **14**, 354–363.

[6] Zayed, M., Wani, S.A. (2023) *A study on generalized degenerate form of 2D Appell polynomials via fractional operators*, Fractal and Fractional, **7(10)**, Article 723.

[7] Zayed, M., Wani, S.A., Quintana, Y. (2023) *Properties of multivariate Hermite polynomials in correlation with Frobenius–Euler polynomials*, Mathematics, **11(16)**, Article 3439.

[8] Wani, S.A. (2024) *Two-iterated degenerate Appell polynomials: properties and applications*, Arab Journal of Basic and Applied Sciences, **31(1)**, 83–92.

[9] Wani, S.A., Abuasbeh, K., Oros, G.I., Trabelsi, S. (2023) *Studies on special polynomials involving degenerate Appell polynomials and fractional derivative*, Symmetry, **15(4)**, Article 840.

[10] Dattoli, G., Ricci, P.E., Cesarano, C., Vázquez, L. (2003) *Special polynomials and fractional calculus*, Mathematical and Computer Modelling, **37**, 729–733.

[11] Dattoli, G., Lorenzutta, S., Mancho, A.M., Torre, A. (1999) *Generalized polynomials and associated operational identities*, Journal of Computational and Applied Mathematics, **108**, 209–218.

[12] Dattoli, G., Ricci, P.E. (2001) *A note on Laguerre polynomials*, International Journal of Nonlinear Sciences and Numerical Simulation, 365–370.

[13] Andrews, L.C. (1985) *Special functions for engineers and applied mathematicians*, Macmillan Publishing Company, New York.

[14] Alatawi, M.S., Khan, W.A. (2022) *New type of degenerate Changhee–Genocchi polynomials*, Axioms, **11**, Article 355.

[15] Khan, W.A., Alatawi, M.S. (2022) *Analytical properties of degenerate Genocchi polynomials of the second kind and some of their applications*, Symmetry, **14**, Article 1500.

[16] Costabile, F.A., Longo, E. (2013) *Δ_h-Appell sequences and related interpolation problem*, Numerical Algorithms, **63**, 165–186.

[17] Wani, S. A., Alazman, I., Alkahtani, B. (2023) *Certain properties and applications of convoluted Δ_h multivariate Hermite and Appell sequences*, Symmetry, **15**, Article 828.

[18] Alazman, I., Alkahtani, B., Wani, S.A. (2023) *Certain properties of Δ_h multivariate Hermite polynomials*, Symmetry, **15**, Article 839.

[19] Alyusof, R., Wani, S.A. (2023) *Certain properties and applications of Δ_h hybrid special polynomials associated with Appell sequences*, Fractal and Fractional, **7**, Article 233.

[20] Steffensen, J.F. (1941) *The poweroid, an extension of the mathematical notion of power*, Acta Mathematica, **73**, 333–366.

[21] Dattoli, G. (2000) *Hermite–Bessel and Laguerre–Bessel functions: a by-product of the monomiality principle*, In *Advanced Special Functions and Applications*, Proc. Melfi School, Aracne, Rome, 147–164.

[22] Dattoli, G. (2000) *Generalized polynomials, operational identities and their applications*, Journal of Computational and Applied Mathematics, **118**, 111–123.

[23] Carlitz, L. (1959) *Eulerian numbers and polynomials*, Mathematics Magazine, **32**, 247–260.

[24] Young, P.T. (2008) *Degenerate Bernoulli polynomials, generalized factorial sums and their applications*, Journal of Number Theory, **128(4)**, 738–758.

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